The Art of Saying “No” – How to Politely Eat Your Way Through an Infinite Meal

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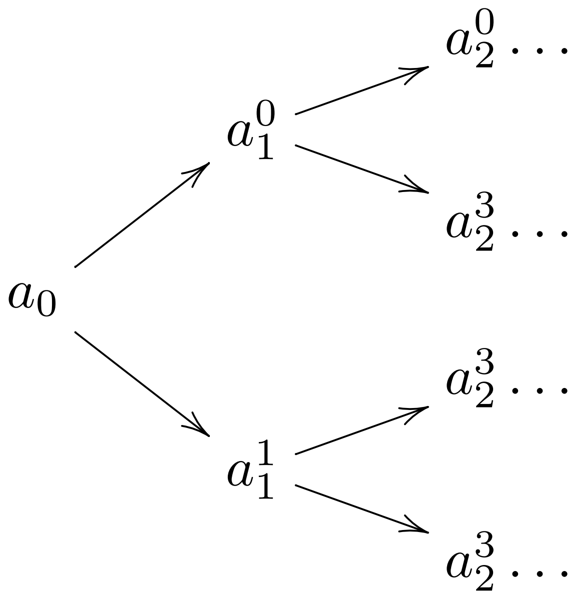
Infinite Objects are Coalgebras

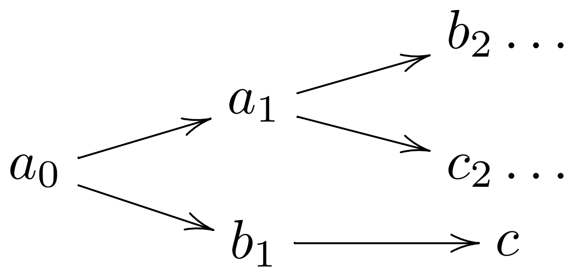
Infinite Streams over A: $\nu X.A \times X$

$$a_0 \longrightarrow a_1 \longrightarrow a_2 \longrightarrow \ldots$$

Infinite Binary Trees over A:
$\nu X.A \times X^2$



Signatures (variable branching):
$\nu X.A \times X^2 + B \times X + C$



Enter Topology . . .

Goal. *Algebraic* Treatment of *continuous* functions $\nu T \to \nu S$

- e.g. representatives of reals: $\{-1, 0, 1\}^\omega \rightsquigarrow [-1, 1]$
- clean (co)inductive definitions and proofs

Discrete Codomain. Continuous Functions $f : \nu X.TX \to B$

- output $b \in B$ after reading *finite amount* of information in $\nu X.TX$

Example. Infinite Streams, or coalgebraically $\nu X.A \times X \to B$

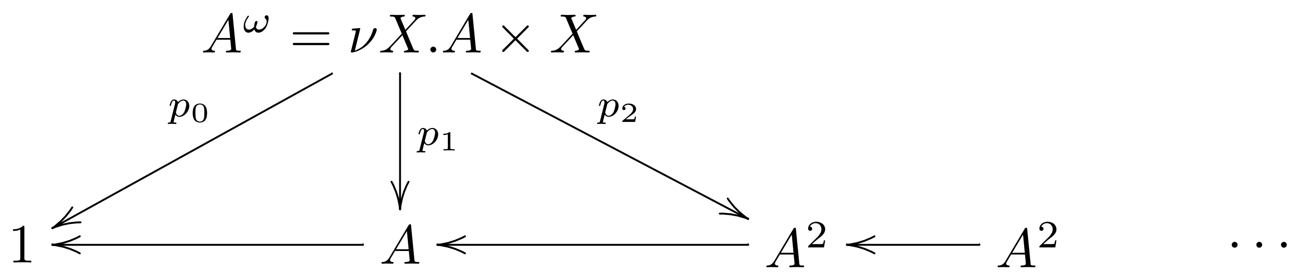
- $f(\alpha)$ depends on finite initial prefix of α

Conceptually. This is the Cantor topology on A^ω (with A discrete)

- generated by $\overline{\alpha} \cdot A^\omega$ where $\overline{\alpha} \in A^*$

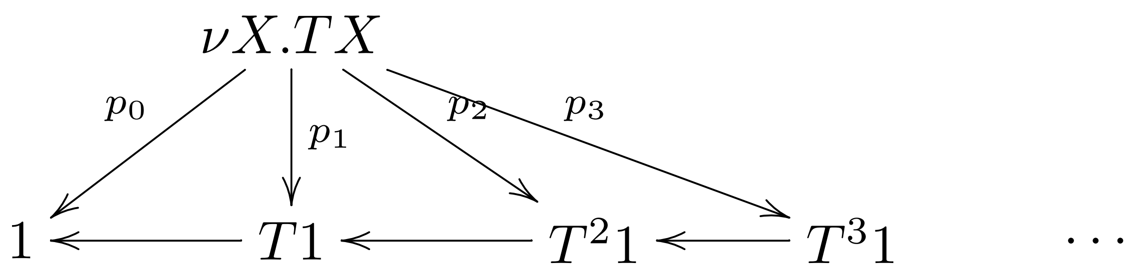
Coalgebraic View

Final Coalgebras arise as *infinite limits*: e.g. streams



Topology generated by $p_i^{-1}(o)$, $o \subseteq A^i$ open

Coalgebraic Generalisation. Suppose $\nu X.TX \xrightarrow{\sigma} T(\nu X.TX)$



where $p_{i+1} = Tp_i \circ \sigma$. Topology generated by $p_i^{-1}(o)$, "$o \subseteq T^i 1$ open"

Continuous Functions: The Case of Streams

Goal. Characterise continuous functions of type $A^\omega \to B$ with B discrete.

Continuity. (A and B discrete) $f : A^\omega \to B$ is continuous ...

- iff f locally constant.

$$(\forall(a_0, a_1, \dots) \in A^\omega)(\exists n \in \omega)\ f \text{ constant on } (a_0, a_1, \dots, a_n) \cdot A^\omega$$

- iff f is in the least class C closed under

$$\frac{f \text{ constant}}{C(f)} \qquad \frac{(\forall a \in A)C(f(a : \_))}{C(f)}$$

Proof ($\Leftarrow$) locally constant functions are so closed.

Proof ($\Rightarrow$) classical logic and dependent choice.

Representation of Continuous Stream Functions

Idea. Proofs of Continuity define the *least class* of functions

$$\frac{f \text{ constant}}{C(f)} \qquad \frac{(\forall a \in A) C(f(a : \_))}{C(f)}$$

and can be represented as an *inductive* data type:

$$R = \mu X.B + (A \to X) \cong B + (A \to R)$$

with two constructors: $\texttt{Ret} : B \to R$ and and $\texttt{Rd} : (A \to R) \to R$

from which a continuous function can be extracted:

$$\begin{array}{llll} \text{eat} : & \mu X.B + (A \to X) & \to A^\omega & \to B \\ \text{eat} & (\texttt{Ret} b) & (a : \alpha) & = b \\ \text{eat} & (\texttt{Rd} f) & (a : \alpha) & = \text{eat}(f\ a)\alpha \end{array}$$

Theorem. If $\to_c$ is continuous functions, then $\text{eat} : R \to (A^\omega \to_c B)$ is onto.

From Streams to Trees

Goal. Classify functions $\text{Tree}(A) \to_c B$ where $\text{Tree}(A) = \nu X.A \times X^2$

Idea. Let R denote the type of representatives with constructors $\texttt{Rd}$ and $\texttt{Ret}$.

$$\begin{array}{llll} \text{eat}: & R & \to \text{Tree}(A) & \to B \\ \text{eat} & (\texttt{Ret}\ b) & (a, l, r) & = b \\ \text{eat} & (\texttt{Rd}\ f) & (a, l, r) & = \text{eat}(f\ a)(l, r) \end{array}$$

Observation. $\text{eat}(f\ a) : \text{Tree}(A)^2 \to B$, so $f(a)$ represents $\text{Tree}(A)^2 \to B$

Mathematical Obfuscation. R_n represents $\text{Tree}(A)^n \to B$

$$\begin{array}{llll} \text{eat}_n: & R_n & \to \text{Tree}(A)^n & \to B \\ \text{eat}_n & (\texttt{Ret}\ b) & (t_1, \ldots, t_n) & = b \\ \text{eat}_n & (\texttt{Rd}_i\ f) & (t_1, \ldots, t_n) & = \text{eat}_{n+1}(f\ a_i)(t_1, \ldots, t_{i-1}, l, r, t_{i+1}, \ldots, t_n) \end{array}$$

where l, r are the left/right subtree of t_i. *Constructors.* $\texttt{Ret}, \texttt{Rd}_1, \ldots, \texttt{Rd}_n$

Escaping the Underworld of Indices

Desired (Inductive) Type with constructors $\mathtt{Ret}, \mathtt{Rd}_1, \ldots, \mathtt{Rd}_n$ as above.

$$R_n \cong B + \sum_{i \in n} (A \to R_{n+1})$$

Realisation Mapping.

$$\begin{array}{llll} \mathsf{eat}_n : & R_n & \to \mathsf{Tree}(A)^n & \to B \\ \mathsf{eat}_n & (\mathtt{Ret}\ b) & (t_1, \ldots, t_n) & = b \\ \mathsf{eat}_n & (\mathtt{Rd}_i\ f) & (t_1, \ldots, t_n) & = \mathsf{eat}_{n+1}(f\ a_i)(t_1, \ldots, t_{i-1}, l, r, t_{i+1}, \ldots, t_n) \end{array}$$

Taking Indices Seriously.

$$R(n) \cong B + \sum_{i \in n} (A \to R(n+1))$$

Observation. Now R has type $\mathsf{Set} \to \mathsf{Set}$ – and we want the *least* such

$$R = \mu F : \mathsf{Set} \to \mathsf{Set}. \Lambda I : \mathsf{Set}.\ B + I \times (A \to F(I+1))$$

Conceptual Digression

Streams. Represent $\mathsf{Stream}(A)^S \to B$ by $R(S)$ where

$$R(S) = \mu X.B + S \times (A \to X)$$

- each $R(S)$ is an initial algebra for a functor of type $\mathsf{Set} \to \mathsf{Set}$
- $\mathsf{eat}(S)$ defined by initiality of $R(S)$ – *separately* for all arities

Linearity: Family of Inductive Types

Trees. Represent $\mathsf{Tree}(A)^S \to_c B$ by $R(S)$ where

$$R = \mu F : \mathsf{Set} \to \mathsf{Set}.\Lambda S : \mathsf{Set}.B + S \times (A \to F(S+1))$$

- R is an initial algebra for a functor $(\mathsf{Set} \to \mathsf{Set}) \to (\mathsf{Set} \to \mathsf{Set})$
- eat is natural and defined by initiality of R – *simultaneously* for all arities

Nonlinearity: Inductive Family of Types

Infinite Objects of Container Type

Container Functors. (Abbot, Altenkirch, Ghani)

$$(S \lhd P)(X) = \sum_{s \in S} X^{P(s)}$$

- S : Set is a set of *shapes*, each of which stores data
- $P : S \to$ Set associates a set of *positions* to every shape

Continuous Functions of type $(\nu X.(S \lhd P)X)^I \to B$

$$R = \mu F : \mathsf{Set} \to \mathsf{Set}.\Lambda I : \mathsf{Set}.B + \sum_{i \in I} \prod_{s \in S} F(I + P(s))$$

Unfolding Isomorphisms.

$$R(I) \cong B + \sum_{i \in I} \prod_{s \in S} R(I + P(s))$$

Intuition.

- if not constant, select tree $(i \in I)$, extract root ($s \in S$), behead and continue

Discrete Codomains are Boring

Next Goal. Represent $A^\omega \to_c B^\omega$

Idea. $f : A^\omega \to B^\omega$ is continuous iff we have an *infinite* proof

$$(R)\frac{\forall a(C(f(a : \_)))}{C(f)} \qquad (W)\frac{C(f)}{C(\lambda\alpha.b : f(\alpha))}$$

where, on any branch in a proof, the right hand rule occurs infinitely often.

Induced Data Type. Wrap up finite occurrences of (R) using a μ

$$R \cong \nu X.\mu Y.B \times X + (A \to Y) \cong B \times R + (A \to R)$$

with constructors $\mathtt{Ret} : B \times R \to R$ and $\mathtt{Rd} : (A \to R) \to R$

Extracted Continuous Function.

$$\begin{array}{llll} \mathsf{eat} : & \nu X.\mu Y.B \times X + (A \to Y) & \to A^\omega & \to B^\omega \\ \mathsf{eat} & (\mathtt{Ret}\ (b, r)) & (a : \alpha) & = b : \mathsf{eat}\ r\ (a : \alpha) \\ \mathsf{eat} & (\mathtt{Rd}\ f) & (a : \alpha) & = \mathsf{eat}\ (f\ a)\ \alpha \end{array}$$

Alternative Computational Representation

We know. Continuous functions of type $A^w \to B$ are represented by

$$A^\omega \to_C B \rightsquigarrow R = \mu X.B + (A \to X)$$

Idea. Re-start the computation as soon as a digit has been produced

$$A^\omega \to_C B^\omega \rightsquigarrow \nu X.\mu Y.B \times X + (A \to Y)$$

with the same computational interpretation

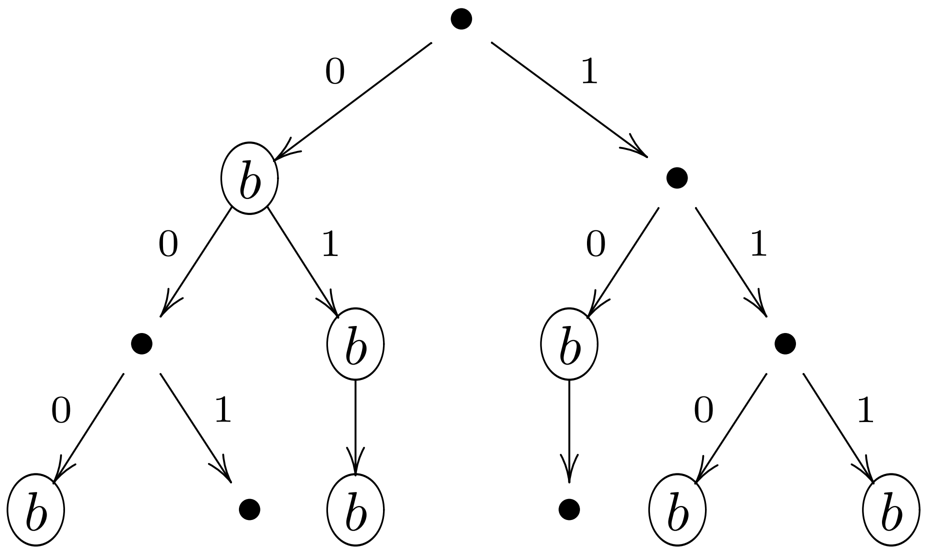
$$\begin{array}{llll} \text{eat}: & \nu X.\mu Y.B \times X + (A \to Y) & \to A^\omega & \to B^\omega \\ \text{eat} & (\texttt{Ret}\ (b, r)) & (a : \alpha) & = b : \text{eat}\ r\ (a : \alpha) \\ \text{eat} & (\texttt{Rd}\ f) & (a : \alpha) & = \text{eat}\ (f\ a)\ \alpha \end{array}$$

Note. Occurrence of $B \times X$ suggests that "codomain slots in"

Stream Functions are Trees

Observation. First-Order Functions $A^\omega \to B^\omega$ are *trees*

$$R = \nu X.\mu Y.B \times X + (A \to Y)$$



Initiality guarantees infinitely many labels on every path

General Codomain

More Ambitious Goal. Represent $A^\omega \to \nu X.(S \lhd P)X = \nu X.\sum_{s\in S} X^{P(s)}$

By Analogy.

$$R = \nu X.\mu Y.\sum_{s\in S} X^{P(s)} + (A \to Y) \cong \sum_{s\in S} R^{P(s)} + (A \to R)$$

with constructors $\mathtt{Ret}_s : (P(s) \to R) \to R$ and $\mathtt{Rd} : (A \to R) \to R$

Associated Functional.

$$\begin{array}{llll} \text{eat} : & \nu X.\mu Y.\sum_{s\in S} X^{P(s)} + (A \to Y) & \to A^\omega & \to \nu Z.(P \lhd S)Z \\ \text{eat} & (\mathtt{Ret}_s\ (r_i)) & (a : \alpha) & = (s, (\text{eat}\ r_i\ (a : \alpha))_{i\in P(s)}) \\ \text{eat} & (\mathtt{Rd}\ f) & (a : \alpha) & = \text{eat}\ (f\ a)\ \alpha \end{array}$$

Observation.

- *codomain* just “slots in”, more general *domains* by same recipie

Induction Meets Coinduction

Example. Continuous Stream Functions

$$f : A^\omega \to_c B^\omega$$

are represented by

$$\underbrace{\nu X.\, \overbrace{\mu Y \;\, B \times X + Y^A}^{T_A(B\times X)}}_{P_A(B)}$$

Lambek's Lemma.

$$P_A(B) = (\nu X)(\mu Y)B \times X + Y^A \cong (\mu Y)B \times P_A(B) + Y^A$$

Pleasant Mathematical Theory.

- supports both *inductive* and *coinductive* definitions and proofs.
- similar for other (co)domains

Inductive Maps Between Coinductive Types

Example. Composition: $P_B(C) \times P_A(B) \to P_A(C)$ where

$$T_A(B) = \mu X.B + (A \to X) \quad \text{and} \quad P_A(B) = \nu X.T_A(B \times X)$$

Operation on *representatives*

$$\begin{array}{ccc}
P_B(C) \times P_A(B) & \longrightarrow & P_A(C) \\
\downarrow & & \downarrow \\
(B^\omega \to_c C^\omega) \times (A^\omega \to_c B^\omega) & \longrightarrow & (A^\omega \to_c C^\omega)
\end{array}$$

As $P_A(B) \cong T_A(B \times P_A(B))$ is bi-inductive: composition

$$\gamma : S = T_B(C \times P_B C) \times T_A(B \times P_A B) \to T_A(C \times S)$$

is an *inductively defined* map between *coinductive* types

More on Composition

Inductive definition of composition

$$\begin{array}{llll}\gamma : S = T_B(C \times P_B C) \times T_A(B \times P_A B) \to T_A(C \times S) & & & \\ \langle\, \mathtt{Ret}\,\langle c, p_{bc}\rangle & , t_{ab} & \rangle & \mapsto \mathtt{Ret}\,\langle c, \mathsf{out}\, p_{bc}, t_{ab}\rangle \\ \langle\, \mathtt{Rd}\,\phi & , \mathtt{Ret}\,\langle b, p_{ab}\rangle & \rangle & \mapsto \gamma\langle \phi\, b, \mathsf{out}\, p_{ab}\rangle \\ \langle\, t_{bc} & , \mathtt{Rd}\,\psi & \rangle & \mapsto \mathtt{Rd}\,\lambda a.\gamma\langle t_{bc}, \psi\, a\rangle \end{array}$$

whose *coinductive* cousin (out : $\nu F \to F(\nu F)$)

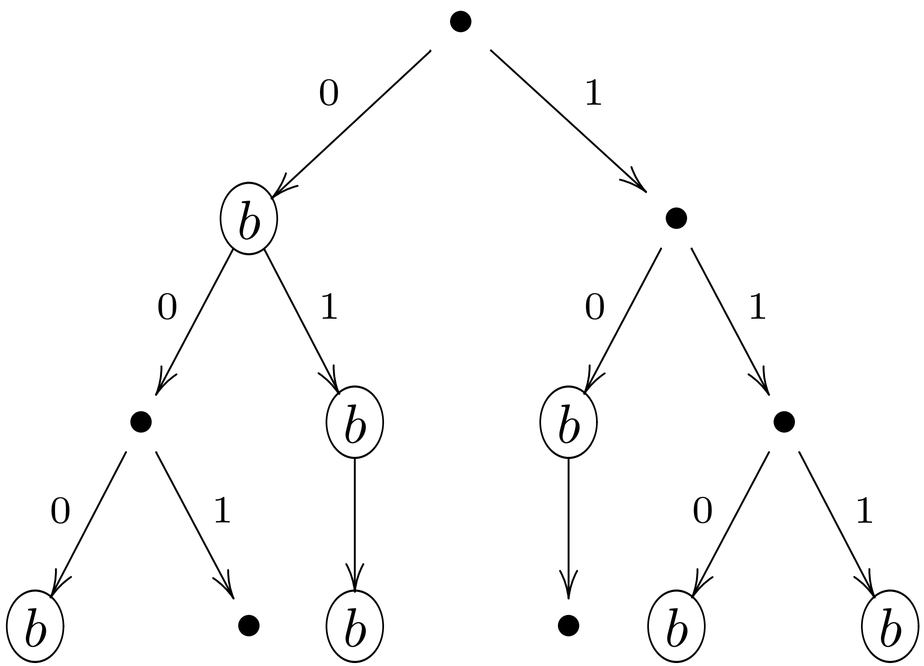
$$\begin{array}{lll}\chi : P_B(C) \times P_A(B) \to P_A(C) & & \\ \langle post, & pre\rangle & \mapsto (\mathsf{unfold}\,\gamma)\langle \mathsf{out}\, post, \mathsf{out}\, pre\rangle \end{array}$$

represents composition.

This is *output centered* – alternatives are possible.

Higher-Order Functions

Observation. First-Order Functions $A^\omega \to B^\omega$ are *trees*



Idea.

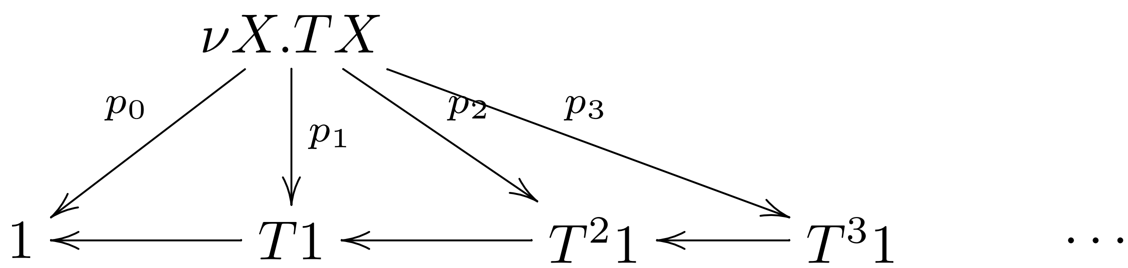
- represent higher-order functions as functions on trees
- *but:* domain doesn't fit into $\nu Z.(S \lhd P)Z = \nu Z.\sum_{s \in S} Z^{P(s)}$

Topological Excursion

Question. What's the natural topology on $A^\omega \to B^\omega$?

Topology on Representatives $R = \nu X.\mu Y.B \times X + (A \to Y)$.

- consider $TX = \mu Y.B \times X + (A \to Y)$ and $\sigma : R \to TR$
- topology given by the inverse limit



where $p_{i+1} = Tp_i \circ \sigma$. Topology generated by $p_i^{-1}(o)$, $o \subseteq T^i 1$ open

Induced Topology on $(A^\omega \to B^\omega)$ is compact-open:

- elements of $T^n 1$ are layers of A-branching trees with labels in B
- single trees define compact-open constraints

Container Magic

Summary so far.

- have representation of functions $\nu Z.(S \lhd P)Z \to X$
- want: representations of $\nu X.\mu Y.(B \times X) + (A \to Y) \to X$

Container Translation. Representations for free – if we solve

$$\mu Y.(B \times X) + (A \to Y) = (S \lhd P)X$$

Theorem. (Abbot/Alternkirch/Ghani) Containers are closed under μ, ν.

More precisely: for every n-ary container

$$C(X_1, \ldots, X_n) = \sum_{s \in S} X_1^{P_1(s)} \times \cdots \times X_n^{P_n(s)}$$

there is an $n-1$-ary container $D(X_1, \ldots, X_{n-1})$ that satisfies

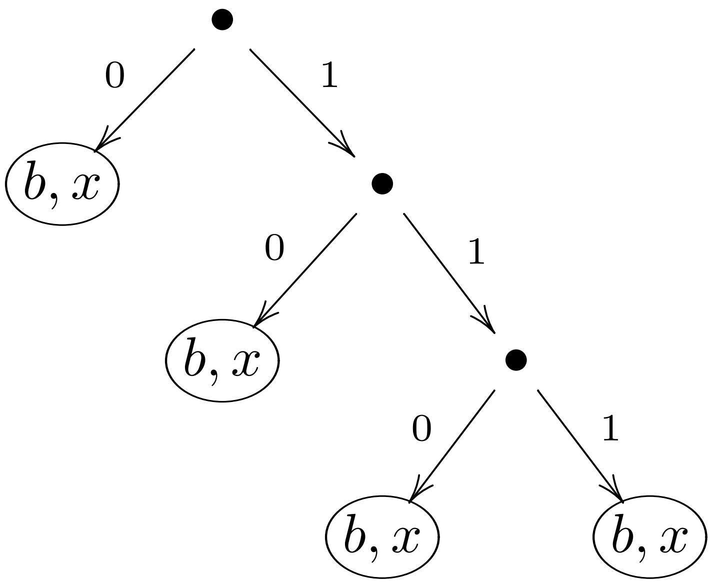
$$D(X_1, \ldots, X_{n-1}) = \mu X_n.C(X_1, \ldots, X_n)$$

Container Translation by Example

Wanted. Solutions of

$$\mu Y.B \times X + (A \to Y) \cong (S \lhd P)X = \sum_{s \in S} X^{P(s)}$$

Observation. We see *trees* with payload at the leaves.



Shapes.

$$S = \mu X.B + (A \to X)$$

Positions.

$P(s) = \{$ paths in S from root to leaves$\}$

Order-Two Example

Representatives. (Recall: $S = \mu X.B + (A \to X)$ and $P(s) =$ paths)

$$R = \nu F.\mu G.\Lambda I.\, C \times F(I) + I \times \prod_{s \in S} G(I + P(s))$$

Unfolding Isomorphisms. (Recall: $R(I)$ represents $(A^\omega \to B^\omega)^I \to C^\omega$)

$$R(I) \cong C \times R(I) + I \times \prod_{s \in S} R(I + P(s))$$

Induced Representation.

$$\begin{array}{llll}
\mathsf{eat}(I): & R & \to T^I & \to \nu Z.C \times Z \\
\mathsf{eat}(I) & (\mathtt{Ret}(c, r)) & (\phi) & = c : (\mathsf{eat}\ r\ \phi) \\
\mathsf{eat}(I) & (\mathtt{Rd}\,(i, f)) & (\phi) & = \mathsf{eat}\ (f(\mathsf{root}\,\phi(i)))\ [\phi, \mathsf{debris}(\phi(i))]
\end{array}$$

Notation. For $t = (r, d) \in T = \nu X.(S \lhd P)X \cong \sum_{s \in S} T^{P(s)}$

$$\mathsf{root}(r, d) = r \qquad \text{and} \qquad \mathsf{debris}(r, d) = d$$

Conclusions

Tree Eating.

- linear structures (streams) $\leadsto$ family of inductive types
- nonlinear structures (trees) $\leadsto$ inductive families of types
- in both cases: sound and complete representation of continuous functions

Higher Order Functions.

- reducible to tree case – but with coding
- possibly very inefficient in practice – try out

Open Questions.

- more combinators (e.g. buffering, currying)
- concrete case studies – in particular integration
- complexity of (higher order) stream functions?