Tensoring Unranked Effects

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Introduction

- Monads encapsulate generic side-effects
  - functional-imperative programming
  - programming semantics
  - program logics
- Monads can be combined:
  - Sum: disjoint union of theories
  - Tensor: commuting union of theories (orthogonal effects)

Introduction (cont'd)

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- In the unranked case, existence of tensors is often unclear
  - E.g., Hyland et al. (2007) prove that tensors of continuations with ranked monads exist
- We prove existence of tensors for uniform monads, which includes
  - Powerset (full or non-empty)
  - Continuations
- Analyse powerset in more detail:
  - Tensor = sets of terms modulo rectangularity
  - Order-theoretic variant of the construction
  - Characterize monads for which tensoring with powerset is conservative

Monads for Computational Effects

Strong monad $\mathbb{T} = (T, \eta, \_^*, t)$:

- TX type of computations over X
- $\eta : X \to TX$ converts values into computations
- $\_^*$ lifts $f : X \to TY$ to $f^* : TX \to TY$
  - Sequential composition $g; f^*$
- $t : X \times TY \to T(X \times Y)$ propagates context

Supports computational metalanguage:

- $do\ x \leftarrow p; q$ sequential composition (uses $\_^*$ and t)
- ret returns a value (η)

Monads: Examples

- State: $TX = S \to (S \times X)$
- Input: $TX = \mu Y.(X + I \to Y)$
- Output: $TX = \mu Y.(X + O \times Y)$
- Non-determinism: $TX = \mathcal{P}(X)$
- Probabilistic non-determinism: $TX = D(X)$
- Continuation passing: $TX = (X \to R) \to R$.

Definition A large Lawvere theory is a category L equipped with an identity-on-objects functor $I : \mathbf{Set}^{op} \to L$ (written $Ie = [e]$) that preserves products.

Large Lawvere theories are equivalent to monads on **Set**:

- L large Lawvere theory $\rightsquigarrow T_L(X) = L(X, 1)$ monad
- T monad $\rightsquigarrow (\mathrm{Kl}(T))^{op}$ large Lawvere theory

Lawvere Theories

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- Objects are sets (think: of variables)
- Morphisms are tuples of terms/ substitutions
- Sums of sets are products
  - for $\kappa_i : 1 \to n$, $\kappa_i(*) = i$: $[\kappa_i] : n \to 1$ projection
  - $n \times k$ is the n-th power of k
    - for $f : k \to l$, have $n \otimes f : n \times k \to n \times l$

Tensors of Lawvere Theories

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Tensor $L_1 \otimes L_2$ is universal w.r.t. having theory morphisms

$$L_1 \to L_1 \otimes L_2 \leftarrow L_2$$

that commute, i.e. (eliding theory morphisms)

$$\begin{array}{ccc}
n_1 \times n_2 & \xrightarrow{n_1 \otimes f_2} & n_1 \times m_2 \\
{\scriptstyle f_1 \otimes n_2}\downarrow & & \downarrow{\scriptstyle f_1 \otimes m_2} \\
m_1 \times n_2 & \xrightarrow{m_1 \otimes f_2} & m_1 \times m_2
\end{array}$$

where $f_1 : n_1 \to m_1$ in L_1, $f_2 : n_2 \to m_2$ in L_2. E.g.

$$\mathrm{lookup}_l(x1 + y1, x2 + y2, x3 + y3) = \mathrm{lookup}_l(x1, x2, x3) + \mathrm{lookup}_l(y1, y2, y3)$$

if there are three values.

Tensors of Monads

…are the same: $T_1 \otimes T_2$ is universal w.r.t. to having monad morphisms $T_1 \to T_1 \otimes T_2 \leftarrow T_1$ that commute; this can now be written as

$$do\ x \leftarrow f_1; y \leftarrow f_2; ret\,(x, y) = do\ y \leftarrow f_2; x \leftarrow f_1; ret\,(x, y)$$

for $f_1 : T_1 X$, $f_2 : T_2 Y$.

Existence of Tensors

- Existence is a size issue: TX / $L(n, m)$ must be sets
- Obviously, the tensor exists if the sum exists
  - the sum, however, need not exist for unranked monads
- Hyland et al. (2007) show that tensors of continuations with ranked monads exist
- Hyland et al. (2006) show that tensors with the state monad always exist

Uniform Lawvere Theories

$\mathsf{c}_L^n : n \to n + c = [id] \times \prod_{f \in c} f$ where $c = hom_L(0,1)$.

Definition A Lawvere theory L is uniform if for every L-morphism $f : n \to m$, there exist a morphism $\hat{f} : k \to 1$ and a set-function $\mathsf{u}_f : k \times m \to n + c$ such that $f = (\hat{f} \otimes m) \circ [\mathsf{u}_f] \circ \mathsf{c}_L^n$.

Theorem The tensor of two Lawvere theories exists if one of them is uniform.

PROOF: Uniformity allows sorting the operations of the uniform theory to the top.

Uniform Theories: Examples

- Non-empty Powerset: m non-empty subsets of n can be obtained from one set (e.g. n itself) by renaming elements.
- Full Powerset: m subsets of n can be obtained from one set (e.g. n itself) by either renaming elements or substituting them by $\emptyset$.
- Continuations: m functions $(n \to R) \to R$ can be obtained from one function $((n + \log_R(m)) \to R) \to R$ by substituting elements of $n + \log_R(m)$ with either elements of n or constants $r \in R$.

Tensoring with Non-empty Powerset

- By the general construction, morphisms $n \to m$ in $L \otimes L_{\mathcal{P}_1}$ are m-tuples of non-empty subsets of $L(n,1)$, modulo something.
- Can improve this to non-empty subsets of $L(n,m)$ modulo something via cartesian product map

$$\mathcal{P}(L(n,1))^m \to \mathcal{P}(L(n,1)^m) \cong \mathcal{P}(L(n,m))$$

  – by AC, this is injective!
- 'something' is rectangular equivalence $\approx_1$, the smallest preorder closed under

$$(\pi) \quad \frac{\forall i.\, [\kappa_i]A \approx_1 [\kappa_i]B}{CA \approx_1 CB}$$

Pointed Theories

Definition L is pointed if L is a pointed category with point $\bot$, and $L(0,1) = \{\bot\}$.

Pointed theories carry a canonical preordering $\sqsubseteq$ defined as the smallest preorder with bottom element $\bot$ and closed under

$$(\pi_{\sqsubseteq})\quad \frac{\forall i.\,[\kappa_i]\circ f \sqsubseteq [\kappa_i]\circ g}{hf \sqsubseteq hg}$$

N.B.: L pointed iff $L \cong L \otimes L_{\bot}$

- $L_{\mathcal{P}}$ is pointed, with points $\lambda.\emptyset$
- Lists are pointed
- Partial state $S \to (S \times X)_{\bot}$ is pointed

Tensoring with Full Powerset

- Clearly, tensoring L with $L_{\mathcal{P}}$ can be conservative only if $L \to L \otimes L_{\perp}$ is conservative; hence assume L is already pointed.
- $L \otimes L_{\mathcal{P}}$ is sets of L-morphisms modulo $\approx_0$ = rectangular equivalence plus $\{\perp\} \approx_0 \emptyset$.
- Have $A \approx_0 B$ iff $\mathrm{cl}(A) = \mathrm{cl}(B)$, where $\mathrm{cl}(A)$ is the smallest downclosed superset of A closed under

$$(\Delta)\quad \frac{\forall i.\, g\Delta_i h \in \mathrm{cl}(A)}{gh \in \mathrm{cl}(A)}$$

with $\Delta_i = \prod \delta_{ij}$, $\delta_{ij} = id$ if $i = j$, $= \perp$ otherwise.

Conservativity

$L \to L \otimes L_{\mathcal{P}}$ preserves the canonical preorder. Note

$$fg = \bigsqcup_i f \Delta_i g \qquad (1)$$

in the tensor.

Theorem a) $L \to L \otimes L_{\mathcal{P}}$ reflects the preordering iff (1) holds in L
b) Under a), $L \to L \otimes L_{\mathcal{P}}$ is injective iff $\sqsubseteq$ is antisymmetric.

Conclusions

- Tensors with uniform monads always exist
- This improves on previous results, in particular for continuations
- Tensoring with powerset has a simple description
  - Monad transformer for non-determinism
- Simple order-theoretic conservativity criterion for pointed case
- Can then use non-deterministic arguments for deterministic effects
- E.g. Fischer/Ladner encoding

$$\begin{aligned} \texttt{if}\ b\ \texttt{then}\ p\ \texttt{else}\ q &\mapsto b?; p + (\neg b)?; q \\ \texttt{while}\ b\ \texttt{do}\ p &\mapsto (b?; p)^\star; (\neg b)? \end{aligned}$$

Future Work

- How much of this works over toposes/domain categories?
  - Existence of tensors for uniform theories probably does work over toposes
- Verification logics for tensors of effects