Dantzig-Wolfe and Lagrangian Decompositions in Integer Linear Programming

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Abstract: We propose in this paper a new Dantzig-Wolfe master model based on Lagrangian Decomposition (LD). We establish the relationship with classical Dantzig-Wolfe decomposition master problem and propose an alternative proof of the dominance of LD on Lagrangian Relaxation (LR) dual bound. As illustration, we give the corresponding models and numerical results for two standard mathematical programs: the 0-1 bidimensional knapsack problem and the generalised assignment problem.

Keywords: Dantzig-Wolfe decomposition; column generation; LR; Lagrangian relaxation; Lagrangian decomposition; 0-1 bidimensional knapsack problem; generalised assignment problem.

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1 Introduction

An integer linear program where constraints are partitioned in two subsets can be formulated as follows:

$$(P)\begin{cases} \max & c^t x \\ \text{s.c.} & Ax = a \\ & Bx = b \\ & x \in X, \end{cases}$$

where $c \in \mathbb{R}^n$, A is a $m \times n$ matrix, B is a $p \times n$ matrix, $a \in \mathbb{R}^m$, $b \in \mathbb{R}^p$ and $X \subseteq \mathbb{N}^n$.

These problems are generally NP-hard and bounds are needed to solve them in generic branch and bound like schemes. To improve the bound based on the continuous relaxation of (P), Lagrangian methods, like Lagrangian

Relaxation (LR) (Geoffrion, 1974), Lagrangian Decomposition (LD) (Guignard and Kim, 1987a, 1987b, Michelon, 1991, Nagih and Plateau, 2000a, 2000b), Lagrangian substitution (Reinoso and Maculan, 1992) and Surrogate Relaxation (SR) (Glover, 1965), are well-known techniques for obtaining bounds in Integer Linear Programming (ILP).

This work recalls the existing link between LR and classical Dantzig-Wolfe Decomposition (DWD) (Dantzig and Wolfe, 1960) and establishes the relationship between LD and DWD to derive a new DW master model.

The equivalence between DWD and LR is well known (Lemaréchal, 2003). Solving a linear program by Column Generation (CG), using DWD, is the same as solving the Lagrangian dual by Kelley's cutting plane method (Kelley, 1960). This work recalls the previous result and extends it to LD, which can be viewed as a specific DWD, to prove the superiority of the new bound obtained.

The paper is organised as follows. Section 2 deals with LR, LD and DWD principles. Section 3 shows the relationship between LD and DWD, and gives a new proof on the LD bound dominance over the LR one. In Section 4 we illustrate with two DW master models on the 0-1 Bi-dimensional Knapsack Problem (0-1_BKP) and the Generalised Assignment Problem (GAP). In Section 5 we present some computational results on the two previous problems.

2 Lagrangian duals and Dantzig-Wolfe decomposition

These approaches can be used in the pre-treatment phase of an exact or heuristic method in order to compute better bounds than linear relaxation. In this section, we recall the principle of Lagrangian duality and its link with DWD and CG.

2.1 Dual Lagrangian relaxation

LR consists in omitting some complicating constraints $(Ax = a)$ and in incorporating them in the objective function using a Lagrangian multiplier $\pi \in \mathbb{R}^m$. We obtain the following relaxation:

$$(\mathrm{LR}(\pi))\begin{cases}\max & c^t x + \pi^t(a - Ax)\\ \text{s.c.} & Bx = b\\ & x \in X.\end{cases}$$

For any $\pi \in \mathbb{R}^m$, the value of $(\mathrm{LR}(\pi))$ is an upper bound on $v(P)$. The best one is given by the LR dual:

$$\begin{aligned}(\mathrm{LRD}) &\equiv \min_{\pi\in\mathbb{R}^m}(\mathrm{LR}(\pi))\\ &\equiv \min_{\pi\in\mathbb{R}^m}\ \max_{\{x\in X, Bx=b\}} c^t x + \pi^t(a - Ax).\end{aligned}$$

Let be $X_B = \{x \in X \mid Bx = b\}$ and $\mathrm{Conv}(X_B)$ its convex hull (boundary of the convex polygon), supposed bounded. We denoted by $x^{(k)}, k \in \{1, \ldots, K\}$ the extreme points of $\mathrm{Conv}(X_B)$. Hence, (LRD) can be reformulated as follows:

$$\text{(LRD)} \equiv \min_{\pi \in \mathbb{R}^m} \max_{k=1,\ldots,K} c^t x^{(k)} + \pi^t (a - A x^{(k)})$$

$$\equiv \begin{cases} \min & z \\ \text{s.t.} & z + \pi^t (A x^{(k)} - a) \geq c^t x^{(k)}, \quad k = 1, \ldots, K \\ & \pi \in \mathbb{R}^m, \quad z \in \mathbb{R}. \end{cases}$$

This new formulation potentially contains an exponential number of constraints, equal to K. Kelley's cutting plans method (Kelley, 1960) considers a reduced set of these constraints that handle a restricted problem. Cuts (constraints) are added at each iteration until the optimum reached.

2.2 Lagrangian Decomposition dual

It is well-known that the efficiency of branch and bound like scheme depends on the quality of the bounds. To improve those provided by LR, Guignard and Kim (1987a, 1987b) have proposed to use LD. In such an approach, copy constraints are added to the formulation (P) to build an equivalent problem:

$$\begin{cases} \max & c^t x \\ \text{s.c.} & Ax = a \\ & By = b \\ & x = y \\ & x \in X, \quad y \in Y, \quad \text{with } Y \supseteq X \end{cases}$$

where the copy variables permits to split the initial problem in two independent sub-problems after applying LR on the copy constraints $x = y$:

$$\text{(LD}(w)) \begin{cases} \max & c^t x + w^t (y - x) \\ \text{s.c.} & Ay = a \\ & Bx = b \\ & x \in X, \quad y \in Y, \end{cases}$$

where $w \in \mathbb{R}^n$ are dual variables associated to the copy constraints. We obtain the two following independent sub-problems:

$$\text{(LD}_y(w)) \begin{cases} \max & w^t y \\ \text{s.c.} & Ay = a \\ & y \in Y \end{cases} \quad \text{and} \quad \text{(LD}_x(w)) \begin{cases} \max & (c - w)^t x \\ \text{s.c.} & Bx = b \\ & x \in X \end{cases}$$

The LD dual is given by

$$\text{(LDD)} \min_{w\in\mathbb{R}^n} \; v(\text{LD}(w))$$

where

$$v(\text{LD}(w)) = \max\{w^t y \mid y \in Y_A\} + \max\{(c-w)^t x \mid x \in X_B\}$$

with

$$Y_A = \{y \mid Ay = a,\ y \in Y\} \quad X_B = \{x \mid Bx = b,\ x \in X\}.$$

This dual can be rewritten as :

$$\text{(LDD)} \begin{cases} \min & \max (c-w)^t x \quad + \quad \max w^t y \\ w \in \mathbb{R}^n & x \in X_B \qquad\qquad\quad y \in Y_A. \end{cases}$$

We assume that the convex hull of the sets Y_A and X_B are bounded. We denote by $x^{(k)}, k \in \{1, \dots, K\}$ the extreme points of X_B and by $y^{(l)}, l \in \{1, \dots, L\}$ those of Y_A. We obtain the following formulation:

$$\text{(LDD)} \begin{cases} \min & \max (c-w)^t x^{(k)} \quad + \quad \max w^t y^{(l)} \\ w \in \mathbb{R}^n & k = 1, \dots, K \qquad\qquad l = 1, \dots, L \end{cases}$$

which can be expressed in this equivalent linear form:

$$\text{(LDD)} \begin{cases} \min & z_1 + z_2 \\ & z_1 \geq (c-w)^t x^{(k)}, \quad k = 1, \dots, K \\ & z_2 \geq w^t y^{(l)}, \quad l = 1, \dots, L \\ & w \in \mathbb{R}^n, \quad z_1, z_2 \in \mathbb{R}. \end{cases}$$

The following theorem give the well-known dominance relationship between (P), (LRD), (LDD) and (LP) which is the linear relaxation of (P).

Theorem 1 (Guignard and Kim, 1987a, 1987b**):** $v(P) \leq v(\text{LDD}) \leq v(LRD) \leq v(LP)$.

2.3 Dantzig-Wolfe decomposition and column generation

The key idea of DWD (Dantzig and Wolfe, 1960) is to reformulate the problem by substituting the original variables with a convex combination of the extreme points of the polyhedron corresponding to a substructure of the formulation.

We know that

$$\forall x \in \text{Conv}(X_B), \quad x = \sum_{k=1}^{K} \lambda_k x^{(k)}$$

with $\sum_{k=1}^{K} \lambda_k = 1$, $\lambda_k \geq 0$, $\forall k \in 1, \ldots, K$.

By substituting in (P) we obtain the master problem of DWD:

$$
(\text{MP}) \begin{cases} \max & \displaystyle\sum_{k=1}^{K} (c^t x^{(k)}) \lambda_k \\ s.c. & \displaystyle\sum_{k=1}^{K} (A x^{(k)}) \lambda_k = a \\ & \displaystyle\sum_{k=1}^{K} \lambda_k = 1 \\ & \lambda_k \geq 0, \qquad k = 1, \ldots, K. \end{cases}
$$

(MP) contains $m + 1$ constraints and (potentially) a huge number of variables (i.e., the number K of extreme points of Conv(X_B)).

Remark 1: Due to the fact that (LRD) is a dual of (MP), $v(\text{LRD}) = v(\text{MP})$ (Lemaréchal, 2003).

CG consists in generating iteratively a subset of the extreme points of X_B to determine an optimal solution of (MP) by solving alternatively:

- a *Restricted Master Problem* of DWD on a subset $\mathcal{K}$ of $\{1, \ldots, K\}$:

$$
(\text{RMP}) \begin{cases} \max & \displaystyle\sum_{k \in \mathcal{K}} (c^t x^{(k)}) \lambda_k \\ s.c. & \displaystyle\sum_{k \in \mathcal{K}} (A x^{(k)}) \lambda_k = a \\ & \displaystyle\sum_{k \in \mathcal{K}} \lambda_k = 1 \\ & \lambda_k \geq 0, \quad k \in \mathcal{K} \end{cases}
$$

- a pricing problem:

$$
(\text{SP}) \begin{cases} \max & c^t x - \pi^t A x - \pi_0 \\ \text{s.c.} & Bx = b \\ & x \in X \end{cases}
$$

  where $(\pi, \pi_0) \in \mathbb{R}^m \times \mathbb{R}$ are the dual variables provided by the resolution of (RMP). The solution of (SP) is incorporated (as a column) in (RMP) if its value is non negative.

This process ends when there is no more variables in $\{1, \ldots, K\} \backslash \mathcal{K}$ with a positive reduced cost.

3 Lagrangian and Dantzig-Wolfe decompositions

This section is dedicated to LD duality. We establish the relationship between LD, DWD and CG. We consider the following DW master problem :

$$\text{(MPD)}\begin{cases}\max & \displaystyle\sum_{k=1}^{K}(cx^{(k)})\lambda_k \\ & \displaystyle\sum_{k=1}^{K}x^{(k)}\lambda_k-\sum_{l=1}^{L}y^{(l)}\gamma_l=0 \\ & \displaystyle\sum_{k=1}^{K}\lambda_k=1 \\ & \displaystyle\sum_{l=1}^{L}\gamma_l=1 \\ & \lambda_k\geq 0,\quad k=1,\dots,K,\quad \gamma_l\geq 0,\ l=1,\dots,L\end{cases},$$

where $x^{(k)}, k\in\{1,\dots,K\}$ are the extreme points of X_B and $y^{(l)}, l\in\{1,\dots,L\}$ those of Y_A.

Lemma 1: *The value of this master problem (MPD) provides a better upper bound on $v(P)$ than the value of the classical DWD (MP).*

Proof:

$$v(\text{MPD})=\begin{cases}\max & \displaystyle\sum_{k=1}^{K}(cx^{(k)})\lambda_k \\ & \displaystyle\sum_{k=1}^{K}x^{(k)}\lambda_k-\sum_{l=1}^{L}y^{(l)}\gamma_l=0 \\ & \displaystyle\sum_{k=1}^{K}\lambda_k=1 \\ & \displaystyle\sum_{l=1}^{L}\gamma_l=1 \\ & \lambda_k\geq 0,\quad k=1,\dots,K,\quad \gamma_l\geq 0,\ l=1,\dots,L.\end{cases}$$

By duality

$$v(\text{MPD})=\begin{cases}\min & z_1+z_2 \\ & z_1+w^tx^{(k)}\geq cx^k,\quad k=1,\dots,K(1) \\ & z_2-w^ty^{(l)}\geq 0,\quad l=1,\dots,L(2) \\ & w\in\mathbb{R}^n,\quad z_1,z_2\in\mathbb{R}\end{cases}$$

If we consider only a subset of the multipliers $w \in \mathbb{R}^n$ such that $w^t = \pi^t A$, where π is a vector of $\mathbb{R}^m$, and substitute in equations (1) and (2) we obtain the following problem:

$$\begin{cases} \min & z_1 + z_2 \\ & z_1 + \pi^t A x^{(k)} \geq c x^k, \quad k = 1, \dots, K \\ & z_2 - \pi^t A y^{(l)} \geq 0, \quad l = 1, \dots, L \\ & w \in \mathbb{R}^n, \quad z_1, z_2 \in \mathbb{R} \end{cases}$$

for which the dual is:

$$\begin{cases} \max & \sum_{k=1}^{K} (c x^{(k)}) \lambda_k \\ & \sum_{k=1}^{K} A x^{(k)} \lambda_k - \sum_{l=1}^{L} A y^{(l)} \gamma_l = 0 \\ & \sum_{k=1}^{K} \lambda_k = 1 \\ & \sum_{l=1}^{L} \gamma_l = 1 \\ & \lambda_k \geq 0, \quad k = 1, \dots, K, \quad \gamma_l \geq 0, \; l = 1, \dots, L. \end{cases}$$

As $y^{(l)}, l \in \{1, \dots, L\}$ are the extreme points of Y_A, we have $Ay^{(l)} = a$, and by $\sum_l \gamma_l = 1$, we obtain the problem (MP). Thus $v(\text{MPD}) \leq v(\text{MP})$. □

Remark 2: If $n > m$, the set $\{\pi^t A, \pi \in \mathbb{R}^m\} \subsetneq \mathbb{R}^n$ and then $v(MPD)$ can be stricly better than $v(\text{MP})$.

Remark 3: As (LDD) (resp. (LRD)) is the dual of (MPD) (resp. (MP)), we can state that

$$v(\text{MPD}) = v(\text{LDD}) = \min_{w \in \mathbb{R}^n} v(\text{LD}(w)) \leq \min_{\pi^t \in \mathbb{R}^m} v(\text{LD}(\pi^t A))$$

and

$$\min_{\pi^t \in \mathbb{R}^m} v(\text{LD}(\pi^t A)) = \min_{\pi \in \mathbb{R}^m} v(\text{LR}(\pi)) = v(\text{LRD}) = v(\text{MP}).$$

This approach supply an alternative proof to the dominance of LD over LR.

4 Decomposition models

This section is devoted to an illustration of this new DWD model on two classical combinatorial optimisation problems : the 0-1 bi-dimensional knapsack problem and the generalised assignment problem.

4.1 The 0-1 bi-dimensional knapsack problem

This problem consists in selecting a subset of given objects (or items) in such a way that the total profit of the selected objects is maximised while two knapsack constraints are satisfied. The formulation of this problem is given by :

$$(0\text{-}1\_BKP)\begin{cases} \max & \sum_{i=1}^{n} c_i x_i \\ \text{s.c.} & \sum_{i=1}^{n} a_i x_i \leq A \\ & \sum_{i=1}^{n} b_i x_i \leq B \\ & x_i \in \{0,1\}, \quad i = 1, \ldots, n \end{cases}$$

where n is the number of objects (or items), the coefficients $a_i (i = 1, \ldots, n)$, $b_i (i = 1, \ldots, n)$ and $c_i (i = 1, \ldots, n)$ are positive integers and A and B are integers such that $max\{a_i : i = 1, \ldots, n\} \leq A < \sum_{i=1,\ldots,n} a_i$ and $max\{b_i : i = 1, \ldots, n\} \leq B < \sum_{i=1,\ldots,n} b_i$.

The classical Dantzig-Wolfe master problem is given by:

$$\begin{cases} \max & \sum_{k=1}^{K} \left(\sum_{i=1}^{n} c_i x_i^{(k)} \right) \lambda_k \\ \text{s.c.} & \sum_{k=1}^{K} \left(\sum_{i=1}^{n} a_i x_i^{(k)} \right) \lambda_k \leq A \\ & \sum_{k=1}^{K} \lambda_k = 1 \\ & \lambda_k \geq 0, \quad k = 1, \ldots, K. \end{cases}$$

where $x^{(k)}, k = 1, \ldots, K$, are the extreme points of $\text{Conv}(\{x_i \in \{0,1\} \mid \sum_{i=1}^{n} b_i x_i \leq B, i = 1, \ldots, n\})$; and the pricing problem is:

$$\begin{cases} \min & \sum_{i=1}^{n} (c_i - \pi a_i) x_i - \pi A \\ \text{s.c.} & \sum_{i=1}^{n} b_i x_i \leq B \\ & x_i \in \{0,1\}, \quad i = 1, \ldots, n. \end{cases}$$

The master problem associated to LD decomposition is given by:

$$\begin{cases} \max \ \sum_{k=1}^{K} \Big(\sum_{i=1}^{n} c_i x_i^{(k)} \Big) \lambda_k \\ \sum_{k=1}^{K} \Big(\sum_{i=1}^{n} x_i^{(k)} \Big) \lambda_k - \sum_{l=1}^{L} \Big(\sum_{i=1}^{n} y_i^{(l)} \Big) \gamma_l = 0 \\ \sum_{k=1}^{K} \lambda_k = 1 \\ \sum_{l=1}^{L} \gamma_l = 1 \\ \lambda_k \geq 0, \quad k = 1, \ldots, K, \ \gamma_l \geq 0, \ l = 1, \ldots, L \end{cases}$$

where $x^{(k)}, k = 1, \ldots, K$ (resp. $y^{(l)}, l = 1, \ldots, L$), are the extreme points of $\mathrm{Conv}(\{x_i \in \{0,1\}, i = 1, \ldots, n \mid \sum_{i=1}^{n} b_i x_i \leq B, i = 1, \ldots, n\})$ (resp. $\mathrm{Conv}(\{y_i \in \{0,1\}, i = 1, \ldots, n \mid \sum_{i=1}^{n} a_i y_i \leq A\})$); and the pricing problems are:

$$\begin{cases} \min \ \sum_{i=1}^{n} u_i y_i \\ \text{s.c.} \ \sum_{i=1}^{n} a_i y_i \leq A \\ y_i \in \{0,1\}, \quad i = 1, \ldots, n \end{cases}$$

and

$$\begin{cases} \min \ \sum_{i=1}^{n} (c_i - u_i) x_i \\ \text{s.c.} \ \sum_{i=1}^{n} b_i x_i \leq B \\ x_i \in \{0,1\}, \quad i = 1, \ldots, n. \end{cases}$$

where $x_i, i = 1, \ldots, n$ and $y_i, i = 1, \ldots, n$ are equal to 1 if object i is filled in the knapsack.

4.2 The generalised assignment problem

It consists of finding a maximum profit assignment of T jobs to I agents such that each job is assigned to precisely one agent subject to capacity restrictions on the agents (Martello and Toth, 1992). The standard integer programming formulation is the following:

$$\begin{cases} \max & \sum_i \sum_t c_{it} x_{it} \\ \text{s.c.} & \sum_i x_{it} = 1, \quad t = 1, \dots, T \\ & \sum_t r_{it} x_{it} \le b_i, \quad i = 1, \dots, I \\ & x_{it} \in \{0,1\}, \quad i = 1, \dots, I, \ t = 1, \dots, T. \end{cases}$$

Two classical Dantzig-Wolfe decompositions can be made, by relaxing the assignment constraints or the capacity constraints.

The first classical Dantzig-Wolfe master problem is given by:

$$\begin{cases} \max & \sum_{k=1}^{K} \Big(\sum_i \sum_t c_{it} x_{it}^{(k)} \Big) \lambda_k \\ \text{s.c.} & \sum_{k=1}^{K} \Big(\sum_i x_{it}^{(k)} \Big) \lambda_k = 1, \quad t = 1, \dots, T \\ & \sum_{k=1}^{K} \lambda_k = 1 \\ & \lambda_k \ge 0, \ k = 1, \dots, K \end{cases}$$

where $x^{(k)}$, $k = 1, \dots, K$, are the extreme points of $\mathrm{Conv}(\{x_{it} \in \{0,1\} \mid \sum_t r_{it} x_{it} \le b_i, i = 1, \dots, I\})$; and the associated pricing problem is:

$$\begin{cases} \min & \sum_i \sum_t (c_{it} - \pi_t) x_{it} - \sum_t \pi_t \\ \text{s.c.} & \sum_t r_{it} x_{it} \le b_i, \qquad i = 1, \dots, I \\ & x_{it} \in \{0,1\}, \quad i = 1, \dots, I, \ t = 1, \dots, T. \end{cases}$$

The second classical Dantzig-Wolfe master problem is given by:

$$\begin{cases} \max & \sum_{l=1}^{L} \Big(\sum_i \sum_t c_{it} y_{it}^{(l)} \Big) \gamma_l \\ \text{s.c.} & \sum_{l=1}^{L} \Big(\sum_t r_{it} y_{it}^{(l)} \Big) \gamma_l \le b_i, \quad i = 1, \dots, I \\ & \sum_{l=1}^{L} \gamma_l = 1 \\ & \gamma_l \ge 0, \quad l = 1, \dots, L \end{cases}$$

where $y^{(l)}, l = 1, \dots, L$ are the extreme points of $\mathrm{Conv}(\{y_{it} \in \{0,1\} | \sum_i y_{it} = 1, t = 1, \dots, T\})$; and the associated pricing problem is:

$$\begin{cases} \min & \sum_i \sum_t (c_{it} - \pi_i) y_{it} - \sum_i \pi_i \\ \text{s.c.} & \sum_i y_{it} = 1, \quad t = 1, \dots, T \\ & y_{it} \in \{0,1\}, \quad i = 1, \dots, I, \ t = 1, \dots, T. \end{cases}$$

The master problem associated to LD is given by:

$$\begin{cases} \max & \sum_{k=1}^{K} \left(\sum_i \sum_t c_{it} x_{it}^{(k)} \right) \lambda_k \\ & \sum_{k=1}^{K} \left(\sum_i \sum_t x_{it}^{(k)} \right) \lambda_k - \sum_{l=1}^{L} \left(\sum_i \sum_t y_{it}^{(l)} \right) \gamma_l = 0 \\ & \sum_{k=1}^{K} \lambda_k = 1 \\ & \sum_{l=1}^{L} \gamma_l = 1 \\ & \lambda_k \geq 0, \quad k = 1, \dots, K, \ \gamma_l \geq 0, \ l = 1, \dots, L \end{cases}$$

where $x^{(k)}, k = 1, \dots, K$ (resp. $y^{(l)}, l = 1, \dots, L$), are the extreme points of $\mathrm{Conv}(\{x_{it} \in \{0,1\} | \sum_t r_{it} x_{it} \leq b_i, i = 1, \dots, I\})$ (resp. $\mathrm{Conv}(\{y_{it} \in \{0,1\} | \sum_i y_{it} = 1, t = 1, \dots, T\})$); and the pricing problems are:

$$\begin{cases} \min & \sum_i \sum_t u_{it} y_{it} \\ \text{s.c.} & \sum_i y_{it} = 1, \quad t = 1, \dots, T \\ & y_{it} \in \{0,1\}, \quad i = 1, \dots, I, \ t = 1, \dots, T \end{cases}$$

and

$$\begin{cases} \min & \sum_i \sum_t (c_{it} - u_{it}) x_{it} \\ \text{s.c.} & \sum_t r_{it} x_{it} \leq b_i, \quad i = 1, \dots, I \\ & x_{it} \in \{0,1\}, \quad i = 1, \dots, I, \ t = 1, \dots, T \end{cases}$$

where $x_{it}, i = 1, \dots, I, t = 1, \dots, T$ and $y_{it}, i = 1, \dots, I, t = 1, \dots, T$ are equal to 1 if job t is assigned to agent i.

5 Numerical experiments

This section is devoted to an experimental comparative study between LD and LR when solved by the CG algorithm. We consider the two optimisation problems defined in the previous section : the 0-1 bidimensional knapsack problem and the generalised assignment problem.

We consider in our tests 6 instances of the 0-1 bi-dimensional knapsack problem from the OR-Library. Table 1 presents a comparative study between CG resolution of LD and LR formulations (denoted CG_LD and CG_LR respectively). The master and pricing problems are solved by CPLEX11.2 solver.

CG_LR and CG_LD optimality are reached for all instances. As expected, LD gives better upper bounds then LR. On average on instances WEINGi, $i = 1, \dots, 6$, %vE associated to LD (resp. RL) is 0.02 (resp. 0.78), but we observe that the average resolution time of CG_LR (0.07 s) is very small compared to CG_LD computation time (10.54 s), this is due to the fact that the computational effort of each CG_LD iteration is greater than the CG_LR one and to the slow convergence of CG_LD compared to CG_LR.

We consider also in our tests 6 instances of the GAP from the OR-Library. All instances gap i, $i = 1, \dots, 6$ have the same sise, 5 agents and 15 jobs. The master and pricing problems are solved by CPLEX11.2 solver. Table 2 shows a comparison between LR and LD algorithms performances, when we apply for LR the second classical Dantzig-Wolfe decomposition, by relaxing the capacity constraints (cf. Section 4.2).

As before, CG_LR and CG_LD optimality are reached for all instances. LD gives better upper bounds then LR. On average on instances gap$i$, $i = 1, \dots, 6$, %vE associated to LD (resp. RL) is 0.13 (resp. 2.85), but we observe that the average resolution time of CG_LR (0.24 s) is still very small compared to CG_LD computation time (282.58 s).

The first classical Dantzig-Wolfe decomposition for LR, by relaxing the assignment constraints (cf. Section 4.2), has been also tested on the same instances, the results show that the bounds are tighter (but they are not better then those obtained by LD) and the CG algorithm takes more iterations and time to converge.

6 Conclusion

This paper focused on Dantzig-Wolfe Decomposition principle. We propose a new Dantzig-Wolfe master problem for ILP, which allows to propose an alternative dominance proof of LD bound over LR bound. As illustration, we have given the two Dantzig-Wolfe decomposition models for the 0-1 Bi-dimensional Knapsack Problem and the Generalised Assignment Problem. The obtained experimental results demonstrate the superiority of the LD bound, but the gain on bound quality impose an additional computation effort. In fact, at each iteration of the CG algorithm for the LD, two pricing problems (generally integer problems) have to be solved. Through this experimental study, we conclude that column generation resolution of LD can be useful if we want to obtain a good initial bound, as for example at the root node of a branch and bound or a branch and price scheme.

Table 1 Lagrangian Relaxation and LD for (0-1_BKP)

	vR	*%vE*	*Iter*	*tG*	*tSP*	*tMP*	*vR*	*%vE*	*Iter*	*tG*	*tSP*	*tMP*
			WEING1						*WEING2*			
CG_LR	141 388.50	0.1	6	0.12	0.12	0.00	130 883.00	0.0	1	0.01	0.01	0.00
CG_LD	141 383.00	0.1	136	9.55	8.72	0.24	130 883.00	0.0	157	13.61	12.56	0.40
			WEING3						*WEING4*			
CG_LR	97 613.92	2.0	5	0.13	0.11	0.00	122 321.58	2.5	7	0.08	0.06	0.01
CG_LD	95 677.00	0.0	142	11.42	10.64	0.25	119 337.00	0.0	156	12.68	11.54	0.33
			WEING5						*WEING6*			
CG_LR	98 796.00	0.0	1	0.01	0.00	0.01	130 697.80	0.1	6	0.05	0.05	0.00
CG_LD	98 796.00	0.0	77	3.51	2.99	0.16	130 623.00	0.0	162	12.47	11.51	0.33

vR: The relaxation value.
%vE: The gap between relaxation and optimal values.
Iter: Number of iterations.
tG: The global resolution time (s).
tSP: The global resolution time of pricing problems (s).
tM: Cumulated master problems resolution time (s).

Table 2 Lagrangian Relaxation and LD for (GAP)

	vR	*%vE*	*Iter*	*tG*	*tSP*	*tMP*	*vR*	*%vE*	*Iter*	*tG*	*tSP*	*tMP*
			gap1						*gap2*			
CG_LR	343.59	2.3	33	0.27	0.16	0.03	339.38	3.8	26	0.22	0.17	0.00
CG_LD	337.00	0.3	1169	383.13	343.61	29.37	327.00	0,0	894	258.41	234.55	15.78
			gap3						*gap4*			
CG_LR	349.68	3.2	33	0.22	0.14	0.01	350.40	2.8	31	0.25	0.17	0.00
CG_LD	339.50	0.1	945	273.18	245.89	19.01	341.00	0.0	878	282.25	258.89	15.74
			gap5						*gap6*			
CG_LR	335.76	3.0	35	0.28	0.17	0.05	351.82	2.0	30	0.22	0.12	0.08
CG_LD	327.25	0.4	595	163.86	149.73	9.05	345.00	0.0	1115	334.65	301.99	23.93

vR: The relaxation value.
%vE: The gap between relaxation and optimal values.
Iter: Number of iterations.
tG: The global resolution time (s).
tSP: The global resolution time of pricing problems (s).
tM: Cumulated master problems resolution time (s).

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