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An Infinitary Affine Lambda-Calculus Isomorphic to the Full Lambda-Calculus

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Abstract

It is well known that the real numbers arise from the metric completion of the rational numbers, with the metric induced by the usual absolute value. We seek a computational version of this phenomenon, with the idea that the role of the rationals should be played by the affine lambda-calculus, whose dynamics is finitary; the full lambda-calculus should then appear as a suitable metric completion of the affine lambda-calculus.

This paper proposes a technical realization of this idea: an affine lambda-calculus is introduced, based on a fragment of intuitionistic multiplicative linear logic; the calculus is endowed with a notion of distance making the set of terms an incomplete metric space; the completion of this space is shown to yield an infinitary affine lambda-calculus, whose quotient under a suitable partial equivalence relation is exactly the full (non-affine) lambda-calculus. We also show how this construction brings interesting insights on some standard rewriting properties of the lambda-calculus (finite developments, confluence, standardization, head normalization and solvability).

1 Introduction

The notion of linearity in computer science, which corresponds to the operational constraint of forcing all arguments of a function to be used exactly once, was brought forth about a quarter century ago by the introduction of linear logic [Gir87]. Since then, it has influenced many aspects of the development of the theory of functional languages: from denotational semantics, categorical semantics, and computational interpretations of classical logic [CHKM10], to higher-order languages for quantum computation [GM10], passing through a number of important pragmatic aspects, such as optimal reduction [AG98], constant-size programming [Hof03] and explicit substitutions [AK10].

From the perspective of the λ-calculus, Girard's translation of intuitionistic logic into linear logic brings to light a decomposition of β-reduction into a purely combinatorial part, in which the argument is simply fed to the function, and a structural one, in which the argument is duplicated or erased as necessary. If we abolish this latter part, we obtain a purely linear λ-calculus; if we allow erasing but not duplication, we obtain the *affine* λ-calculus.

As a rewriting system, the affine (or linear) fragment of the λ-calculus is extremely simple: its theory of residues is almost trivial, so all the fundamental properties of β-reduction, such as finiteness of developments, confluence, and standardization, become immediate to prove. Additionally, the calculus is strongly normalizing even in absence of types. This is because, as mentioned above, only the combinatorial part of β-reduction is left, so that the affine λ-calculus is really just a calculus of permutations: all that we are allowed to do with an atomic object (*i.e.*, a variable) is displace it, or erase it. Obviously, the expressive power of such a calculus is drastically reduced; that is why linear logic comes with additional constructors which allow, albeit in a controlled way, duplication of atomic objects, so that the full power of the λ-calculus may be recovered.

In this paper, we push forward the (*per se* rather obvious) idea that a non-linear, duplicable object is equivalent to infinitely many linear, non-duplicable objects, *i.e.*, we replace potential (non-linear) infinity with actual (linear) infinity. Of course, the rigorous manipulation of infinity requires some form of *topology*, and our idea will be satisfactorily realized only if non-linearity may be shown to arise from linearity in a topologically natural manner.

We may draw here an analogy with Cantor's definition of the real numbers as equivalence classes of Cauchy sequences of rational numbers. This analogy comes from an old remark of Girard [Gir87], who noticed how the purely linear fragment of linear logic seems to be, at least morally, "dense" in full linear logic (see Sect. 4.4). Our starting point will therefore be a λ-calculus directly drawn from linear logic. Technically, for the acquainted reader, we take a term calculus corresponding to the proofs of the fragment of intuitionistic multiplicative linear logic whose formulas are generated by $A, B ::= X \mid A_1 \otimes \cdots \otimes A_n \multimap B$, $n \in \mathbb{N}$ (with 1 being the nullary tensor).

Such a calculus is *polyadic*, *i.e.*, abstractions are of the form $\lambda\mathbf{x}.t$, with $\mathbf{x}$ a sequence of variables (each occurring at most once), and applications of the form $t\mathbf{u}$, with $\mathbf{u}$ a sequence of terms (all having disjoint free variables, also disjoint from those of t). For defining reduction, we face the problem of matching the arities of $\mathbf{x}$ and $\mathbf{u}$ in $(\lambda\mathbf{x}.t)\mathbf{u}$. This is usually solved by types, but we prefer to develop an untyped theory, and we solve this issue by introducing a special term $\bot$ which is substituted to $\mathbf{x}(i)$ in case $\mathbf{u}(i)$ does not exist. Then, the definition of our terms will be something like

$$t ::= \bot \mid x \mid \lambda\mathbf{x}.t \mid t\mathbf{u},$$

where x ranges over a set of variables $\mathcal{V}$, $\mathbf{x}$ is an injective function in $\mathcal{V}^{\mathbb{N}}$, and $\mathbf{u}$ is a function from $\mathbb{N}$ to terms which is almost everywhere $\bot$; all this, of course, with the restriction that every variable, free or bound, appears at most once. Reduction is defined by the usual rule

$$(\lambda\mathbf{x}.t)\mathbf{u} \to t[\mathbf{u}/\mathbf{x}],$$

which now makes sense because the operation of substituting $\mathbf{u}(i)$ to the only free occurrence (if any) of $\mathbf{x}(i)$ in t is defined for all $i \in \mathbb{N}$. We denote by $\Lambda_{\mathrm{p}}^{\mathrm{aff}}$ the calculus just introduced (the subscript p is for "polyadic", and the superscript reminds us that the calculus is affine). In the end, $\Lambda_{\mathrm{p}}^{\mathrm{aff}}$ is very similar to Boudol's λ-calculus with multiplicities [Bou93], with sequences instead of multisets.

The next step is turning $\Lambda_{\mathrm{p}}^{\mathrm{aff}}$ into a metric space. Considering the fact that we are manipulating sequences, using a metric of pointwise convergence is the simplest choice. Actually, we slightly modify the metric so that only terms whose height is the same may be "close". The intuition behind the definition of the metric may be given with an example. Consider the terms $\Delta_n = \lambda x_0 \dots x_n . x_0 \langle x_1, \dots, x_n \rangle$, where we used the notation $\langle \dots \rangle$ to explicitly denote sequences. These correspond to programs which take a list of terms, extract the head, and apply it to the rest of the list. Our metric, which is basically pointwise convergence, will give us that the sequence $(\Delta_n)_{n \in \mathbb{N}}$ is Cauchy. And yet, no finite term may be the limit, because the length of the sequence given in argument to x_0 is unbounded. Clearly, the sequence is tending to the infinite term $\Delta = \lambda x_0 x_1 x_2 \dots . x_0 \langle x_1, x_2, \dots \rangle$.

We have just proved, informally, that the metric space $\Lambda_{\mathrm{p}}^{\mathrm{aff}}$ is not complete. Its completion, which we denote by $\Lambda_{\infty}^{\mathrm{aff}}$, contains terms whose applications may accept infinite arguments, *i.e.*, the sequence $\mathbf{u}$ in $t\mathbf{u}$ is no longer forced to be almost everywhere $\bot$. Of course, this also means that the abstraction $\lambda \mathbf{x}.t$ is allowed to bind infinitely many variables.

Observe that $\Lambda_{\infty}^{\mathrm{aff}}$ is still affine: every variable appears at most once. However, as expected, the calculus is no longer strongly normalizing: consider $\Omega = \Delta \langle \Delta, \Delta, \Delta, \dots \rangle$; the term Δ, described above, takes an *infinite* list, extracts the head, and applies it to the rest of the (infinite) list. Then, obviously, Ω reduces to itself.

We start recognizing Girard's translation of intuitionistic logic in linear logic: a non-linear argument becomes infinitely many linear arguments. However, $\Lambda_{\infty}^{\mathrm{aff}}$ is still not the λ-calculus; it is too big, it contains a continuum of terms. The next step then is to restrict the behavior of infinitary application, so that, in $t\mathbf{u}$, all terms $\mathbf{u}(i)$ actually "look alike". In the context of denotational semantics of linear logic, this is called *uniformity* (see, for instance, [Gir01, Mel04]).

Just as in [Gir01], we introduce uniformity by means of a partial equivalence relation (PER) on $\Lambda_{\mathrm{p}}^{\mathrm{aff}}$, called *renaming equivalence* and denoted by $\approx$; a term t is deemed *uniform* if $t \approx t$. Intuitively, renaming equivalence relates terms that "look alike" under any permutation of their application sequences. This is why $\approx$ is only partial: for example, if u is not a variable, then $u \not\approx x$ and the term $t = z\langle x, u, u, \dots \rangle$ will not be uniform because, for instance, $z\langle u, x, u, \dots \rangle$ does not "look like" t. On the other hand, $\lambda x_0 x_1 x_2 \dots . x_1 \langle x_0, x_2, \dots \rangle \approx \Delta$ (hence both are uniform). This makes sense because, in a "uniform world", the variables x_0, x_1 will be replaced by equivalent ("look-alike") terms, so their exchange has no consequence (consider, for instance, the definition of Ω above).

The set $\Lambda_{\infty}^{\mathrm{u}}$ of uniform affine terms has a problem though: it is not stable under reduction. Indeed, suppose u is a term s.t. $u \to u'$. Then, $z\langle u, u, u, \dots \rangle \to z\langle u', u, u, \dots \rangle$, which is no longer uniform, since u' has no reason to "look like" u. The obvious solution is to consider *infinitary reduction*, *i.e.*, define the reduction $z\langle u, u, u, \dots \rangle \Rightarrow z\langle u', u', u', \dots \rangle$ as one step. This actually works quite smoothly, without even needing the techniques developed in the context of infinitary term rewriting [DKP91, KKSdV95]. It is easy to show that $\Rightarrow$ is compatible with $\approx$, *i.e.*, equivalent terms reduce to equivalent terms. The set $\Lambda_{\infty}^{\mathrm{u}}$ with the notion of reduction $\Rightarrow$ is thus a well defined calculus.

The final result may be stated as follows:

$$(\Lambda, \to_{\beta}) \cong (\Lambda_{\infty}^{\mathrm{aff}} / {\approx}, \Rightarrow),$$

that is, the usual, full λ-calculus, with β-reduction, is isomorphic to the infinitary affine calculus, modulo $\approx$, with infinitary reduction $\Rightarrow$ (note that, by definition, the quotient automatically discards non-uniform terms). The isomorphism must be understood in the Curry-Howard sense: there is a bijection between Λ and $\Lambda_\infty^{\mathrm{aff}}/\approx$ which preserves the basic constructions of the calculi (abstraction and application), and which commutes with their respective notions of reduction.

The above result is probably not very surprising for the reader familiar with linear logic (and especially with Ehrhard's recent work on differential linear logic and the Taylor expansion of λ-terms [ER08]) or with Boudol's work on the λ-calculus with multiplicities [Bou93]. Other readers may instead have recognized the ideas developed here to be already present in games semantics [AJM00, Mel04]. However, we claim that the above isomorphism, when technically formulated as we do in this paper, brings some interesting insights and novel perspectives on well known properties of reduction in the λ-calculus, such as the finiteness of developments, confluence, standardization, solvability, and head normalization. We discuss this in Sect. 4.

The rest of the paper is devoted to formally presenting these results. However, instead of starting from $\Lambda_{\mathrm{p}}^{\mathrm{aff}}$ and building its completion, we begin in Sect. 2 with a much bigger calculus, named Λ_∞, which is infinitary and not affine. In fact, the definition of metric and the essential properties of completeness and density do not need affinity, which would only be an unnecessary (and annoying) technical restriction at this stage. Then, in Sect. 3, we define $\Lambda_{\mathrm{p}}^{\mathrm{aff}}$ and $\Lambda_\infty^{\mathrm{aff}}$ as affine subcalculi of Λ_∞, and verify that the latter is the metric completion of the former. After that, we follow the pattern delineated in this introduction.

2 A Polyadic Infinitary Lambda-Calculus

2.1 Terms and Reduction

Definition 1 (Terms) *We fix a denumerably infinite set $\mathcal{V}$ of* variables*, ranged over by x, y, z and not including $\bot$, and denote by $\mathcal{V}_{\mathrm{inj}}^{\mathbb{N}}$ the subset of $\mathcal{V}^{\mathbb{N}}$ of functions which are injective. Then, we define*

$$\begin{aligned}\Lambda_\infty^0 &= \mathcal{V} \cup \{\bot\},\\ \Lambda_\infty^{h+1} &= \Lambda_\infty^h \cup \{\lambda\mathbf{x}.t \mid t \in \Lambda_\infty^h,\, \mathbf{x} \in \mathcal{V}_{\mathrm{inj}}^{\mathbb{N}}\} \cup \\ &\quad \cup \{t\mathbf{u} \mid t \in \Lambda_\infty^h,\, \mathbf{u} \in (\Lambda_\infty^h)^{\mathbb{N}}\},\end{aligned}$$

and let the set of terms *be $\Lambda_\infty = \bigcup_{h\in\mathbb{N}} \Lambda_\infty^h$. The* height *of a term t is the least h such that $t \in \Lambda_\infty^h$. We denote by $\mathcal{B}(\Lambda_\infty)$ the subset of $\Lambda_\infty^{\mathbb{N}}$ of sequences of bounded height, i.e., if $\mathbf{u} \in \mathcal{B}(\Lambda_\infty)$, then there exists $h \in \mathbb{N}$ such that, for all $i \in \mathbb{N}$, the height of $\mathbf{u}(i)$ is at most h.*

The fact that terms, albeit infinitary, have a finite height allows us to manipulate them using induction (on their height). Therefore, the standard notions of free and bound variable, α-equivalence, and of capture-free substitution are defined as usual. In particular, given $s \in \Lambda_\infty$, $\mathbf{u} \in \mathcal{B}(\Lambda_\infty)$, and $\mathbf{x} \in \mathcal{V}_{\mathrm{inj}}^{\mathbb{N}}$, we define $s[\mathbf{u}/\mathbf{x}]$ by induction:

- $x[\mathbf{u}/\mathbf{x}] = \begin{cases} \mathbf{u}(i) & \text{if } \exists i \in \mathbb{N}.\, \mathbf{x}(i) = x; \\ x & \text{if } \forall i \in \mathbb{N},\, \mathbf{x}(i) \neq x; \end{cases}$
- $(\lambda \mathbf{y}.t)[\mathbf{u}/\mathbf{x}] = \lambda \mathbf{y}.t[\mathbf{u}/\mathbf{x}]$;
- $(t\mathbf{v})[\mathbf{u}/\mathbf{x}] = t[\mathbf{u}/\mathbf{x}]\mathbf{v}'$, where $\forall i \in \mathbb{N}$, $\mathbf{v}'(i) = \mathbf{v}(i)[\mathbf{u}/\mathbf{x}]$.

Of course, this operation is not effective in general, as it may require an infinite amount of work. We denote by $\mathrm{fv}(t)$ the set of free variables of t. As usual, terms are always considered up to α-equivalence. A *context* is a function from Λ_∞ to itself, defined inductively as follows: the identity function is a context; if C is a context, then:

- for all $\mathbf{x} \in \mathcal{V}^{\mathbb{N}}_{\mathrm{inj}}$, $t \mapsto \lambda \mathbf{x}.C(t)$ is a context;
- for all $\mathbf{u} \in \mathcal{B}(\Lambda_\infty)$, $t \mapsto C(t)\mathbf{u}$ is a context;
- for all $n \in \mathbb{N}$, $u, v_0, \ldots, v_{n-1} \in \Lambda_\infty$, and $\mathbf{v} \in \mathcal{B}(\Lambda_\infty)$, $t \mapsto u\mathbf{v}^n_C[t]$ is a context, where $\mathbf{v}^n_C[t] \in \mathcal{B}(\Lambda_\infty)$ is defined as follows:

$$\mathbf{v}^n_C[t](i) = \begin{cases} v_i & \text{if } 0 \leq i < n; \\ C(t) & \text{if } i = n; \\ \mathbf{v}(i-n-1) & \text{if } i > n. \end{cases}$$

One-step reduction is defined by

1. $(\lambda \mathbf{x}.t)\mathbf{u} \to t[\mathbf{u}/\mathbf{x}]$;
2. if $t \to t'$, then $C(t) \to C(t')$, for every context C.

Reduction, denoted by $\to^*$, is its reflexive-transitive closure.

2.2 Infinitary Terms as the Metric Completion of Finite Terms

In what follows, if X, Y are sets s.t. $\bot \in X$, and if $f \in X^Y$, we define the *support* of f as $\mathrm{supp}\, f = \{y \in Y \mid f(y) \neq \bot\}$.

Definition 2 (Finite term) *If S is a set s.t. $\bot \in S$, we denote by $S^{(\mathbb{N})}$ the subset of $S^{\mathbb{N}}$ of functions with finite support. The elements of $S^{(\mathbb{N})}$ are finite sequences, and we denote them by $\langle s_0, \ldots, s_{n-1}\rangle$, with $s_{n-1} \neq \bot$ and $s_{n+p} = \bot$ for all $p \in \mathbb{N}$.* Finite terms *are generated by the following grammar:*

$$t, u ::= \bot \mid x \mid \lambda x_0 \ldots x_{n-1}.t \mid t\langle u_0, \ldots, u_{n-1}\rangle.$$

We denote by Λ_{p} the set of finite terms. Obviously, $\Lambda_{\mathrm{p}} \subseteq \Lambda_\infty$: the embedding is trivial for variables and applications; for abstractions, we embed $\lambda x_0 \ldots x_{n-1}.t$ as $\lambda \mathbf{x}.t$, with $\mathbf{x}(i) = x_i$ for $0 \leq i < n$, while the values for $i \geq n$ are arbitrary (and irrelevant by α-equivalence).

The calculus Λ_{p} is nothing but a polyadic λ-calculus (the subscript p is for "polyadic"), with a special term $\bot$ which behaves as a variable impervious to substitution; α-equivalence, substitution, and reduction are all effective operations on Λ_{p}, as in the usual λ-calculus. We shall see that this finite calculus is actually enough to describe Λ_∞, via a metric completion process.

The standard way of defining a distance between terms is to consider them as trees [AN80, Cou83, DKP91, CO90, KKSdV95, KKSdV97].

Definition 3 (Partial trees) *A* position *is an element of* $\mathbb{P} = \mathbb{N}^*$*, the set of finite words over* $\mathbb{N}$*; an* alphabet *is a set* Σ *s.t.* $\bot \notin \Sigma$*, and a* partial tree *is an element of* $\mathbb{T}(\Sigma) = (\Sigma \cup \{\bot\})^{\mathbb{P}}$*. A partial tree is* finite *if its support is finite; it has* finite height *if the words in its support have bounded length. We denote by* $\mathbb{T}_0(\Sigma)$ *and* $\mathbb{T}_h(\Sigma)$ *the sets of all finite partial trees and of all partial trees of finite height, resp., on the alphabet* Σ*.*

If we set $\Sigma_\lambda = \mathcal{V} \cup \{\lambda\mathbf{x} \mid \mathbf{x} \in \mathcal{V}_{\text{inj}}^{\mathbb{N}}\} \cup \{@\}$, terms may be injectively mapped into $\mathbb{T}_h(\Sigma_\lambda)$ in the obvious way, so that we may speak of a position a within a term t, and freely use the notation $t(a)$ to denote the symbol present at that position.

If we consider $\Sigma \cup \{\bot\}$ to be equipped with the discrete metric (as will always be the case in this paper), the function space $\mathbb{T}(\Sigma)$ may be endowed with the topology of *simple* (or pointwise) convergence, which is metrizable as follows. If $a = n_1 \cdots n_k \in \mathbb{P}$, we set $\alpha(a) = 2^{-n_1-1} \cdots 2^{-n_k-1}$, and define

$$d_s(t, t') = \sup\{\alpha(a) \mid a \in \mathbb{P},\, t(a) \neq t'(a)\}.$$

It is immediate to see that d_s is a bounded ultrametric, whose uniformity admits as base of entourages the sets

$$\mathcal{U}_\varepsilon = \{(t, t') \in \mathbb{T}(\Sigma)^2 \mid \forall a \in A_\varepsilon,\ t(a) = t'(a)\},$$

where $0 < \varepsilon \leq 1$ and $A_\varepsilon = \{a \in \mathbb{P} \mid \alpha(a) \geq \varepsilon\}$. Note that A_ε grows bigger as ε becomes smaller, the extreme cases being $A_1 = \{\epsilon\}$ (the empty word) and $A_0 = \mathbb{P}$. Hence, closer and closer trees coincide on more and more positions, which means that d_s yields the topology of simple convergence (whence the subscript s). It is not hard to see (*cf.* [AN80, Cou83]) that $(\mathbb{T}(\Sigma), d_s)$ is a complete metric space, in which $\mathbb{T}_0(\Sigma)$ is dense, which means that the former is the metric completion of the latter.

It is easy to modify the metric on d_s so that the completion contains only trees whose height is finite. Consider the trivial pseudometric ρ on $\mathbb{T}_0(\Sigma)$ such that $\rho(t, t') = 0$ if t, t' have the same height, and $\rho(t, t') = 1$ otherwise. Then, $\max(d_s, \rho)$ is a bounded ultrametric, according to which a sequence $(t_n)_{n \in \mathbb{N}} \in \mathbb{T}_0(\Sigma)$ is Cauchy iff it is Cauchy for d_s and, for n sufficiently large, every t_n has the same height. Therefore, the completion of $(\mathbb{T}_0(\Sigma), \max(d_s, \rho))$ is the space $\mathbb{T}_h(\Sigma)$ of trees with finite height, but possibly infinite width.

The above metric immediately applies to terms, regarded as elements of $\mathbb{T}_h(\Sigma_\lambda)$, provided we use the following version of Barendregt's convention: we partition $\mathcal{V}$ into two sets $\mathcal{V}_b$ and $\mathcal{V}_f$, and we fix an injective function $\nu : \mathbb{P} \times \mathbb{N} \to \mathcal{V}_b$; then, we stipulate that all bound (resp. free) variables in terms belong to $\mathcal{V}_b$ (resp. $\mathcal{V}_f$); furthermore, given a term t and a position a, we assume that, whenever $t(a) = \lambda\mathbf{x}$, we have, for all $i \in \mathbb{N}$, $\mathbf{x}(i) = \nu(a, i)$.

Actually, we may directly define, by induction on terms, a closely related family of ultrametrics:

Definition 4 (Inductive ultrametric on terms) *We say that two terms are* isocephalous *if they are either equal, or both abstractions, or both applications. Then, given* $0 < \gamma_1, \gamma_2 \leq 1$*, we define an ultrametric* d *by induction on terms:* d *is discrete on* Λ^0_∞*; given* $t, t' \in \Lambda^{h+1}_\infty$*,* $d(t, t') = 1$ *as soon as* t, t' *are not* *isocephalous, or have different heights; if they are isocephalous and of equal height, we have two cases:*

$$d(\lambda\mathbf{x}.t_1, \lambda\mathbf{x}.t_1') = \gamma_1 d(t_1, t_1'),$$
$$d(t_1\mathbf{u}, t_1'\mathbf{u}') = \gamma_2 \max\left(d(t_1, t_1'), \sup_{i\in\mathbb{N}} \frac{d(\mathbf{u}(i), \mathbf{u}'(i))}{2^{i+1}}\right).$$

Note that, in the abstraction case, we used α-equivalence to rewrite t and t' so that the abstracted variables coincide. This, together with assuming the range of $\mathbf{x}$ to be "fresh", amounts to the Barendregt convention used above.[1]

It is not hard to see that, setting $\gamma_1 = \gamma_2 = \frac{1}{2}$, one has $d = \max(d_s, \rho)$. Furthermore, all of the above metrics are uniformly equivalent, so the actual value of γ_1, γ_2 does not matter. The interest is that the inductive definition is sometimes more convenient than the definition on trees. In the following, we shall use d to denote the inductive metric obtained by setting $\gamma_1 = \gamma_2 = 1$.

Lemma 1 *Regarded as a subset of $(\mathbb{T}_h(\Sigma_\lambda), \max(d_s, \rho))$, Λ_∞ is closed.*

PROOF. Let t be a tree which is not a term. This means that t contains an "anomaly", such as a node labelled by a variable which is not a leaf, or a node labelled by $\lambda\mathbf{x}$ which has more than one sibling. This anomaly occurs at some position $a \in \mathbb{P}$. Now, take any ε such that $a \in A_\varepsilon$ (this certainly exists); every term in the open ball of radius ε centered at t contains the same anomaly, which proves that $\mathbb{T}_h(\Sigma_\lambda) \setminus \Lambda_\infty$ is open, hence Λ_∞ is closed. □

Lemma 2 (Approximations) *For every $t \in \Lambda_\infty$ and $n \in \mathbb{N}$, there exists $\lfloor t \rfloor_n \in \Lambda_{\mathrm{p}}$ such that*

$$d(\lfloor t \rfloor_n, t) < 2^{-n}.$$

PROOF. We define $\lfloor t \rfloor_n$ by induction on t:

- If $t \in \Lambda_\infty^0$, $\lfloor t \rfloor_n = t$, for all $n \in \mathbb{N}$. Note that $\lfloor t \rfloor_n$ is obviously finite.
- If $t = \lambda\mathbf{x}.t'$, then $\lfloor t \rfloor_n = \lambda\mathbf{x}.\lfloor t' \rfloor_n$. By induction, $\lfloor t \rfloor_n$ is finite.
- If $t = t'\mathbf{u}$, then $\lfloor t \rfloor_n = \lfloor t' \rfloor_n \langle \lfloor \mathbf{u}(0) \rfloor_n, \ldots, \lfloor \mathbf{u}(n-1) \rfloor_n \rangle$, which, by induction, is obviously finite.

The inequality $d(\lfloor t \rfloor_n, t) < 2^{-n}$ may now be proved by a straightforward induction on t. □

Proposition 3 *(Λ_∞, d) is the metric completion of $(\Lambda_{\mathrm{p}}, d)$.*

PROOF. We know that $(\mathbb{T}_h(\Sigma_\lambda), \max(d_s, \rho))$ is complete; then, by Lemma 1, and by the fact that d is uniformly equivalent to $\max(d_s, \rho)$, (Λ_∞, d) is itself complete. Lemma 2 shows that Λ_{p} is dense in Λ_∞, so we conclude by uniqueness of the completion. □

[1] These fastidious issues disappear if we consider so-called *nameless terms*, *i.e.*, using de Bruijn indices [dB72]. However, for the sake of readability, we chose to stick to the more friendly "named" notation.

2.3 Cauchy-Continuity of Reduction

It is methodologically interesting to observe that the reduction relation on Λ_∞, which is not effective and may therefore have a dubious computational value, may actually be inferred, in a unique way, from reduction in Λ_{p}.

Let $a \in \mathbb{P}$; we define a function $R_a : \Lambda_{\mathrm{p}} \to \Lambda_{\mathrm{p}}$ as follows: $R_a(t) = t'$ if there is a redex at position a in t (*i.e.*, regarding t as a tree, $t(a) = @$ and $t(a0) = \lambda\mathbf{x}$), and we obtain t' by reducing it; otherwise, $R_a(t) = t$. We shall see that each R_a is *Cauchy-continuous*; then, by a standard result of analysis, each R_a uniquely extends to a continuous function on the completion Λ_∞, *i.e.*, the Cauchy-continuity of reduction on Λ_{p} automatically implies the existence of a (unique) notion of reduction on Λ_∞ (note that simple continuity would not be enough).[2] Of course we already know this (we can define R_a directly on Λ_∞), but what we want to stress here is that, *even if we were not able to explicitly describe the terms of* Λ_∞, we would still know how to reduce them.

In the rest of this section, we shall concentrate on finite terms. Observe that contexts (the definition of which may be adapted to the finite case in the obvious way) are injective, so every context C establishes a bijection between Λ_{p} and $C(\Lambda_{\mathrm{p}})$. This bijection may actually be seen to be a uniform homeomorphism, when $C(\Lambda_{\mathrm{p}})$ is considered as a metric subspace of Λ_{p}. In other words, Λ_{p} is "isotropic": its uniform structure does not change when we embed it in itself, considering terms as subterms in any possible position.

Lemma 4 (Isotropy) *Every context C induces a uniform homeomorphism between Λ_{p} and $C(\Lambda_{\mathrm{p}})$.*

PROOF. First, by a straightforward induction on C, using the inductive definition of d, one can prove that contexts are short maps, *i.e.*, for every t, t', $d(C[t], C[t']) \leq d(t, t')$. This shows that every C is a uniformly continuous injection. We are therefore left with proving that the inverse function $C^{-1} : C(\Lambda_{\mathrm{p}}) \to \Lambda_{\mathrm{p}}$ is also uniformly continuous. This is done again by induction on C. The base case (the identity context) is trivial. For the inductive cases, we only consider $C = u_0\langle u_1, \ldots, u_{i-1}, C', u_{i+1}, \ldots, u_n\rangle$ for some context C'; the other cases are analogous. Let $\varepsilon > 0$; we must find a $\delta > 0$ such that, for all t, t', $d(C[t], C[t']) < \delta$ implies $d(t, t') < \varepsilon$. By induction hypothesis, we have a δ_0 such that $d(C'[t], C'[t']) < \delta_0$ implies $d(t, t') < \varepsilon$; but the inductive definition of d gives us $d(C[t], C[t']) = 2^{-i-1} d(C'[t], C'[t'])$, so obviously $\delta = 2^{-i-1}\delta_0$ yields the desired property. □

Observe how the metric d induces a metric on $\Lambda_{\mathrm{p}}^{(\mathbb{N})}$, by setting $\mathbf{d}(\mathbf{u}, \mathbf{u}') = d(\bot\mathbf{u}, \bot\mathbf{u}')$ (actually, $\bot$ may be replaced by any other term, yielding the same metric). Note that this is *not* equivalent to the metric yielding the topology of pointwise convergence; in fact, this latter would admit Cauchy sequences $(\mathbf{u}_n)_{n\in\mathbb{N}}$ in which the height of the terms in $\mathbf{u}_n$ is unbounded.

[2] Let X, Y be metric spaces. A function $f : X \to Y$ is *Cauchy-continuous* if it preserves Cauchy sequences, *i.e.*, whenever $(x_n)_{n\in\mathbb{N}}$ is Cauchy in X, $(f(x_n))_{n\in\mathbb{N}}$ is Cauchy in Y. Uniform continuity implies Cauchy-continuity, which in turn implies continuity. The converse implications fail in general; however, continuity implies Cauchy-continuity if X is complete, while all three notions are equivalent if X is compact. Let $\overline{X}, \overline{Y}$ denote the completions of $X.Y$. A continuous function $f : X \to Y$ admits a continuous extension $\overline{f} : \overline{X} \to \overline{Y}$ (by "extension" we mean $\overline{f}(x) = f(x)$ for all $x \in X$) iff it is Cauchy-continuous. In that case, by virtue of its continuity and of the density of X in $\overline{X}$, $\overline{f}$ is *a fortiori* unique.

Lemma 5 (Application is a uniform homeomorphism) *The application map* $(t, \mathbf{u}) \mapsto t\mathbf{u}$ *induces a uniform homeomorphism from the product of* $(\Lambda_{\mathrm{p}}, d)$ *and* $(\Lambda_{\mathrm{p}}^{(\mathbb{N})}, \mathbf{d})$ *(with the product uniformity) to* $\Lambda_{\mathrm{p}}^{@}$*, the subspace of terms which are applications.*

Proof. Analogous to the the proof of Lemma 4. □

In the following, when we say that a certain property "eventually" holds for a sequence $(t_n)_{n\in\mathbb{N}}$, we mean that there exists $n \in \mathbb{N}$ s.t. all t_{n+p} have that property, for all $p \in \mathbb{N}$.

Lemma 6 *For every Cauchy sequence* $(t_n)_{n\in\mathbb{N}}$ *in* $(\Lambda_{\mathrm{p}}, d)$*, there exists* $h \in \mathbb{N}$ *such that eventually all* t_n *have height* h *and are isocephalous. Then, one of the following holds:*

1. *eventually,* $t_n = \bot$*, or* $t_n = x$ *for some* $x \in \mathcal{V}$*;*
2. *eventually,* $t_n = \lambda\mathbf{x}.t'_n$*, and* $(t'_n)_{n\in\mathbb{N}}$ *is Cauchy;*
3. *eventually,* $t_n = t'_n\mathbf{u}_n$*, and both* $(t'_n)_{n\in\mathbb{N}}$*,* $(\mathbf{u}_n)_{n\in\mathbb{N}}$ *are Cauchy.*

Conversely, if $(t_n)_{n\in\mathbb{N}}$ *and* $(\mathbf{u}_n)_{n\in\mathbb{N}}$ *are Cauchy, then:*

4) *for all* $\mathbf{x} \in \mathcal{V}_{\mathrm{inj}}^{\mathbb{N}}$*,* $(\lambda\mathbf{x}.t_n)_{n\in\mathbb{N}}$ *is Cauchy;*

5) $(t_n\mathbf{u}_n)_{n\in\mathbb{N}}$ *is Cauchy.*

Proof. That a Cauchy sequence is eventually isocephalous is obvious; then, point 1 is the case $h = 0$, and points 2 and 3 cover the case $h > 0$. Points 2 and 4 (resp. 3 and 5) are immediate consequences of Lemma 4 (resp. Lemma 5). □

Lemma 7 (Substitution is Cauchy-continuous) *If* $(t_n)_{n\in\mathbb{N}}$ *and* $(\mathbf{u}_n)_{n\in\mathbb{N}}$ *are Cauchy sequences in* $(\Lambda_{\mathrm{p}}, d)$ *and* $(\Lambda_{\mathrm{p}}^{(\mathbb{N})}, \mathbf{d})$*, resp., then* $(t_n[\mathbf{u}_n/\mathbf{x}])_{n\in\mathbb{N}}$ *is Cauchy in* $(\Lambda_{\mathrm{p}}, d)$*, for all* $\mathbf{x} \in \mathcal{V}_{\mathrm{inj}}^{\mathbb{N}}$.

Proof. By Lemma 6, we may associate a height h with $(t_n)_{n\in\mathbb{N}}$; the result is then proved by a straightforward induction on h, applying the various points of Lemma 6. □

Proposition 8 *For every* $a \in \mathbb{P}$*,* R_a *is Cauchy-continuous.*

Proof. By the Isotropy Lemma 4, it is enough to prove the result for R_ϵ (with ϵ denoting the empty word). Let $(s_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. By Lemma 6, either eventually all s_n contain a redex at position ϵ, or eventually none of them does. In the latter case, we may conclude; in the former, we have, eventually, $s_n = (\lambda\mathbf{x}.t_n)\mathbf{u}_n$, with $(t_n)_{n\in\mathbb{N}}$ and $(\mathbf{u}_n)_{n\in\mathbb{N}}$ both Cauchy, by Lemma 6. Then, $R_\epsilon(s_n) = t_n[\mathbf{u}_n/\mathbf{x}]$, so we may conclude by Lemma 7. □

We remark that Proposition 8 cannot be improved, *i.e.*, reduction is not uniformly continuous. Indeed, define $s_n = (\lambda x_0 \ldots x_{n-1}x_n.x_n)$ and $\mathbf{u}_n = \langle\bot, \ldots, \bot, y\rangle$, with n occurrences of $\bot$; it is immediate to check that $\lim \mathbf{u}_n = \langle\rangle$ (whereas $(s_n)_{n\in\mathbb{N}}$ is not Cauchy). Then, if we let $t_n = s_n\mathbf{u}_n$ and $t'_n = s_n\langle\rangle$, for every $\delta > 0$, there exists $n \in \mathbb{N}$ such that $d(t_n, t'_n) < \delta$; and yet, $d(R_\epsilon(t_n), R_\epsilon(t'_n)) = d(y, \bot) = 1$.

All the proofs of this section are still valid if we replace Λ_{p} and $\Lambda_{\mathrm{p}}^{(\mathbb{N})}$ with Λ_∞ and $\mathcal{B}(\Lambda_\infty)$, resp., showing that reduction on infinitary terms as we naively defined it in Sect. 2.1 is (Cauchy-)continuous. Therefore, if we demand topological compatibility with the usual, finitary reduction, that naive definition is actually the only possible one.

3 The Affine Subcalculi

3.1 Affine Terms

Definition 5 (Affine terms) *A term $t \in \Lambda_\infty$ is* affine *if every variable, free or bound, appears at most once in t. We denote by $\Lambda_\infty^{\mathrm{aff}}$, $\Lambda_{\mathrm{p}}^{\mathrm{aff}}$, and $\mathcal{B}(\Lambda_\infty^{\mathrm{aff}})$ the set of all affine terms, of all finite affine terms, and of all sequences of bounded height of affine terms, resp.*

Note that affinity is obviously preserved by reduction.

Proposition 9 *$(\Lambda_\infty^{\mathrm{aff}}, d)$ is the metric completion of $(\Lambda_{\mathrm{p}}^{\mathrm{aff}}, d)$.*

Proof. Observe that, whenever $t \in \Lambda_\infty^{\mathrm{aff}}$, the approximations $\lfloor t \rfloor_n$ defined in the proof of Lemma 2 are all affine. This shows that $\Lambda_{\mathrm{p}}^{\mathrm{aff}}$ is dense in $\Lambda_\infty^{\mathrm{aff}}$. Then, by completeness of Λ_∞, it is enough to prove that $\Lambda_\infty^{\mathrm{aff}}$ is a closed subset of it. This may be done essentially by the same argument given in the proof of Lemma 1. □

As a finite (polyadic) affine calculus, $\Lambda_{\mathrm{p}}^{\mathrm{aff}}$ enjoys strong normalization and a strong form of confluence (the diamond property of $\to^{=}$, *i.e.*, reduction in at most one step). The former is proved by a size-decreasing argument; the latter is a consequence of the absence of duplication.

Although infinitary, the calculus $\Lambda_\infty^{\mathrm{aff}}$ is still affine, and enjoys the same strong form of confluence (which is notoriously false in the λ-calculus). As in Sect. 2.3, we give a topological proof of this property, which does not suppose we know how to explicitly describe infinitary affine terms (and their reduction).

Proposition 10 *In $\Lambda_\infty^{\mathrm{aff}}$, the relation $\to^{=}$, i.e., reduction in at most one step, enjoys the diamond property.*

Proof. We start with a little analysis of how residues behave in $\Lambda_{\mathrm{p}}^{\mathrm{aff}}$. Let r', r'' be distinct redexes at position a', a'' in the same term. We want to locate the position a''_1 of the (at most unique) residue of r'' after reducing r'. We say that r' *contains* r'' if $a' \leq a''$ in the prefix order. Assuming that r'' does not contain r' (which, by symmetry, is a redundant situation), we have four possibilities:

i) r' does not contain r''; then, $a''_1 = a''$;

ii) $a'' = a'00b$; then, $a''_1 = a'b$;

iii) $a'' = a'ib$, with $i > 0$, and r'' has no residue at all;

iv) $a'' = a'ib$, with $i > 0$, and r'' has a (unique) residue; then, $a''_1 = a'b'b$, for some $b' \in \mathbb{P}$ which depends on the position of the substituted variable.

In all four cases, r' has exactly one residue after reducing r'', and its position is always a'.

Now, let $t \in \Lambda_\infty^{\mathrm{aff}}$ be such that $t' \,{}^{=}\!\!\leftarrow t \to^{=} t''$. We may suppose that both reductions are strict, otherwise confluence is immediate. With the notations of Sect. 2.3, we have $R_{a'}(t) = t'$ and $R_{a''}(t) = t''$, for some $a', a'' \in \mathbb{P}$. Of course we may assume $a' \neq a''$, otherwise $t' = t''$. By Proposition 9, there is a sequence of finite affine terms $(t_n)_{n\in\mathbb{N}}$ such that $\lim t_n = t$; moreover, every t_n eventually contains redexes at a', a'', and, if we let $t'_n = R_{a'}(t_n)$ and $t''_n = R_{a''}(t_n)$, by continuity of reduction we have $t' = \lim t'_n$ and $t'' = \lim t''_n$.

Note that, in everyone of the four cases listed above, when we consider the reduction $t_n \to t'_n$, for n sufficiently large the position in t'_n of the residue (if any) of the redex located at a'' does not depend on n: for the first three cases, this is evident; for the fourth case, the position depends on where the substituted variable is, but this position will be the same for n sufficiently large (remember that closer and closer terms coincide on more and more positions). Therefore, for all n sufficiently large:

- if we are in cases i, ii, or iv, $R_{a''_1}(t'_n) = R_{a'}(t''_n) = u_n$, and we set $u = \lim u_n$;
- if we are in case iii, $R_{a'}(t''_n) = t'_n$, and we set $u = t'$.

In both cases, $t' \to^= u \;{}^=\!\!\leftarrow t''$. □

On the other hand, strong normalization is a typical example of a "discontinuous" property, which does not extend to $\Lambda^{\mathrm{aff}}_\infty$. Indeed, the function $\mathrm{NF} : \Lambda^{\mathrm{aff}}_{\mathrm{p}} \to \Lambda^{\mathrm{aff}}_{\mathrm{p}}$ mapping each term to its unique normal form is not continuous, and has therefore no hope of being extended to $\Lambda^{\mathrm{aff}}_\infty$.

3.2 Uniformity

In what follows, we fix a denumerably infinite subset $\mathfrak{V} \subseteq \mathcal{V}^{\mathbb{N}}_{\mathrm{inj}}$, the elements of which are called *supervariables*, such that the ranges of the functions in $\mathfrak{V}$ form a partition of $\mathcal{V}$, *i.e.*, for all $\mathbf{x}, \mathbf{y} \in \mathfrak{V}$ and $i, j \in \mathbb{N}$, $\mathbf{x} \neq \mathbf{y}$ implies $\mathbf{x}(i) \neq \mathbf{y}(j)$, and for all $x \in \mathcal{V}$, there exist $\mathbf{x} \in \mathfrak{V}$ and $i \in \mathbb{N}$ such that $\mathbf{x}(i) = x$, in which case we say that x *belongs* to $\mathbf{x}$.

Definition 6 (Uniformity) *We define* renaming equivalence*, denoted by $\approx$, as the smallest partial equivalence relation on $\Lambda^{\mathrm{aff}}_\infty$ such that:*

- *if x, x' belong to the same supervariable, then $x \approx x'$;*
- *if $t \approx t'$, then $\lambda\mathbf{x}.t \approx \lambda\mathbf{x}.t'$ for every supervariable $\mathbf{x}$;*
- *if $t \approx t'$ and $\mathbf{u}, \mathbf{u}'$ are uniform sequences of terms such that $\mathbf{u}(0) \approx \mathbf{u}'(0)$, then $t\mathbf{u} \approx t'\mathbf{u}'$, where by* uniform sequence *we mean that, for all $i, j \in \mathbb{N}$, $\mathbf{u}(i) \approx \mathbf{u}(j)$ (and similarly for $\mathbf{u}'$).*

A term t is uniform *if $t \approx t$. We denote by $\Lambda^{\mathrm{u}}_\infty$ the set of uniform terms, and by $\mathcal{U}(\Lambda^{\mathrm{u}}_\infty)$ the subset of $\mathcal{B}(\Lambda^{\mathrm{aff}}_\infty)$ consisting of uniform sequences of uniform terms. If $\mathbf{u}, \mathbf{u}' \in \mathcal{U}(\Lambda^{\mathrm{u}}_\infty)$ are such that $\mathbf{u}(0) \approx \mathbf{u}'(0)$ (and hence, by symmetry and transitivity, $\mathbf{u}(i) \approx \mathbf{u}'(j)$ for all $i, j \in \mathbb{N}$), we write $\mathbf{u} \approx \mathbf{u}'$.*

The finite term $z\langle x_0\rangle$ is not uniform, because, in the sequence $\langle x_0\rangle$, $x_0 \not\approx \bot$ (indeed, $\bot$ is not uniform). If the range of the supervariable $\mathbf{x}$ is $x_0, x_1, x_2, \ldots$, the infinitary term $u = z\langle x_0, x_1, x_2, \ldots\rangle$ is uniform, whereas $\lambda x_0 x_2 x_4 \ldots .u$ is not uniform, because the abstraction does not bind the whole range of the supervariable; on the contrary, $\lambda\mathbf{x}.u$ is uniform.

Note that affine terms are obviously not closed under substitution, *i.e.*, $t \in \Lambda^{\mathrm{aff}}_\infty$ and $\mathbf{u} \in \mathcal{B}(\Lambda^{\mathrm{aff}}_\infty)$ implies $t[\mathbf{u}/\mathbf{x}] \in \Lambda^{\mathrm{aff}}_\infty$ only if the free variables of t and $\mathbf{u}$ do not intersect. In the following, we tacitly suppose this to be always the case.

Lemma 11 *Let $t, t' \in \Lambda^{\mathrm{aff}}_\infty$ and $\mathbf{u}, \mathbf{u}' \in \mathcal{U}(\Lambda^{\mathrm{u}}_\infty)$ be s.t. $t \approx t'$ and $\mathbf{u} \approx \mathbf{u}'$. Then, for all $\mathbf{x} \in \mathfrak{V}$, $t[\mathbf{u}/\mathbf{x}] \approx t'[\mathbf{u}'/\mathbf{x}]$.*

Proof. A straightforward induction on t. The case $t = x$ (which implies $t' = x'$) is the only interesting one. The hypothesis $x \approx x'$ implies that either none of x, x' is in the range of $\mathbf{x}$, in which case we conclude immediately, or there exist $i, i' \in \mathbb{N}$ s.t. $x = \mathbf{x}(i)$ and $x' = \mathbf{x}(i')$, in which case the hypothesis that $\mathbf{u} \approx \mathbf{u}'$ allows us to conclude. □

Lemma 12 *For all $t \in \Lambda^{\mathrm{u}}_{\infty}$, $\mathbf{u} \in \mathcal{U}(\Lambda^{\mathrm{u}}_{\infty})$, and $\mathbf{x} \in \mathfrak{V}$, $t[\mathbf{u}/\mathbf{x}] \in \Lambda^{\mathrm{u}}_{\infty}$.*

Proof. We set $t' = t[\mathbf{u}/\mathbf{x}]$, and reason by induction on t. Let $t = x$. If x does not belong to $\mathbf{x}$, we conclude trivially; if $x = \mathbf{x}(i)$ for some $i \in \mathbb{N}$, $t' = \mathbf{u}(i)$, which is uniform by hypothesis. The case $t = \lambda\mathbf{y}.t_1$ is immediate. Suppose now that $t = t_1\mathbf{v}$; we have $t' = t'_1\mathbf{v}'$, where $t'_1 = t_1[\mathbf{u}/\mathbf{x}]$ and $\mathbf{v}'(i) = \mathbf{v}(i)[\mathbf{u}/\mathbf{x}]$ for all $i \in \mathbb{N}$. The fact that t'_1 and all the terms in $\mathbf{v}'$ are uniform is a consequence of the induction hypothesis, and the uniformity of $\mathbf{v}'$ is an immediate consequence of Lemma 11. □

In the following, we write $t \to_h t'$ if $t \to t'$ by reducing the head redex (which is defined as usual).

Lemma 13 *Let $t \in \Lambda^{\mathrm{u}}_{\infty}$. Then:*

- *$t \to_h t'$ implies $t' \in \Lambda^{\mathrm{u}}_{\infty}$;*
- *furthermore, for all $u \approx t$, $u \to_h u' \approx t'$.*

Proof. Both points are by induction on t; point 1 uses Lemma 12, point 2 Lemma 11. □

Lemma 13 does not extend to reduction in general; for instance, if u is a closed uniform term s.t. $u \to_h u'$, then $z\langle u, u, u, \ldots\rangle \to z\langle u', u, u, \ldots\rangle$, which is not uniform. However, we do know that $z\langle u', u', u', \ldots\rangle$ is uniform; the idea then is to define a notion of reduction which allows infinitely many parallel steps, so as to preserve uniformity.

Definition 7 (Infinitary reduction) *We define the relations $\Rightarrow_k$ on $\Lambda^{\mathrm{u}}_{\infty}$, with $k \in \mathbb{N}$, as follows:*

- *$(\lambda\mathbf{x}.t)\mathbf{u} \Rightarrow_0 t[\mathbf{u}/\mathbf{x}]$;*
- *if $t \Rightarrow_k t'$, then $\lambda\mathbf{x}.t \Rightarrow_k \lambda\mathbf{x}.t'$;*
- *if $t \Rightarrow_k t'$, then $t\mathbf{u} \Rightarrow_k t'\mathbf{u}$;*
- *if $\mathbf{u} \in \mathcal{U}(\Lambda^{\mathrm{u}}_{\infty})$, and $\mathbf{u}(0) \Rightarrow_k u'_0$, by uniformity the "same" reduction may be performed in all $\mathbf{u}(i)$, $i \in \mathbb{N}$, obtaining the term u'_i. If we define $\mathbf{u}'(i) = u'_i$ for all $i \in \mathbb{N}$, we set $t\mathbf{u} \Rightarrow_{k+1} t\mathbf{u}'$.*

We denote by $\Rightarrow$ the union of all $\Rightarrow_k$, for $k \in \mathbb{N}$.

For instance, if $I = \lambda x_0x_1x_2\ldots.x_0$, and $t = I\langle I, I, I, \ldots\rangle$, we have

$$z\langle y_0\langle t, t, \ldots\rangle, y_1\langle t, t, \ldots\rangle, \ldots\rangle \Rightarrow_2 z\langle y_0\langle I, I, \ldots\rangle, y_1\langle I, I, \ldots\rangle, \ldots\rangle.$$

Note that $\Rightarrow_k$ is infinitary iff $k > 0$. Indeed, $\Rightarrow_0$ is head reduction, which is *not* infinitary.

Proposition 14 *Let $t \in \Lambda^{\mathrm{u}}_{\infty}$. Then:*

- $t \Rightarrow t'$ *implies* $t' \in \Lambda^{\mathrm{u}}_{\infty}$*;*
- *furthermore, for all* $u \approx t$*,* $u \Rightarrow u' \approx t'$*.*

PROOF. By definition, $t \Rightarrow t'$ iff $t \Rightarrow_k t'$ for some $k \in \mathbb{N}$. The proof is by induction on k. Lemma 13 is the base case; the rest is straightforward. □

3.3 The Isomorphism with the Full Lambda-Calculus

In what follows, we denote by Λ the set of usual λ-terms, ranged over by M, N, L. We denote by $\to_\beta$ usual β-reduction. The set of λ-calculus variables is assumed to be $\mathcal{V}$.

We fix an injection $\ulcorner \cdot \urcorner : \mathbb{P} \to \mathbb{N}$. We also fix a bijection between λ-calculus variables and supervariables, and make the notational convention that the variables x, y, z are mapped to the supervariables $\mathbf{x}, \mathbf{y}, \mathbf{z}$, resp. Then, for all $a \in \mathbb{P}$, we define by induction the map $[\![\cdot]\!]_a : \Lambda \to \Lambda_\infty$, as follows:

$$\begin{aligned} [\![x]\!]_a &= \mathbf{x}(\ulcorner a \urcorner) \\ [\![\lambda x.M]\!]_a &= \lambda \mathbf{x}.[\![M]\!]_a \\ [\![MN]\!]_a &= [\![M]\!]_{a0}\langle [\![N]\!]_{a1}, [\![N]\!]_{a2}, [\![N]\!]_{a3}, \ldots \rangle \end{aligned}$$

Lemma 15 *Let* $M \in \Lambda$ *and* $a, a' \in \mathbb{P}$*.*

1. $[\![M]\!]_a \in \Lambda^{\mathrm{aff}}_{\infty}$*;*
2. $[\![M]\!]_a \approx [\![M]\!]_{a'}$*; in particular,* $[\![M]\!]_a \in \Lambda^{\mathrm{u}}_{\infty}$*.*

PROOF. We claim that, given $M, M' \in \Lambda$ and $a, a' \in \mathbb{P}$, if a, a' are incomparable in the prefix order, then $\mathrm{fv}([\![M]\!]_a) \cap \mathrm{fv}([\![M']\!]_{a'}) = \emptyset$; by looking at how application is treated in the definition of $[\![\cdot]\!]_a$, this is enough to prove point 1. Now, by inspecting that same definition, we see that every variable in $\mathrm{fv}([\![M]\!]_a)$ must be of the form $\mathbf{x}(\ulcorner ab \urcorner)$, whereas every variable appearing in $\mathrm{fv}([\![M']\!]_{a'})$ must be of the form $\mathbf{x}'(\ulcorner a'b' \urcorner)$, for some $\mathbf{x}, \mathbf{x}' \in \mathfrak{V}$ and $b, b' \in \mathbb{P}$. The fact that a, a' are incomparable means that, for all b, b', $ab \neq a'b'$, so the claim follows from the injectivity of $\ulcorner \cdot \urcorner$ and of supervariables (as functions from $\mathbb{N}$ to $\mathcal{V}$), and the fact that their ranges form a partition of $\mathcal{V}$.

Now that we know that $[\![M]\!]_a$ is actually affine, point 2 may be proved by a straightforward induction on M. □

We define a function $(\!|\cdot|\!) : \Lambda^{\mathrm{u}}_{\infty} \to \Lambda$, as follows:

$$\begin{aligned} (\!|\mathbf{x}(i)|\!) &= x \quad \text{(for all } i \in \mathbb{N}) \\ (\!|\lambda \mathbf{x}.t|\!) &= \lambda x.(\!|t|\!) \\ (\!|t\mathbf{u}|\!) &= (\!|t|\!)(\!|\mathbf{u}(0)|\!) \end{aligned}$$

In the first case, we implicitly use that fact that the ranges of supervariables form a partition of $\mathcal{V}$, hence every variable is equal to $\mathbf{x}(i)$ for some $\mathbf{x} \in \mathfrak{V}$ and $i \in \mathbb{N}$.

Lemma 16 *For all* $t, t' \in \Lambda^{\mathrm{u}}_{\infty}$*,* $t \approx t'$ *implies* $(\!|t|\!) = (\!|t'|\!)$

PROOF. Immediate from the definition. □

Lemma 17 *Let $(b_i)_{i\in\mathbb{N}}$ be a sequence of positions which are pairwise incompatible in the prefix order. Given $N \in \Lambda$, we define $\overrightarrow{[\![N]\!]}_{b_i} = \langle [\![N]\!]_{b_0}, [\![N]\!]_{b_1}, [\![N]\!]_{b_2}, \ldots\rangle \in \mathcal{U}(\Lambda_\infty^{\mathrm{u}})$. Then, for all $M, N \in \Lambda$, for every variable x, for every $a \in \mathbb{P}$ and every $(b_i)_{i\in\mathbb{N}} \in \mathbb{P}$ as above, $[\![M[N/x]]\!]_a \approx [\![M]\!]_a[\overrightarrow{[\![N]\!]}_{b_i}/\mathbf{x}]$.*

Proof. By induction on M. If $M = y \neq x$, we conclude immediately; if $M = x$, we use point 2 of Lemma 15. The case $M = \lambda x.M_1$ is immediate. Let $M = M_1 L$. We have, using the induction hypothesis,

$$\begin{aligned}
[\![M[N/x]]\!]_a &= [\![M_1[N/x]L[N/x]]\!]_a \\
&= [\![M_1[N/x]]\!]_{a0}\overrightarrow{[\![L[N/x]]\!]}_{a(i+1)} \\
&\approx [\![M_1]\!]_{a0}[\overrightarrow{[\![N]\!]}_{b_i}/\mathbf{x}]\langle [\![L]\!]_{a1}[\overrightarrow{[\![N]\!]}_{b_i}/\mathbf{x}], [\![L]\!]_{a2}[\overrightarrow{[\![N]\!]}_{b_i}/\mathbf{x}], \ldots\rangle \\
&= [\![M_1]\!]_{a0}\overrightarrow{[\![L]\!]}_{a(i+1)}[\overrightarrow{[\![N]\!]}_{b_i}/\mathbf{x}] \\
&= [\![M_1 L]\!]_a[\overrightarrow{[\![N]\!]}_{b_i}/\mathbf{x}].
\end{aligned}$$

□

Lemma 18 *For all $t \in \Lambda_\infty^{\mathrm{u}}$, $\mathbf{u} \in \mathcal{U}(\Lambda_\infty^{\mathrm{u}})$ and $\mathbf{x} \in \mathfrak{V}$, $(\!|t[\mathbf{u}/\mathbf{x}]|\!) = (\!|t|\!)[(\!|\mathbf{u}(0)|\!)/x]$.*

Proof. By induction on t. Let $t = \mathbf{y}(i)$, for some $\mathbf{y} \in \mathfrak{V}$ and $i \in \mathbb{N}$. If $\mathbf{y} \neq \mathbf{x}$, we conclude immediately; otherwise, we use Lemma 16. The inductive cases are straightforward. □

The *applicative depth* of a redex in the λ-calculus is defined by induction: $(\lambda x.M)N$ is at applicative depth 0; if a redex is at applicative depth k in M, then its applicative depth is k in $\lambda x.M$ and MN, and $k+1$ in NM. In the following, we write $M \to_{\beta k} M'$ to denote the fact that $M \to_\beta M'$ by reducing a redex at applicative depth k (*e.g.*, $\to_{\beta 0}$ is head reduction).

Theorem 19 *For all $M \in \Lambda$, $t \in \Lambda_\infty^{\mathrm{u}}$, and $a \in \mathbb{P}$:*

1. *$(\!|[\![M]\!]_a|\!) = M$;*
2. *$[\![(\!|t|\!)]\!]_a \approx t$;*
3. *$M \to_{\beta k} M'$ implies $[\![M]\!]_a \Rightarrow_k t' \approx [\![M']\!]_a$;*
4. *$t \Rightarrow_k t'$ implies $(\!|t|\!) \to_{\beta k} (\!|t'|\!)$.*

Proof. All points are proved by straightforward inductions. Lemma 17 is used in (3), and Lemma 18 in (4). □

Corollary 20 *In the Curry-Howard sense, $(\Lambda, \to_\beta)$ is isomorphic to $(\Lambda_\infty^{\mathrm{aff}}/\approx, \Rightarrow)$.*

4 Discussion and Perspectives

We shall now discuss some applications of the results presented above, along with some comments on related work in the existing literature, and future perspectives.

4.1 On Some Properties of Beta-Reduction

The isomorphism of Corollary 20 gives interesting insights on some well known properties of reduction in the λ-calculus.

Lemma 21 *Let $t_0 \in \Lambda_\infty^{\mathrm{aff}}$, and suppose there is an infinite reduction sequence $t_0 \to t_1 \to t_2 \to \cdots$ such that every step reduces a residue of a redex of t_0. Then:*

1. *for every finite set of positions $P \subseteq \mathbb{P}$, there exists $n \in \mathbb{N}$ such that the position of the redex reduced in $t_n \to t_{n+1}$ is not in P;*
2. *the height of all t_n, for $n \in \mathbb{N}$, is bounded.*

Proof. Essentially, both points are consequences of the fact that, since the height of t_0 is finite, no infinite chain of redexes successively containing another may exist in t_0.

For point 1, suppose, for the sake of contradiction, that the infinite reduction does not fire redexes outside of a finite set $P \subseteq \mathbb{P}$. Since the reduction is infinite, we have that infinitely many redexes in the sequence have position a, for some $a \in P$, which we may choose of minimal length. Obviously, only one redex at position a may exist in t_0, so all the other redexes are residues of redexes of t_0 that have "moved" during reduction. But, by considering the analysis of the positions of residues made at the beginning of the proof of Proposition 10, which applies without change to $\Lambda_\infty^{\mathrm{aff}}$, we see that the only way of "moving" a redex from position a'' is by firing a redex at position a' such that $a' \leq a''$; moreover, in that case, the position of the residue is still of the form $a'b$. Since a is of minimal length, we have a redex which is fired at a and which "moves" another redex from aa_1 to a, which in turn "moves" another redex from aa_2 to a, and so on. These redexes are all already present in t_0, and their positions, which are of the form $aa_1 \ldots a_n$ with $a_1, \ldots, a_n$ non-empty, contradict the fact that t_0 is of finite height.

For point 2, let a_i be the position of the redex fired in $t_i \to t_{i+1}$. Supposing that the length of a_i is large at will brings us to a contradiction, using similar arguments as above. Then, the length of a_i is bounded, and all a_i (except perhaps finitely many) are pairwise incomparable, which means that from some t_n onward the redexes are all independent, so the height is obviously bounded (by twice the height of t_n). □

Theorem 22 (Complete developments) *Let A be a set of redex positions of a term $t \in \Lambda_\infty^{\mathrm{aff}}$. Fix a total ordering $a_0 < a_1 < a_2 < \ldots$ of A, of order type at most ω, and define the sequence $(t'_n)_{n \in \mathbb{N}}$ by induction, as follows: $t'_0 = t$; t'_{n+1} is the reduct of t'_n obtained by firing the only residue (if any) of a_n in t'_n; if there is no such residue, $t'_{n+1} = t'_n$. Then:*

1. *the sequence $(t'_n)_{n \in \mathbb{N}}$ is Cauchy;*
2. *any ordering yields an equivalent Cauchy sequence.*

Proof. We may suppose $(t'_n)_{n \in \mathbb{N}}$ to be non-stationary, otherwise it is trivially Cauchy. Then, we may apply point 1 of Lemma 21: if we fix a finite $P \subseteq \mathbb{P}$, for n sufficiently large, the "activity" in the reduction sequence $t'_n \to t'_{n+1} \to t'_{n+2} \to \cdots$ occurs all outside of P, which means that, for all $p \in \mathbb{N}$, $t'_n(a) = t'_{n+p}(a)$

for all $a \in \mathbb{P}$. Since P is large at will, $d_s(t'_n, t'_{n+p})$ is small at will. Point 2 of Lemma 21 guarantees that the height of the terms in $(t'_n)_{n\in\mathbb{N}}$ is stationary, which concludes the proof of point 1.

Take now a different total ordering of A, yielding the sequence $(t''_n)_{n\in\mathbb{N}}$. Since this sequence too is Cauchy, for all $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that, for all $p \in \mathbb{N}$, $\max(d(t'_n, t'_{n+p}), d(t''_n, t''_{n+p})) < \varepsilon$. But both t'_n, t''_n are reducts of t, so by Proposition 10, there exists u s.t. $t'_n \to^* u \;{}^*\!\!\leftarrow t''_n$. Moreover, thanks to point 2 of Lemma 21, taking n sufficiently large guarantees us that all reducts of t'_n and t''_n to obtain u have the same height (the height of u). Furthermore, observe that these reductions do not alter t'_n and t''_n at positions within a certain finite $P \subseteq \mathbb{P}$, which is enough to guarantee that $\max(d(t'_n, u), d(u, t''_n)) < \varepsilon$. But d is an ultrametric, so $d(t'_n, t''_n) < \varepsilon$, which proves the equivalence of the two Cauchy sequences. □

By Theorem 22, given $t \in \Lambda^{\mathrm{aff}}_\infty$ and a set of redex positions A of t, me may define $t \to_A t'$ by setting $t' = \lim t'_n$, where $(t'_n)_{n\in\mathbb{N}}$ is any sequence induced by a total ordering of A. Note that, for uniform terms, the reductions $\Rightarrow_k$ of Definition 7 are particular cases of this infinitary reduction.

The finiteness of developments, confluence, and standardization property of the λ-calculus may all be seen as consequences of Theorem 22. In fact, Theorem 22 implies a strong form of the finite developments property (called FD! by Barendregt [Bar84]); this, as is well known, may be used to prove the Church-Rosser theorem and the standardization theorem (see Chapters 11.2 and 11.4 of [Bar84]).

Corollary 23 (FD!) *Let $M \in \Lambda$.*

1. *All developments of M are finite, and can be extended to a complete development.*
2. *All complete developments of M end with the same term.*

The result follows from Theorem 22 by considering the fact that a family of redexes of M induces a (perhaps infinite) family of redexes of $[\![M]\!]$. We give a hopefully illustrating example. Consider $M = (\lambda x.zxx)(II)$. The two complete developments of its two redexes are $M \to_\beta z(II)(II) \to_\beta\to_\beta zII$ and $M \to_\beta (\lambda x.zxx)I \to_\beta zII$. Translated in $\Lambda^{\mathrm{u}}_\infty$, using the notations $\mathbf{x}(i) = x_i$ and $u = I\langle I, I, \ldots\rangle$, this gives $t = (\lambda\mathbf{x}.z\langle x_0, x_2, \ldots\rangle\langle x_1, x_3, \ldots\rangle)\langle u, u, \ldots\rangle$. Note how, in the reduction $t \Rightarrow_0 z\langle u, u, \ldots\rangle\langle u, u, \ldots\rangle$, *no copy of any redex is made*: we have just distributed an infinite sequence of u's to *two* infinite sequences. Then, the two $\Rightarrow_1$ reductions that lead to $z\langle I, I, \ldots\rangle\langle I, I, \ldots\rangle$ are nothing but a "splitting" of the infinite reduction $t \Rightarrow_1 (\lambda\mathbf{x}.z\langle x_0, x_2, \ldots\rangle\langle x_1, x_3, \ldots\rangle)\langle I, I, \ldots\rangle$ which starts the other development. Theorem 22 says that the "splitting" is always possible, and leads to the same result; indeed, the infinite reductions may be performed sequentially, in any order (but the intermediate results are not uniform).

In the end, Theorem 22 may be thought as saying that the λ-calculus is confluent because, morally, it is strongly confluent: the failure of the diamond property is a sort of accident, caused by the fact that some reductions are "split in two". This idea is already present in Tait's proof of the Church-Rosser theorem, implicitly using the FD! property (Chapter 3.2 of [Bar84]).

4.2 Head Normalization and Solvability

Another classic result of the λ-calculus is the following:

Theorem 24 (Wadsworth [Wad71]) *For every $M \in \Lambda$, M is solvable iff M has a hnf.*

Once the standardization theorem is proved, this is an immediate corollary of the fact that, if MN has a hnf, then M has a hnf. We shall sketch a proof of this by a density argument.

First, let $\mathrm{HNF} : \Lambda_{\mathrm{p}}^{\mathrm{aff}} \to \Lambda_{\mathrm{p}}^{\mathrm{aff}}$ be the function assigning to each term its normal form by head reduction (remember that $\Lambda_{\mathrm{p}}^{\mathrm{aff}}$ is strongly normalizing). Then, define a hnf $t \in \Lambda_{\infty}^{\mathrm{aff}}$ to be *proper* if the head is a variable (and not $\perp$).

Lemma 25 *In $\Lambda_{\mathrm{p}}^{\mathrm{aff}}$, $\mathrm{HNF}(t\mathbf{u})$ proper implies $\mathrm{HNF}(t)$ proper, and this latter hnf is reached in at most as many steps as the former.*

Proof. Immediate, by contraposition. □

Now, looking at Theorem 19, we see that head reduction in Λ and $\Lambda_{\infty}^{\mathrm{u}}$ perfectly match, *i.e.*, no infinite reduction is needed in $\Lambda_{\infty}^{\mathrm{u}}$. So let MN have a hnf P, and take $(t_n\mathbf{u}_n)_{n\in\mathbb{N}} \in \Lambda_{\mathrm{p}}^{\mathrm{aff}}$ s.t. $\lim t_n = [\![M]\!]$ and $\lim t_n\mathbf{u}_n = [\![MN]\!]$ (we do not specify the parameter $a \in \mathbb{P}$, since it is irrelevant). We have $[\![MN]\!] \to_h^* [\![P]\!]$ in k steps, which is a proper hnf, so $\mathrm{HNF}(t_n\mathbf{u}_n)$ is eventually proper, so by Lemma 25 $\mathrm{HNF}(t_n)$ is eventually proper and is reached in at most k steps, which means that $[\![M]\!]$ has a hnf, hence M also has a hnf, namely $(\!|\lim \mathrm{HNF}(t_n)|\!)$.

When we expand the details, the above proof cannot be claimed to be shorter or simpler than Wadsworth's original proof (see Chapter 8.3 of [Bar84]). However, we may say that it is *conceptually* simpler, in that it applies a general strategy: prove a property in $\Lambda_{\mathrm{p}}^{\mathrm{aff}}$, and verify that it "passes to the limit".

As already noted above, not all properties of $\Lambda_{\mathrm{p}}^{\mathrm{aff}}$ "pass to the limit". In fact, we have already mentioned at the end of Sect. 3.1 that the normal form map NF is not continuous. The same may be said about the function HNF introduced above. However, the continuity of a function depends on the topology we put on its domain and codomain. We may then ask the following question: under what conditions does HNF become Cauchy-continuous on $\Lambda_{\mathrm{p}}^{\mathrm{aff}}$, and hence uniquely extendable to $\Lambda_{\infty}^{\mathrm{aff}}$? Under some reasonable assumptions about the uniform structure of $\Lambda_{\mathrm{p}}^{\mathrm{aff}}$, we are able to give an exact answer.

Definition 8 (λ-regular uniformity) *Below, we denote by $\Lambda_{\mathrm{p}}^{\lambda}$ (resp. $\Lambda_{\mathrm{p}}^{@}$) the set of finite terms which are abstractions (resp. applications), with the subspace uniformity. A uniform structure on Λ_{p} is* λ-regular *if it satisfies the following:*

- *for all $\mathbf{x} \in \mathcal{V}_{\mathrm{inj}}^{\mathbb{N}}$, the injection $\lambda_{\mathbf{x}} : \Lambda_{\mathrm{p}} \to \Lambda_{\mathrm{p}}^{\lambda}$ mapping t to $\lambda\mathbf{x}.t$, and its inverse $\lambda_{\mathbf{x}}^{-1}$, are Cauchy-continuous;*
- *the injection $@ : \Lambda_{\mathrm{p}} \times \Lambda_{\mathrm{p}}^{(\mathbb{N})} \to \Lambda_{\mathrm{p}}^{@}$ mapping $(t, \mathbf{u})$ to $t\mathbf{u}$, and its inverse $@^{-1}$, are Cauchy continuous, when we equip $\Lambda_{\mathrm{p}}^{(\mathbb{N})}$ with the uniformity induced by the injection $\mathbf{u} \mapsto \perp\mathbf{u}$, and the product space with the product uniformity.*

The uniform structure is furthermore said to be height-bounded *if all Cauchy sequences w.r.t. to it have bounded height.*

For instance, the uniformity induced by the metric d is λ-regular, as shown by Lemma 4, Lemma 5. It is also obviously height-bounded.

The fact that Λ_p^{aff} carries a λ-regular, height-bounded uniformity is actually equivalent to Lemma 6, if we further assume that $\mathcal{V} \cup \{\bot\}$ is discrete. From that, Lemma 7 can be proved, and the head reduction map H, sending each term to its one-step head-reduct, may be shown to be Cauchy-continuous (however, we do not believe this to be enough to show Proposition 8). Hence, we know how to extend head-reduction to the metric completion $\overline{\Lambda_p^{aff}}$, and we may therefore speak of head-normalization for terms in the completed space.

Theorem 26 *Suppose that Λ_p^{aff} is equipped with a λ-regular, height-bounded uniform structure (as a subspace of Λ_p). Then,* HNF *is Cauchy-continuous on Λ_p^{aff} iff every term in the metric completion $\overline{\Lambda_p^{aff}}$ is head-normalizing.*

The backward implication is easy, and holds for every uniformity: by definition, one step of head-reduction $H(t)$ in $\overline{\Lambda_p^{aff}}$ is defined by taking a sequence $(t_n)_{n\in\mathbb{N}} \in \Lambda_p^{aff}$ s.t. $\lim t_n = t$, and setting $H(t) = \lim H(t_n)$. Now, let $(t_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\overline{\Lambda_p^{aff}}$; we have to show that $(\mathrm{HNF}(t_n))_{n\in\mathbb{N}}$ is Cauchy. Since $\lim t_n$ head-normalizes in k steps, by the above definition every t_n head-normalizes in at most k steps, so $\mathrm{HNF}(t_n) = H^k(t_n)$, and we conclude by stability of Cauchy-continuity under composition.

The proof of the forward implication is less trivial, and uses the following program transformation:

$$\begin{aligned} x^\dagger &= x \\ (\lambda \mathbf{x}.t)^\dagger &= \lambda k.\lambda \mathbf{x}.t^\dagger\langle k\rangle, \\ (t\mathbf{u})^\dagger &= \lambda k.t^\dagger\langle k\langle\rangle\rangle \mathbf{u}^\dagger, \end{aligned}$$

where $\mathbf{u}^\dagger$ denotes the sequence obtained by applying $(\cdot)^\dagger$ to all terms of $\mathbf{u}$. The following fundamental properties may be proved by straightforward inductions:

Lemma 27 *Let $t \in \Lambda_p^{aff}$.*

1. *$t \to_h t'$ implies that $t^\dagger\langle k\rangle$ and $t'^\dagger\langle k\langle\rangle\rangle$ head-reduce to the same term;*
2. *hence, $t \to_h^* \mathrm{HNF}(t)$ in n steps implies*
$$\mathrm{HNF}(t^\dagger\langle k\rangle) = \mathrm{HNF}(\mathrm{HNF}(t)^\dagger\langle k\langle\rangle^n\rangle),$$
where $k\langle\rangle^n$ denotes the n-fold application $k\langle\rangle\ldots\langle\rangle$;
3. *if t is a hnf, then $t^\dagger\langle k\langle\rangle^n\rangle$ has a hnf whose height is at least n.*

We reason by contraposition: we suppose the existence of $t \in \overline{\Lambda_p^{aff}}$ whose head reduction does not terminate, and infer that HNF is not Cauchy-continuous. Take a sequence $(t_n)_{n\in\mathbb{N}}$ such that $\lim t_n = t$, and consider $(t_n^\dagger\langle k\rangle)_{n\in\mathbb{N}}$. By definition of $(\cdot)^\dagger$ and λ-regularity, this sequence in still Cauchy. Now, the lengths of the reductions $t_n \to_h^* \mathrm{HNF}(t_n)$ must be described by an unbounded function $\ell : \mathbb{N} \to \mathbb{N}$, for otherwise the head reduction of t would terminate. By point 2 of Lemma 27, $\mathrm{HNF}(t_n^\dagger\langle k\rangle) = \mathrm{HNF}(\mathrm{HNF}(t_n)^\dagger\langle k\langle\rangle^{\ell(n)}\rangle)$, so by point 3 of Lemma 27 the sequence $(\mathrm{HNF}(t_n^\dagger\langle k\rangle))_{n\in\mathbb{N}}$ has unbounded height, and, by height-boundedness, cannot be Cauchy.

Theorem 26 is a sort of "sanity check": it would be embarrassing if HNF(t) were defined even if t had no hnf. If the uniformity on $\Lambda_{\mathrm{p}}^{\mathrm{aff}}$ is well behaved, this cannot happen.

4.3 Comparison with Other Infinitary Lambda-Calculi

The investigation of infinitary rewriting was initiated by Dershowitz et al. [DKP91], and is still a growing research field. A survey on infinitary term rewriting systems and infinitary λ-calculi may be found in Chapter 12 of [Ter03].

The work which is most related to the present one is the paper by Kenneway, Klop, Sleep and de Vries (KKSV for short) on the infinitary λ-calculus [KKSdV97]. There, the authors define 8 different metrics on usual λ-terms, which may be referenced by a string *abc* with $a, b, c \in \{0, 1\}$. Each bit controls whether the height of terms is allowed to grow indefinitely on one of the basic constructors: $a = 1$ allows indefinite growth on abstractions; $b = 1$ (resp. $c = 1$) allows indefinite growth on applications in the function (resp. argument) position. Each metric yields a different completion of the space of finite terms. For instance, 000 is the discrete metric, 001 is a metric allowing Böhm-like trees in the completion (terms which may be infinite on the argument side of application, such as $I(I(I\ldots)))$, and the completion w.r.t. 111 contains terms whose height is infinite in all possible ways.

This brief description highlights an immediate difference between KKSV's work and our own: we study terms which are possibly infinite in *width*, but finite in height; KKSV's focus is exactly dual. This is because KKSV are interested in studying the notion of infinitary rewriting, namely reductions of possibly infinite length, with no concern about affinity or duplication. On the other hand, our aim is to describe *finite* reductions in the λ-calculus in terms of non-duplicating reductions. Of course, in order to do that, we too are led to consider reductions of infinite length; but these are of a much simpler nature than the ones considered by KKSV. For instance, Theorem 22 is false in KKSV's infinitary calculi, where complete developments do not always exist (the infinite family of redexes in the term $I(I(I\ldots))$ mentioned above has no complete development according to KKSV's notion of reduction). As a consequence, infinitary rewriting is not confluent in general; in some cases, it is confluent up to a certain notion of equivalence of *meaningless* terms. The main technical point is the absence, in our calculus, of infinite chains of redex containment, which are possible if terms are allowed to have infinite height.

Another interesting remark is that, by weakening our metric, we may easily introduce terms with infinite height in the completion of $\Lambda_{\mathrm{p}}^{\mathrm{aff}}$. For instance, the completion of $(\Lambda_{\mathrm{p}}^{\mathrm{aff}}, d_s)$ contains affine terms which are infinite in every way, in height as in width, as in KKSV's metric 111. Then, presumably, one might be able to develop all of KKSV's theory starting from $\Lambda_{\mathrm{p}}^{\mathrm{aff}}$, instead of the full λ-calculus. More in general, studying terms of infinite width (as opposed to height) seems to be a novelty in the context of infinitary rewriting, which may be worth exploring.

4.4 The Proof-Theoretic Perspective

This work originated from the following result, first mentioned by Girard [Gir87], and recently used as the underlying idea of a categorical construction for the free exponential comonad in models of linear logic [MTT09]. Given $n \in \mathbb{N}$ and a linear logic formula A, define $!_n A = (A \,\&\, 1) \otimes \cdots \otimes (A \,\&\, 1)$, n times, and $?_n A = (!_n A^\perp)^\perp$. Let $\sigma_! A$ denote a formula obtained from A by replacing every occurrence of subformula of the form $!B$ in A with $!_n B$ for some n (not necessarily the same for every occurrence), and similarly for $\sigma_? A$.

Theorem 28 (Approximation) *Let A be a provable formula of propositional linear logic. Then, for every $\sigma_!$, there exists $\sigma_?$ s.t. $\sigma_? \sigma_! A$ is provable in the multiplicative additive fragment.*

In some sense, the Approximation Theorem says that multiplicative additive propositional linear logic is "dense" in propositional linear logic. We remind that the multiplicative additive fragment is the "purely linear" part of linear logic, where structural rules are completely forbidden.

Our motivation was to give a formal status to the above intuition. The current results concern *untyped* calculi, and are therefore not satisfactory from the point of view of proof theory. However, the same ideas presented here may be applied, *mutatis mutandis*, to Girard's *ludics* [Gir01], an untyped, game-theoretic framework from which multiplicative additive linear logic may be recovered. Our preliminary results give a new solution to the question of modeling the exponential modalities in ludics, alternative to the one proposed by Basaldella and Faggian [BF09]. For the reader familiar with the terminology of ludics, our solution is based on infinite *ramifications*, which are consistent with the fact that, read bottom-up, an exponential rule creates unboundedly many *loci* for unlimited (affine) use during the rest of the proof.

4.5 Other Directions for Future Work

Results like Theorem 26 open intriguing questions about the existence of other metrics than the one considered here. For the time being, we have conducted a preliminary attempt at classifying the metrics satisfying Definition 8, and none of the metrics found so far has remarkable properties. However, we are nowhere near a complete classification (if this is possible at all), so we have no conclusive remark to make yet.

In general, there is a want of algebraic structure in the λ-calculus, on which the uniform structure may rest upon. This makes exploring the above question even more difficult, and brings up a whole new direction of research, aiming at formulating calculi with more structured algebraic properties.

An example might be Ehrhard and Regnier's resource λ-calculus (which is strongly related to Boudol's λ-calculus with multiplicities). It is plausible that our results may be reformulated using resource λ-terms, but exactly how remains to be seen. Likewise, the relationship between the Taylor expansion of λ-terms [ER08] and our work is still to be understood.

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