Universitat Politècnica de Catalunya
Departament de Llenguatges i Sistemes Informàtics

Satisfiability of Graph Constraints

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What is a Graph Constraint?

A graph constraint (Ehrig, Habel, Heckel, Penneman, Taentzar, ...) is the description of a pattern that must be (must not be) present on a given graph.

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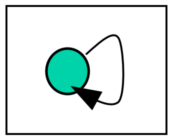
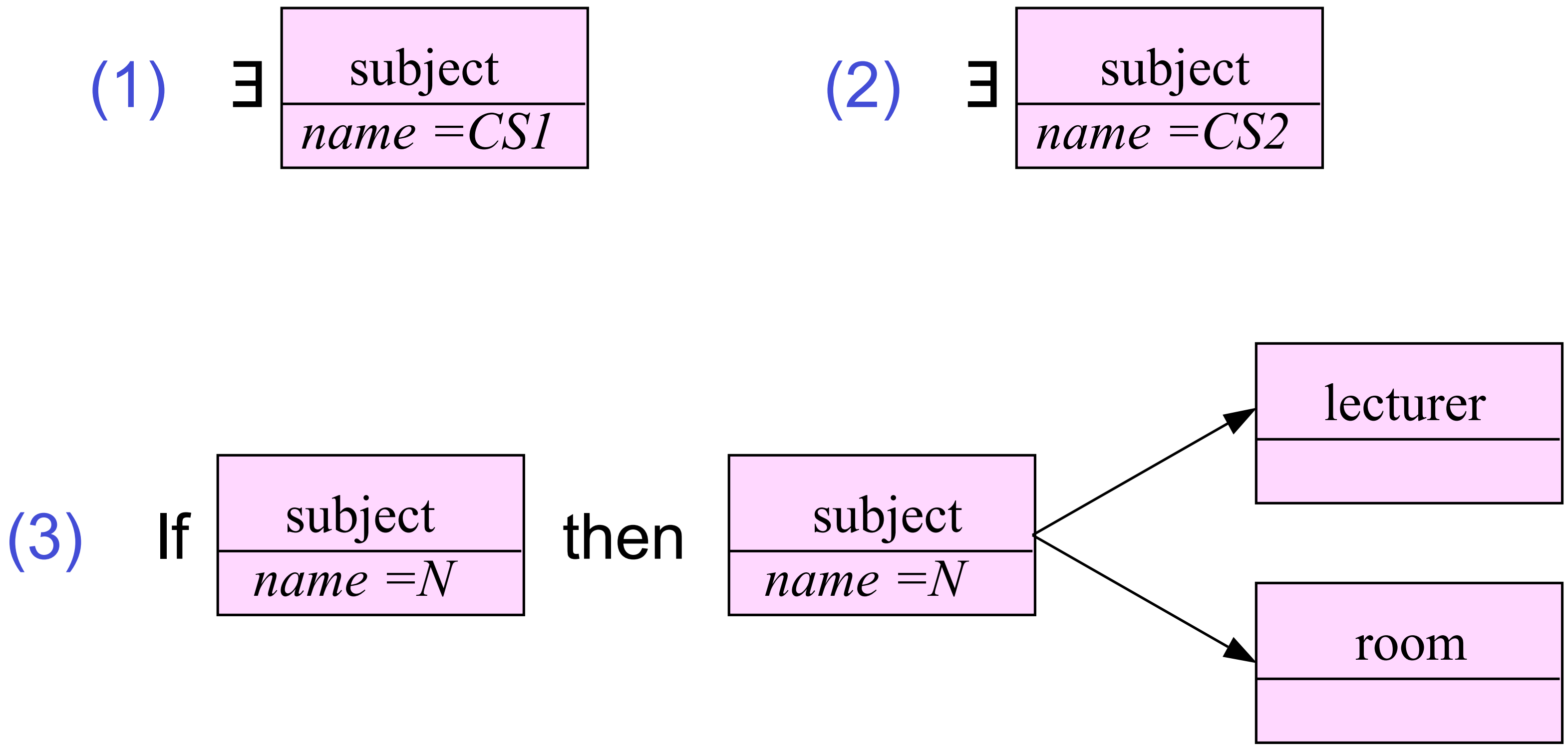
For instance:

$\neg\exists$ 

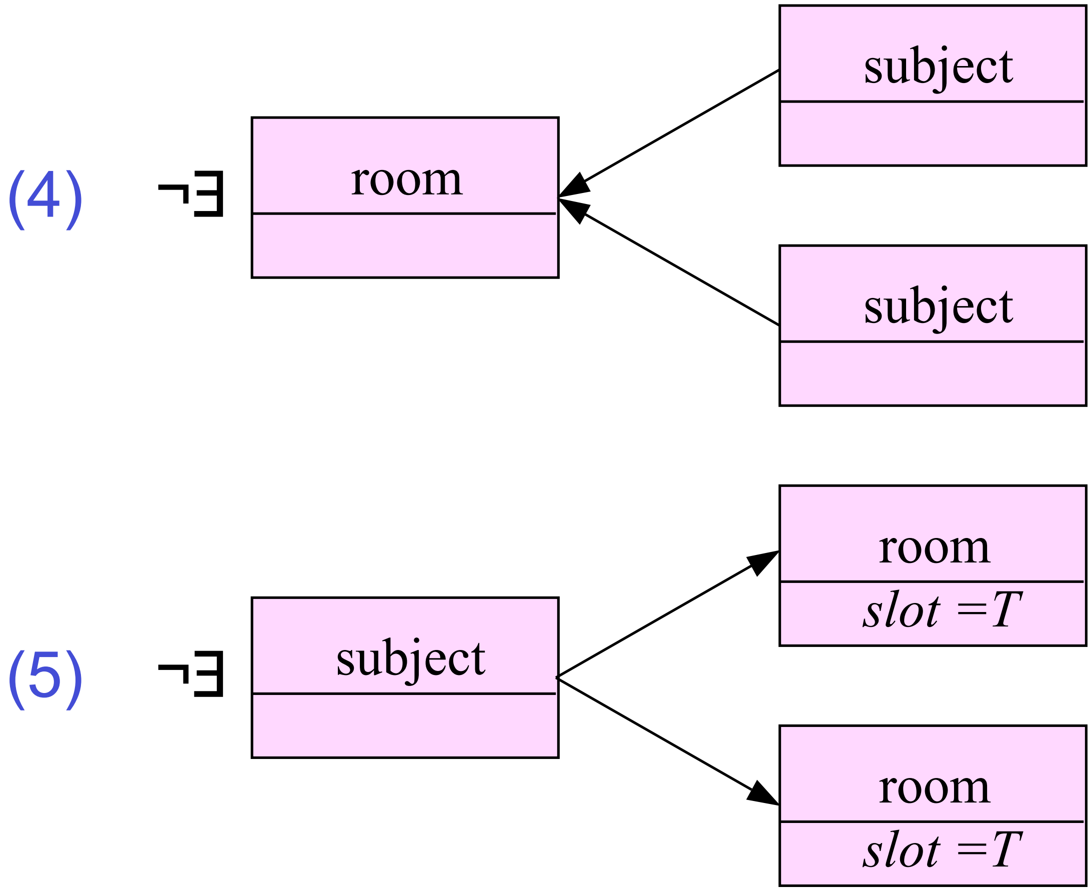
Motivation

- Specification and validation of XML-like classes of documents
- Specification and validation of graph-like models

An example

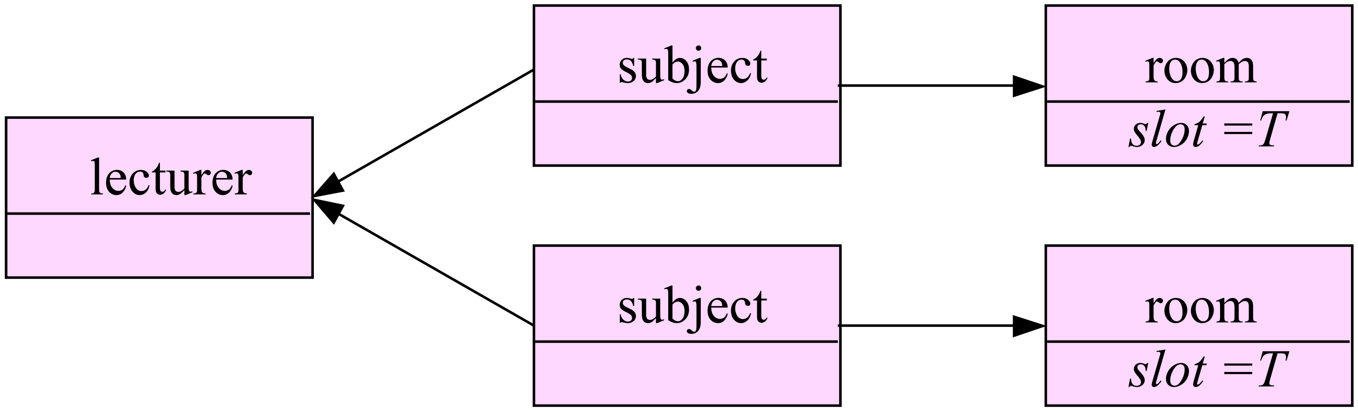

(1) ∃
subject
name =CS1
(2) ∃
subject
name =CS2
(3) If
subject
name =N
then
subject
name =N
lecturer
room

An example

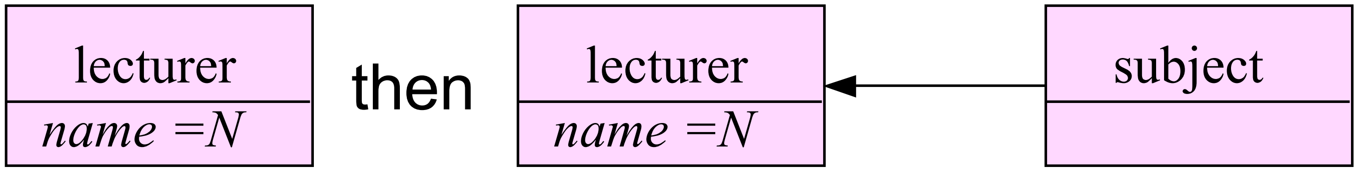


An example

(6) ¬∃



(7) ¬If



The problem:

Given a set of graph constraints $\mathcal{C}$, does it exist a graph G that satisfies $\mathcal{C}$?

What constraints?

Basic constraints:

- $\exists C$

Atomic constraints:

- $\forall (c{:}X \rightarrow C)$

What constraints?

Plus ¬, ∨

Basic constraints:

- $\exists C$

Atomic constraints:

- $\forall(c{:}X \rightarrow C)$

What constraints?

Plus ¬, ∨

Basic constraints:

- ∃C

Atomic constraints:

- ∀(c:X → C)

If c is a simple (resp. basic) then ¬c is called a negative constraint and c a positive constraint

Satisfaction

Basic constraints:

$G \models \exists C$ if there is a monomorphism $h:C \to G$

Atomic constraints

$G \models \forall(g:X \to C)$ if for every monomorphism $h:X \to G$, there exists a monomorphism $f:C \to G$ such that $f \cdot c = h$.

Satisfaction

Basic constraints:

$G \models \exists C$ if there is a monomorphism $h:C \to G$

Atomic constraints

$G \models \forall(g:X \to C)$ if for every monomorphism $h:X \to G$, there exists a monomorphism $f:C \to G$ such that $f \cdot c = h$.

G is a finite graph?

Basic Assumption

In the following, we assume that we want to know if a given set of positive and negative basic (resp. atomic, nested) constraints is satisfiable.

Refutation

Given a set of constraints $\mathcal{C}$ and a set of inference rules a refutation procedure is a sequence:

$$\mathcal{C} = \mathcal{C}_1 \Rightarrow \mathcal{C}_2 \Rightarrow \ldots \Rightarrow \mathcal{C}_k \Rightarrow \ldots$$

such that, $\mathcal{C}$ just includes the true clause (i.e. $\mathcal{C} = \{\exists\emptyset\}$), and for every i, $\mathcal{C}_{i+1}$ is obtained from $\mathcal{C}_i$ by the application of a rule from the given set and $\mathcal{C}_{i+1} \neq \mathcal{C}_i$.

A refutation procedure is fair if every inference that can by applied at a given moment is eventually applied.

A procedure is sound if whenever it generates a false constraint this implies that $\mathcal{C}$ is unsatisfiable

Refutation

To prove soundness it is enough to prove that the rules are sound. This means that if C_2 is obtained from C_1 by the application of a rule then:

$$G \models C_1 \text{ implies that } G \models C_2$$

A refutation procedure is (refutationally) complete if whenever C is unsatisfiable it generates a false constraint.

Inference rules for basic constraints

1)

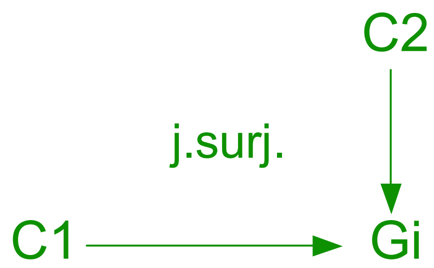
$$\frac{\exists C1 \vee \Gamma 1 \qquad \neg \exists C2}{\Gamma 1}$$

If there exists a monomorphism $C2 \rightarrow C1$

2)

$$\frac{\exists C1 \vee \Gamma 1 \qquad \exists C2}{\exists G1 \vee \ldots \vee \exists Gk \vee \Gamma 1}$$

If there are no monomorphisms $C2 \rightarrow C1$



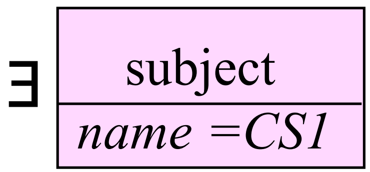
Example

(1) ∃ subject
name =CS1

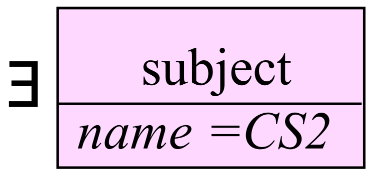
(2) ∃ subject
name =CS2

Example

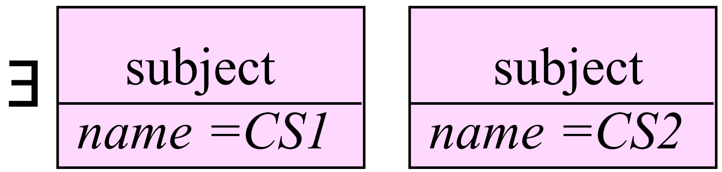
(1)


∃
subject
name =CS1

(2)


∃
subject
name =CS2

(8)

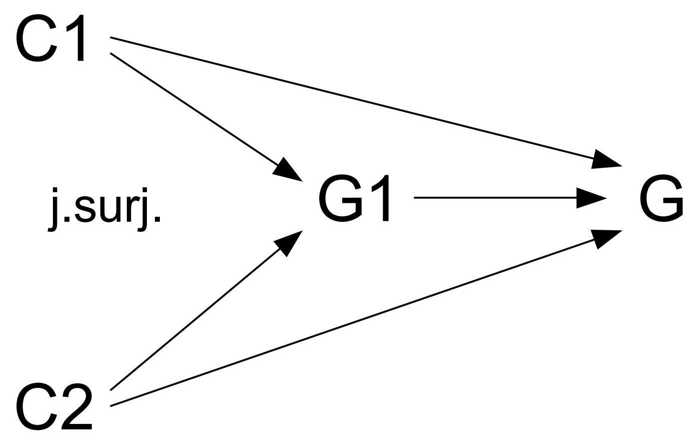

∃
subject
name =CS1
subject
name =CS2

Soundness of the inference rules

2) Suppose that

$$G \models \exists C1 \vee \Gamma 1 \text{ and } G \models \exists C2$$

The case $G \models \Gamma 1$ is trivial. If $G \models \exists C1$ there should be G1:



Then $G \models \exists G1$

Completeness of the inference rules

Each inference using rule 2) can be seen as a step in the construction of a model satisfying the given constraints.

Rule 1) "eliminates" invalid constructions. At the end, either we have "eliminated" all the constructions, i.e. we have generated the empty clause or we have a valid minimal model,

Soundness, completeness and termination

- A refutation procedure always terminates.

- A set of constraints C is unsatifiable iff a refutation procedure generates the empty clause.

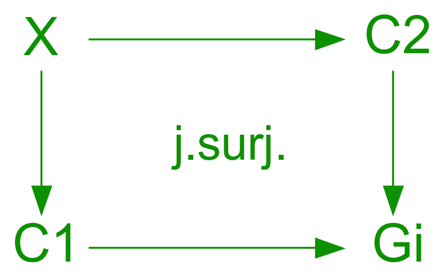
- A set of constraints C is satifiable iff a refutation procedure does not generate the empty clause.

Rules for basic and positive atomic constraints

3)

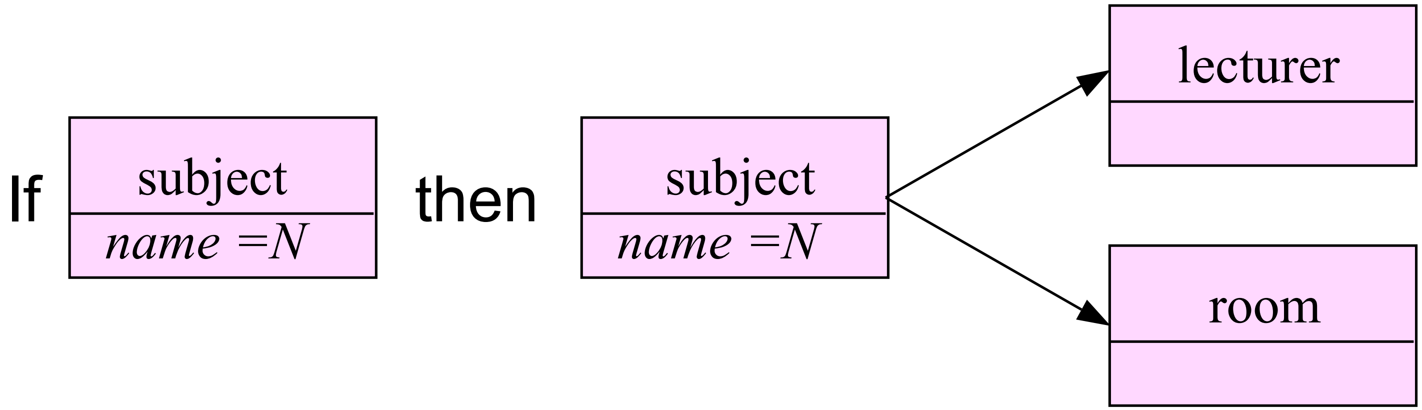
$$\frac{\exists C1 \vee \Gamma \qquad \forall(g':X \rightarrow C2)}{\exists G1 \vee \ldots \vee \exists Gk \vee \Gamma}$$

If X → C1 and:

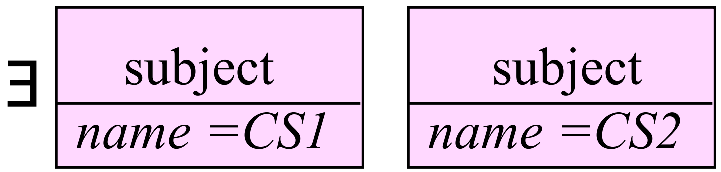


Example

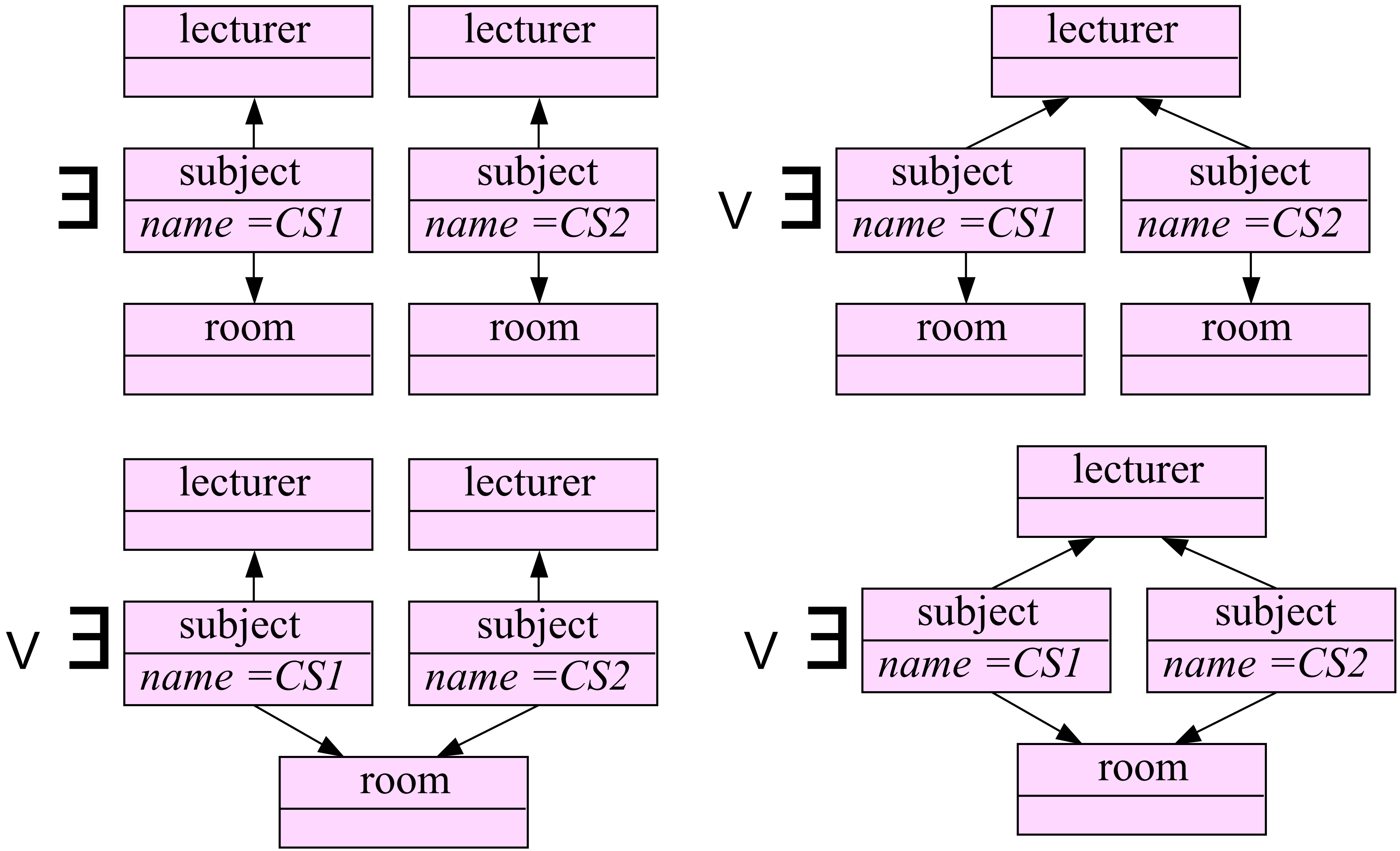
(3)


If
subject
name =N
then
subject
name =N
lecturer
room

(8)


∃
subject
name =CS1
subject
name =CS2

Example


∃
lecturer
subject
name =CS1
room
lecturer
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∨ ∃
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name =CS2
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∨ ∃
lecturer
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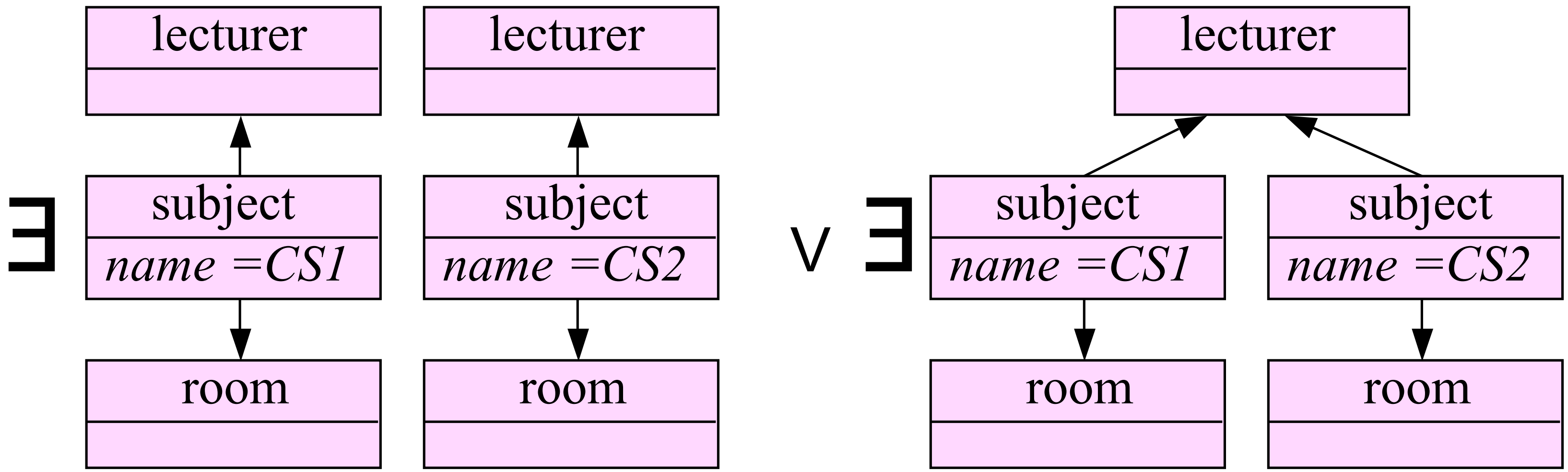
∨ ∃
lecturer
subject
name =CS1
subject
name =CS2
room

Example


∃
lecturer
subject
name =CS1
room
lecturer
subject
name =CS2
room
∨ ∃
lecturer
subject
name =CS1
room
subject
name =CS2
room

Completeness of the inference rules

The construction of a valid model is similar to the previous case, but now it may not end. But now we may have the following situation:

$$G1 \rightarrow G2 \rightarrow G3 \rightarrow .... \rightarrow Gn \rightarrow ...$$

Completeness of the inference rules

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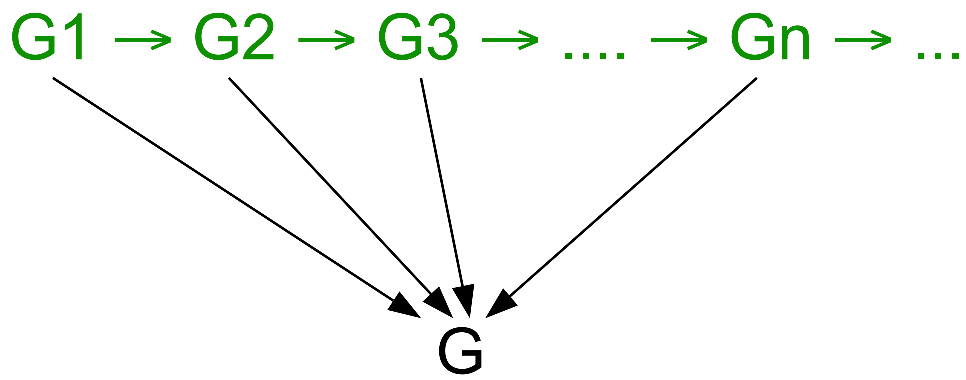
$$G1 \rightarrow G2 \rightarrow G3 \rightarrow .... \rightarrow Gn \rightarrow ...$$

In this case the model of the constraints would be the colimit:



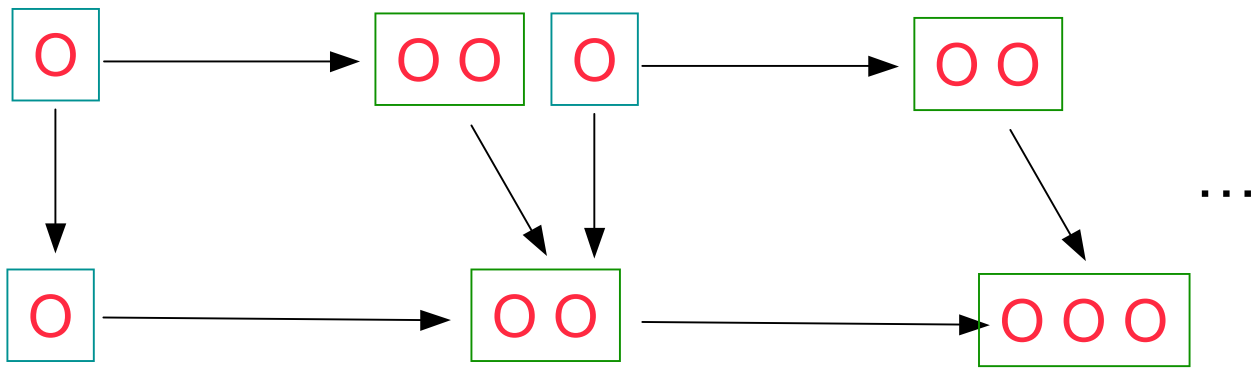
Soundness, completeness and termination

- A refutation procedure may not terminate.

- A set of constraints C is unsatifiable iff a refutation procedure generates the empty clause.

(Non) Termination

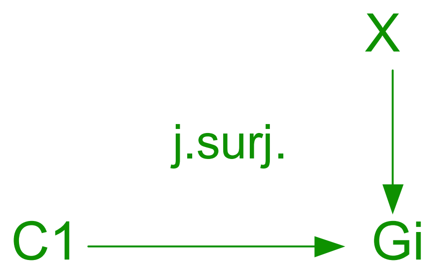
- Given $\exists$(O) $\vee$ Γ and $\forall$(O $\rightarrow$ O O), for instance, we may have the following situation:



Inference rules for atomic constraints

4)

$$\frac{\exists C1 \vee \Gamma 1 \qquad \neg\forall(X \rightarrow C2) = c}{\exists G1 \vee \ldots \vee \exists Gk \vee \Gamma 1}$$



Does not work: we cannot ensure that, in the limit, G satisfies

$$\neg\forall(X \rightarrow C2) = c$$

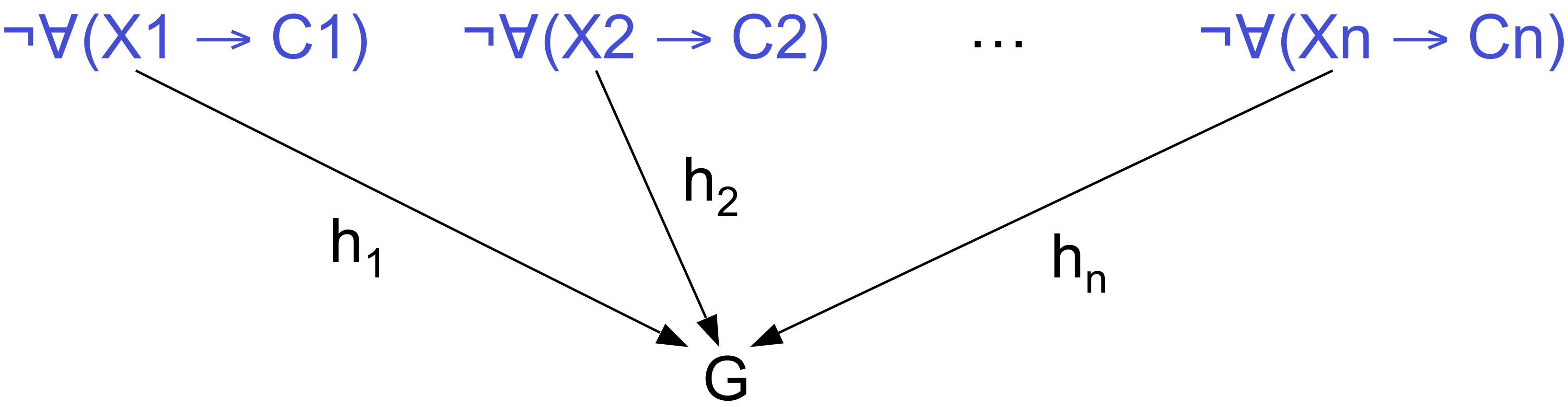
Inference rules for atomic constraints

Moreover, we would need some addicional inference rule to generate the empty clause, for instance, when we have the following constraints

(1) $\forall(X \rightarrow C)$ (2) $\neg\forall(X \rightarrow C)$

Contextual literals

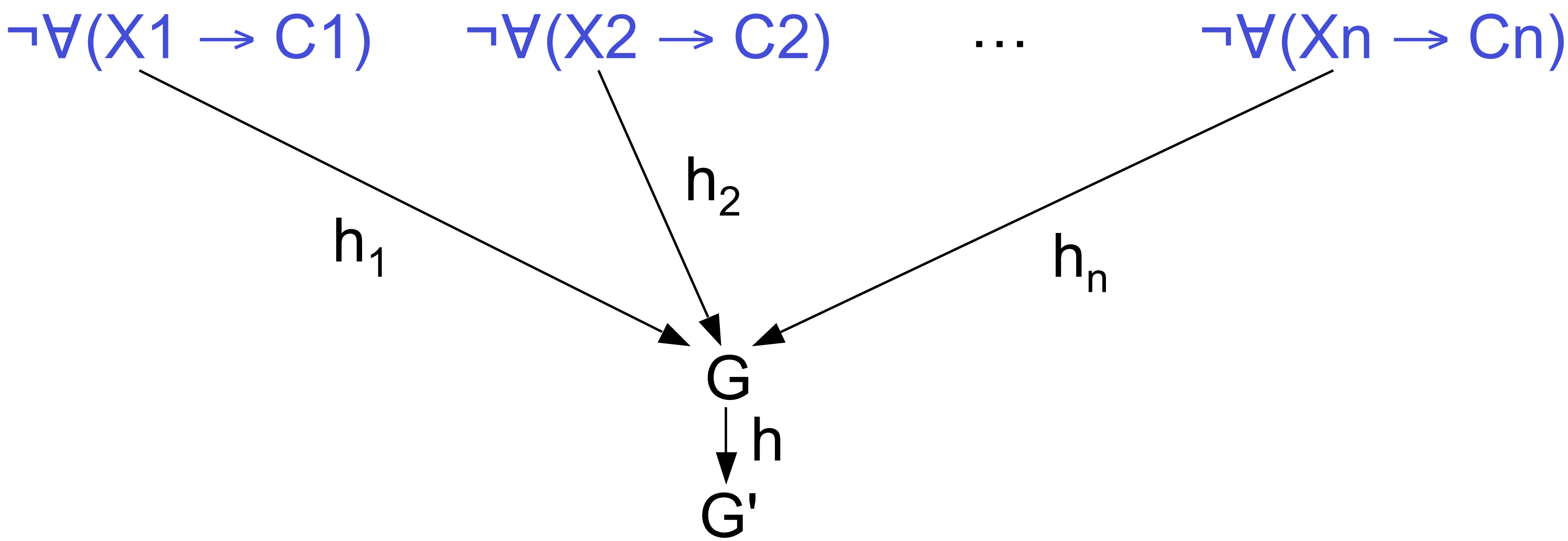
A contextual literal $\exists G[\mathcal{C}]$ is given by:



Where $\mathcal{C} = \{\langle\neg\forall(X1 \rightarrow C1),h_1\rangle\ldots\langle\neg\forall(X1 \rightarrow C1),h_n\rangle\}$ is the context. G is assumed to satisfy all the constraints in the context via the corresponding morphisms

Contextual literals

G' satisfies ∃G[$\mathcal{C}$] if there is a monomorphism h:



and G' satisfies all the constraints in the context via the corresponding composition of morphisms

Inference rules for atomic constraints

1)

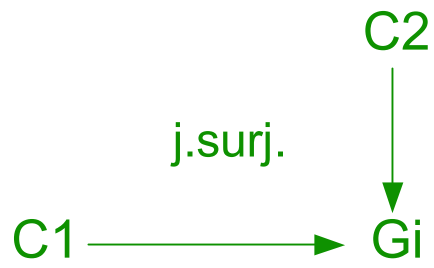
$$\frac{\exists C1\ [\mathcal{C}] \lor \Gamma 1 \quad \neg\exists C2}{\Gamma 1}$$

If there exists a monomorphism $C2 \rightarrow C1$

2)

$$\frac{\exists C1[\mathcal{C}] \lor \Gamma 1 \quad \exists C2}{\exists G1[\mathcal{C}] \lor \dots \lor \exists Gk[\mathcal{C}] \lor \Gamma 1}$$

If there are no monomorphisms $C2 \rightarrow C1$



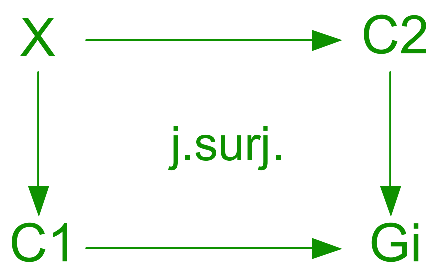
$Gi \models \mathcal{C}$

Inference rules for atomic constraints

3)

$$\frac{\exists C1[\mathcal{C}] \vee \Gamma \qquad \forall(g':X \rightarrow C2)}{\exists G1[\mathcal{C}] \vee \ldots \vee \exists Gk[\mathcal{C}] \vee \Gamma}$$

If $X \rightarrow C1$ and:



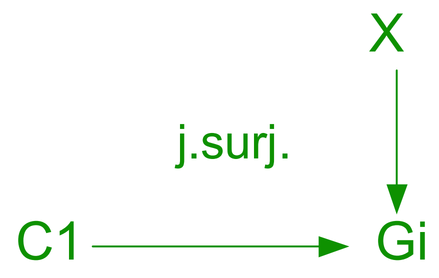
$Gi \models \mathcal{C}$

Inference rules for atomic constraints

4)

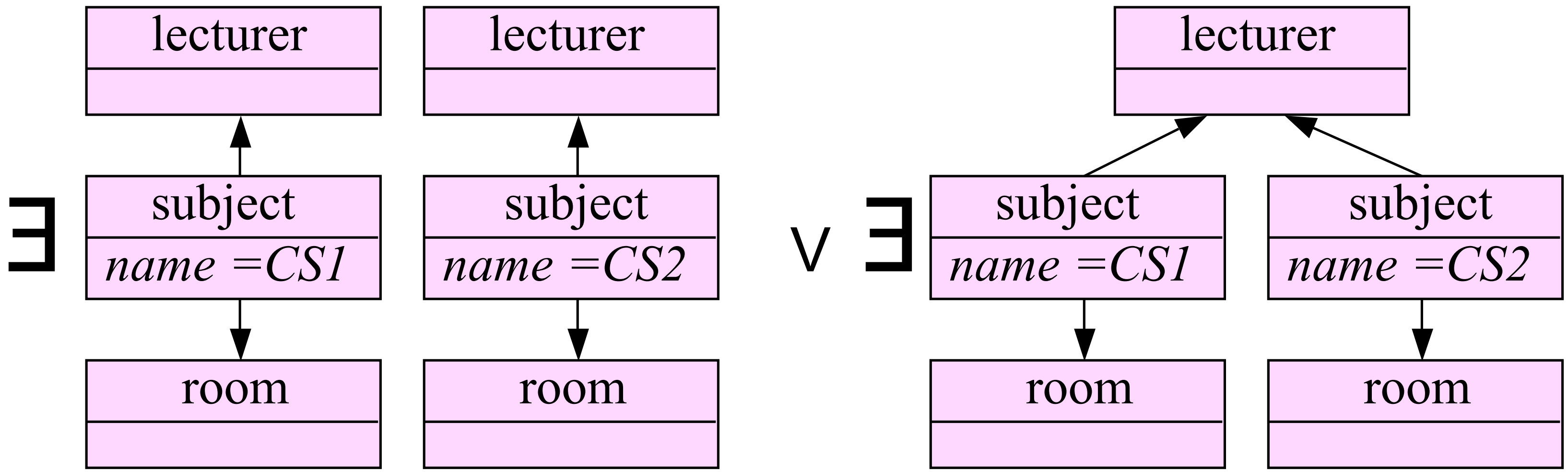
If $c \notin \mathcal{C}$ and:

$$\frac{\exists C1[\mathcal{C}] \vee \Gamma 1 \qquad \neg\forall(X \rightarrow C2) = c}{\exists G1[\mathcal{C} \cup \{c\}] \vee \ldots \vee \exists Gk[\mathcal{C} \cup \{c\}] \vee \Gamma 1}$$

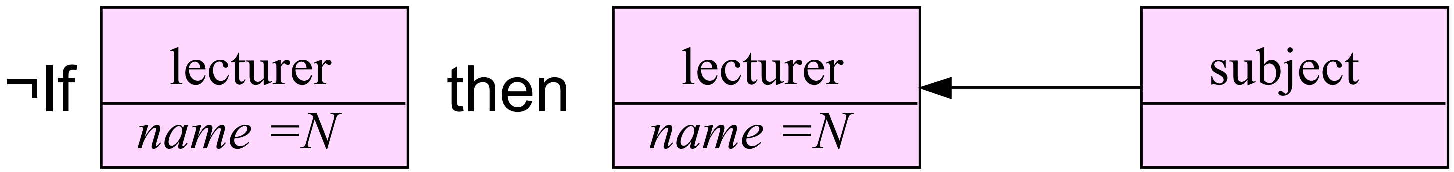


$Gi \models \mathcal{C} \cup \{c\}$

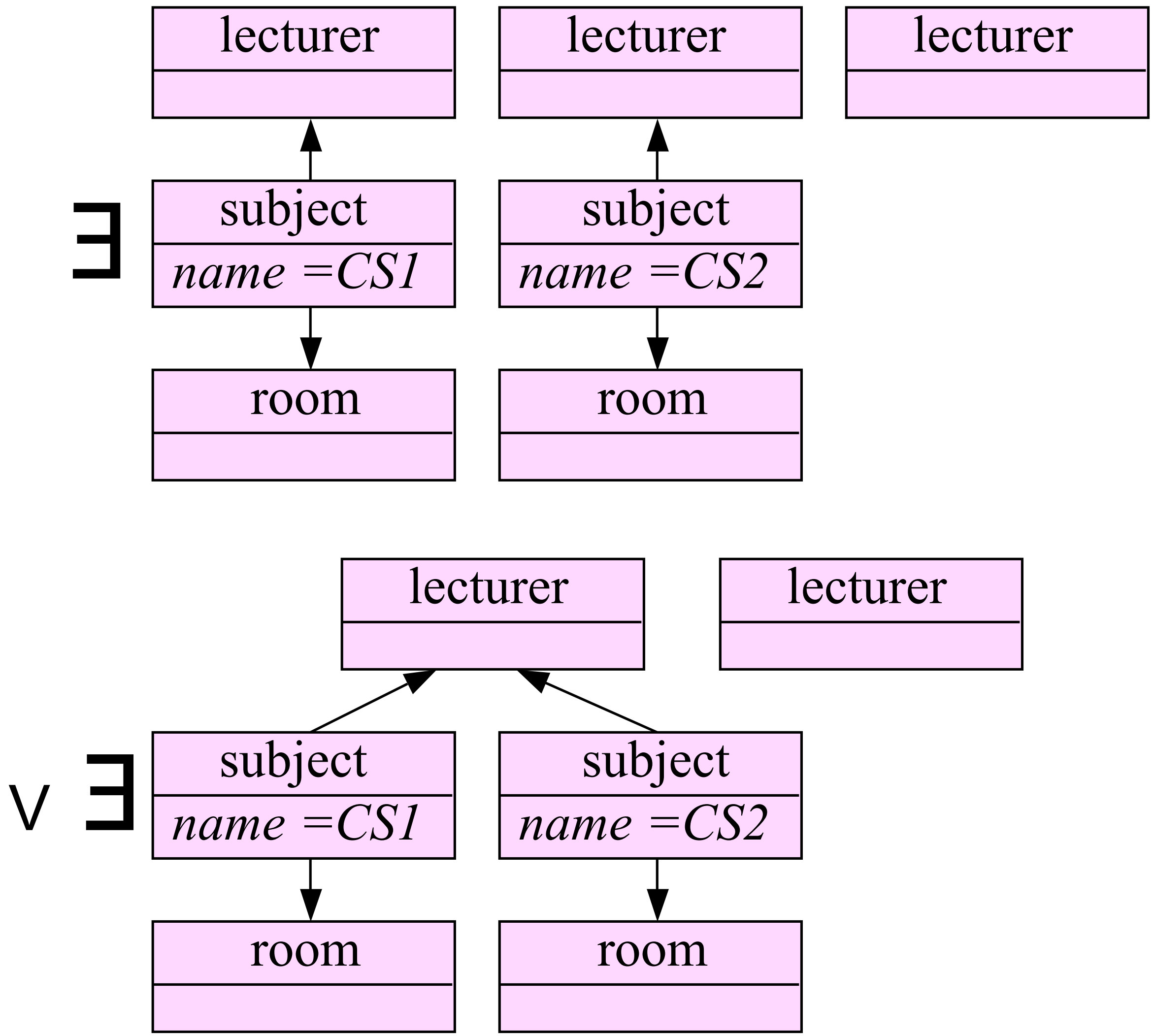
Example


∃
lecturer
subject
name =CS1
room
lecturer
subject
name =CS2
room
∨ ∃
lecturer
subject
name =CS1
room
subject
name =CS2
room

(7)


¬If
lecturer
name =N
then
lecturer
name =N
subject

Example


lecturer
lecturer
lecturer
∃
subject
name =CS1
subject
name =CS2
room
room
lecturer
lecturer
∨ ∃
subject
name =CS1
subject
name =CS2
room
room

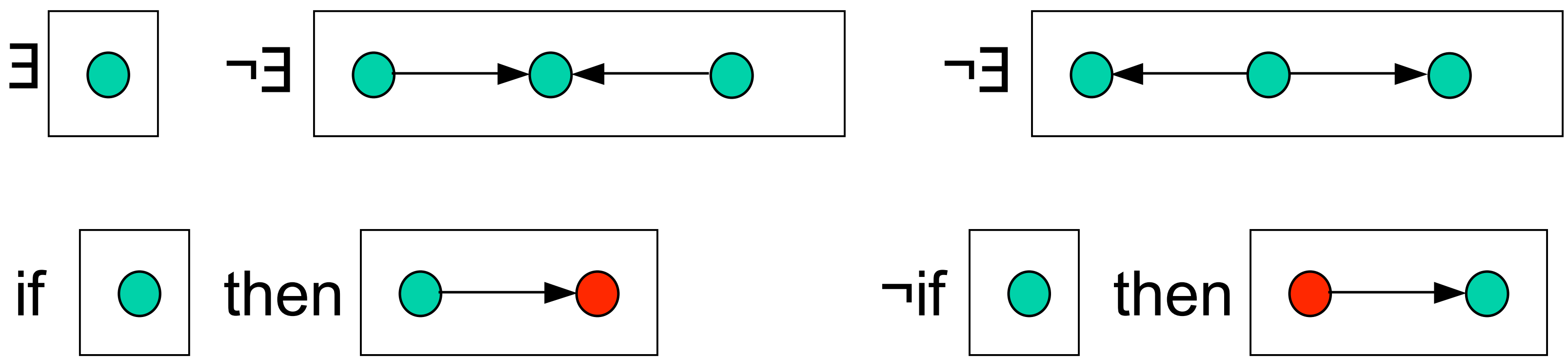
Soundness, completeness and termination

- A refutation procedure may not terminate.

- A set of constraints C is unsatifiable iff a refutation procedure generates the empty clause.

Constraints may be satisfied only by infinite graphs

The set of constraints:

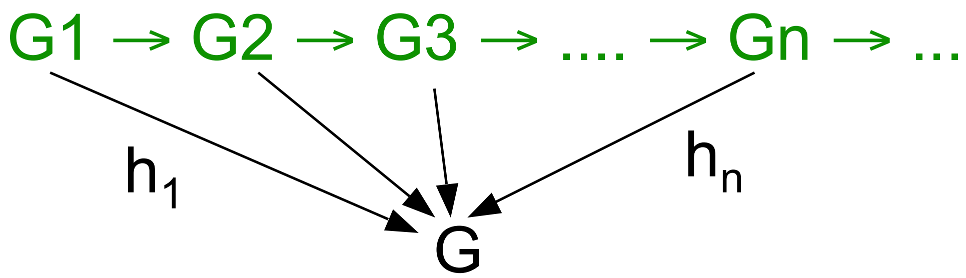


Is not satisfied by any finite graph, but is satisfied by:

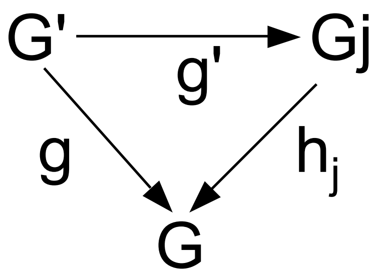
Generality of the results

The previous results apply to any (weakly) adhesive category satisfying:

- (Finite) pair factorization.
- Existence of infinite colimits:



such that for every h:G' → G there is a g': G' → Gj such that:



"Simplification" Rules

5)

$$\frac{\forall(g{:}X \rightarrow C) \qquad \neg\exists C1}{\neg\exists X}$$

If $C1 \rightarrow C$

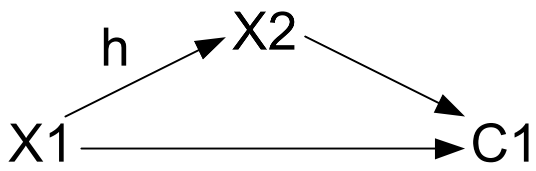
6)

$$\frac{\neg\forall(X \rightarrow C) \qquad \neg\exists C1}{\blacksquare}$$

If $C1 \rightarrow X$

7)

$$\frac{\forall(g1{:}X1 \rightarrow C1) \quad \forall(g2{:}X2 \rightarrow C2)}{\forall(g2\bullet h{:}X1 \rightarrow C2)}$$



Conclusion

We have seen sound and complete procedures for several classes of graph constraints.

Open problems:

- (General) Attributed constraints
- Nested constraints
- "Efficiency"