Abramsky-Jagadeesan-Malacaria strategies and the geometry of interaction

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Abstract:

Abramsky, Jagadeesan and Malacaria (henceforth abbreviated as AJM) gave in [AJM95] a model for PCF in which types are interpreted as games and terms as history-free strategies. This model offers a full completness result: each valid strategy is equivalent to a strategy induced by a term. Our aim is to make clear the link between this model and geometry of interaction, which was already mentioned in [AJM95] We define a matrix of partial functions associated to a MELL proof-net (eventually including PCF constants), following Girard's geometry of interaction. By relaxing AJM's definition of plays we show that this matrix induces a valid strategy. Moreover in the case of a proof-net translated from a PCF^- term (PCF deprived of the fixed-point operators) we show that this interpretation is precisely the same as the AJM one.

1 Games, positions and strategies:

Let's consider MELL formulas built on two atomic formulas B (booleans) and N (naturals) where negations apply only on atomic formulas.

We call *depth* of a subformula F of A the number of subformulas of A containing F as a subformula and starting with an exponential !/?.

From now on we will call subformula of A a couple $F\vec{\imath}$ where F is a proper subformula of A and $\vec{\imath}$ is a sequence of integers of length the depth of F in A.

A move on the game A associated to the formula A is a triplet $(\alpha, \vec{\imath}, c)$ where :

- $\alpha, \vec{\imath}$ is an atomic subformula of A, t.i.t.s. α is an occurence of $B, B^{\perp}, N$ or $N^{\perp}$ in A.
- c is the *content* of the move:
  - if $\alpha = B$ or $B^{\perp}$, $c \in \{*, tt, ff\}$.
  - if $\alpha = N$ or $N^{\perp}$, $c \in \{*, n \in \mathcal{N}\}$.

The contents $*$ are *questions*, the others are *answers*.

The author of a move is *Opponent* if $\alpha = B$ or N and $c = *$, or if $\alpha = B^{\perp}$ or $N^{\perp}$ and c is an answer; it is *Player* otherwise.

The set of moves on A (resp. of moves by Opponent, Player) is denoted by M_A (resp. M_A^O, M_A^P).

Therefore we have:

- $M_{A_1 \otimes A_2} = M_{A_1 \wp A_2} = M_{A_1} + M_{A_2}$ (disjoint union)
- $M_{!A_1} = M_{?A_1} = \{(\alpha, j.\vec{\imath}, c)/(\alpha, \vec{\imath}, c) \in M_{A_1}\}$

j will be called the *index* of the move $(\alpha, j.\vec{\imath}, c)$ in $M_{!A_1}$.

In the sequel if $F\vec{\imath}$ is a subformula of A and s belongs to M_A^* (the set of finite sequences over M_A), $s|F\vec{\imath}$ will denote the sequence obtained by restricting s to its moves of the form $(\alpha, \vec{\imath}.\vec{\jmath}, c)$, where $(\alpha, \vec{\jmath}, c) \in M_F$, and replacing such a move by $(\alpha, \vec{\jmath}, c)$. So we have $s|F\vec{\imath} \in {M_F}^*$.

Henceafter we will specify that the move m is by Opponent (resp. Player) by noting $\underset{O}{m}$ (resp. $\underset{P}{m}$).

We define a set of relaxed positions on A (RP_A) in which the two players may start (contrary to the set of positions in [], denoted by P_A). RP_A is a prefix-closed subset of M_A^* whose elements s satisfy the following conditions:

- alternacy: Opponent and Player play alternatively in s.
- bracketing condition: for any prefix t of s the number of answers in t is inferior to the number of questions.

The bracketing condition allows us to associate to each answer a unique question, the most recently asked and yet unanswered question.

RP_A is defined inductively in the following way:

- $RP_B = \{\epsilon, *, *tt, *ff\}$
- $RP_N = \{\epsilon, *, *n \setminus n \in \mathcal{N}\}$
- $RP_{A^\perp} = RP_A$
- $RP_{A_1 \otimes A_2}$: an element of $M^*_{A_1 \otimes A_2}$ belongs to $RP_{A_1 \otimes A_2}$ if it satisfies:
  1. $s|A_1 \in RP_{A_1}$ and $s|A_2 \in RP_{A_2}$.
  2. If two successive moves in s don't belong to the same component A_1 or A_2 it is the opponent who switched component.
  3. An answer in s played in A_1 (resp. A_2) is associated to a question played in A_1 (resp. A_2).
- $RP_{!A}$: an element s of $M^*_{!A}$ belongs to $RP_{!A}$ if it satisfies:
  1. $\forall i \in \mathcal{N}$, $s|A, i \in RP_A$.
  2. if two successive moves in s have different indexes it is the Opponent who switched indexes.
  3. An answer in s has the same index as the associated question.

4. If s is of the form

$$s = ...\underset{\mathrm{O}}{m_1}\underset{\mathrm{P}}{m_2}...\underset{\mathrm{O}}{m_1'}\underset{\mathrm{P}}{m_3}...$$

$$\begin{array}{ll} \text{where} & m_1 = (\alpha_1, j.\vec{\imath}, c_1) \\ & m_1' = (\alpha_1, j'.\vec{\imath}, c_1) \\ & m_2 = (\alpha_2, j.\vec{k}, c_2) \\ \text{then} & m_3 = (\alpha_2, j'.\vec{k}, c_2) \end{array}$$

This means that Player plays the same way in the different indexes.

Notice that $RP_{A_1 \wp A_2} = RP_{({A_1}^{\perp} \otimes {A_2}^{\perp})^{\perp}}$ (resp. $RP_{?A_1} = RP_{(!{A_1}^{\perp})^{\perp}}$) can be defined equivalently directly from RP_{A_1} and RP_{A_2} (resp. RP_{A_1}) in the same way as $RP_{A_1 \otimes A_2}$ (resp.$RP_{!A_1}$) simply by inverting "Opponent" and "Player" in conditions 1, 2, 3 (resp. 1,2,3,4). Notice that we have:

$$RP_{(A_1 \wp A_2) \wp A_3} = RP_{A_1 \wp (A_2 \wp A_3)}$$

$$RP_{(A_1 \otimes A_2) \otimes A_3} = RP_{A_1 \otimes (A_2 \otimes A_3)}$$

We have the following properties :

Proposition 1 *An element s of M_A^* belongs to RP_A iff:*

a) For any atomic subformula $\alpha, \vec{\imath}$ of A we have $s|(\alpha, \vec{\imath}) = \epsilon, *$ or $*a$, where a is an answer in M_α.

b) The associations question-answer in s match with the associations question-answer in the restrictions $s|\alpha, \vec{\imath}$. T.i.t.s. an answer $(\alpha, \vec{\imath}, a)$ in s is associated to a question of the form $(\alpha, \vec{\imath}, *)$.

c) For any subformula $F, \vec{\imath}$ of A we have :

- *if $F = F_1 \otimes F_2$: in $s|F\vec{\imath}$ only the Opponent can switch components.*
- *if $F = !F_1$: in $s|F\vec{\imath}$ only the Opponent can switch index and the Player plays the same way in the different indexes.*
- *dual conditions (we invert Opponent and Player) for $F = F_1 \wp F_2$ and $F = ?F_1$.*

This is easily proved by induction on A.

Proposition 2 *(link between RP_A and the set of positions P_A defined in [])*

Let A be a formula built with the connectives $\otimes, \multimap, !$ and $\overline{A}$ the equivalent formula built with $\otimes, \wp, !, ?$ and where negations apply only to atomic subformulas. If s belongs to M_A^* we have:

$s \in P_A$ iff

- *$s \in RP_{\overline{A}}$*
- *for any subformula $F, \vec{\imath}$ of A, $s|M_{F,\vec{\imath}}$ starts with a move by Opponent in M_F.*

Strategies considered here will only be history-free strategies, so we can define them using functions of strategies. A partial function $f : M_A^O \to M_A^P$ is a *function of strategy* if it satisfies:

$\forall s\ a \in RP_A$ if $f(a) = b$ then $sab \in RP_A$.
O

A subset σ of RP_A is a (relaxed history-free) strategy for Player if there exists a function of strategy f on A such that $\sigma = traces(f)$, where $traces(f)$ is defined inductively by :

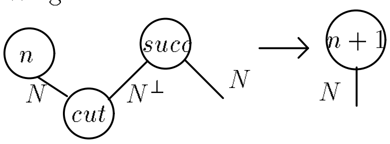
$$traces(f) = \{\epsilon\} \cup \{sab/\ s \in traces(f),\ sa \in RP_A\ f(a) = b\}.$$

2 Translation of the PCF^- terms into proof-nets :

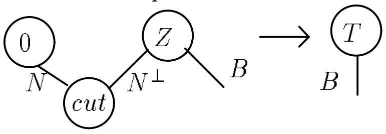
We only consider the case of PCF deprived of the fixed-point operators, which will be denoted PCF^-. We need to introduce new nodes corresponding to the constants of PCF^-:

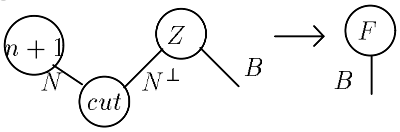
- unary nodes : T,F,n (for each integer n) of respective conclusion B,B and N.
- nodes of arity two : *succ* and *pre* whose edges are typed by $N^\perp$,N ; Z whose edges are typed by $N^\perp$,B.
- nodes of arity four : if_B whose edges are typed by $B^\perp$,$?B_1^\perp$,$?B_2^\perp$,B and if_N whose edges are typed by $B^\perp$,$?N_1^\perp$,$?N_2^\perp$,N.

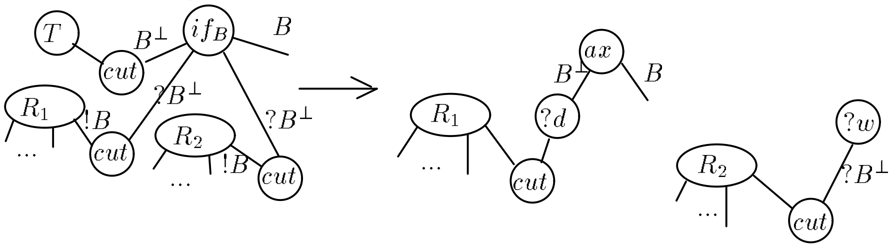
We give for these nodes the following reduction rules:



The rule for pre is defined in a similar way...



and 



The second rule for if_B is obtained by replacing T by F and inverting R_1 and R_2 in the right-hand proof-net.

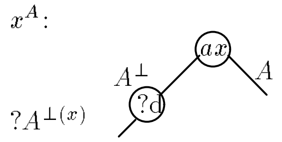
The rules for if_N are similar.

Now we can associate to each term a proof-net , using the following translation of types into linear formulas:

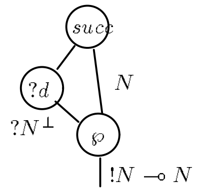
$A^* = A$	if $A = B$ or N
$(A_1 \to A_2)^* = (!A_1^*) \multimap A_2^*$	otherwise .

We will omit in the sequel the sign $*$. The translation is defined inductively by :

- x^A:

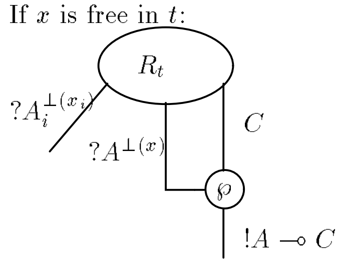


- The constants $T : B$, $F : B$, $n : N$ are translated by the corresponding nodes.
- The translations of $succ : N \to N$, $pre : N \to N$, $Z : N \to B$, if_B : $B, B, B \to B$ and $if_N : B, N, N \to N$ are obtained from the corresponding nodes by doing appropriate derelictions and $\wp$-rules on the conclusions in order to get a unique conclusion typed by A^* where A is the type of the constant. For instance $succ$ is translated by :



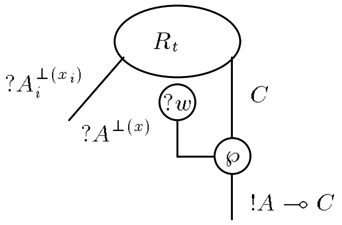
- $\lambda x^A t^B : A \to B$:

If x is free in t:

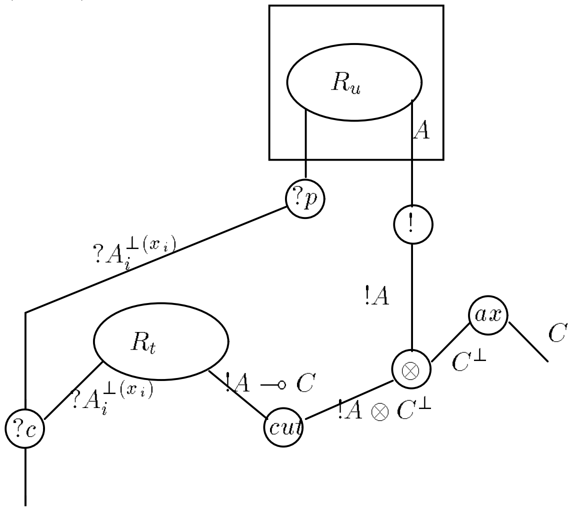


Otherwise:

otherwise :



– $(t^{A \to B})u^A$:



(The contractions are done on the conclusions corresponding to common free variables of t and u).

The reduction in the proof-nets then simulates the PCF-reduction in the following sense:

Lemma 1 *If t and t' are PCF^- terms such that $t \to t'$ then the proof-net translation R_t of t can be reduced into a proof-net R' obtained from $R_{t'}$ by a certain number of weakening rules.*

The weakening nodes that we must add to $R_{t'}$ correspond to the free variables of t that have been erased during the reduction. See [Dan90] for a discussion about the representation of lambda-terms by proof-nets.

3 Matrix of partial functions associated to a proof-net:

To a proof-net R of conclusions $A_1, ..., A_n$ we associate a matrix $\mathcal{M}_R$ of order n and of coefficient indexed by i, j a partial function $f^R_{A_i,A_j} : M_{A_i} \to M_{A_j}$ (we will write simply f_{A_i,A_j} when there is no ambiguity). It will satisfy the following *determinism condition* : (1) $\forall j \neq j'$ $\mathrm{dom}(f_{A_i,A_j}) \cap \mathrm{dom}(f_{A_i,A_{j'}}) = \emptyset$.

We will give "elementary" matrices (copy-cat matrices and matrices corresponding to PCF constants) as well as constructions on matrices ($\otimes$, $\wp$, cut, dereliction, contraction, weakening) that will enable us to associate inductively to each proof-net a matrix. We shall give these constructions and the interpretation of proof-nets simultaneously.

3.1 MELL proof-nets :

For these proof-nets the following *reversibility condition* is satisfied :
(2) $\forall i, j$ f_{A_i,A_j} is injective ,$\mathrm{dom}\, f_{A_i,A_j} = \mathrm{Im}\, f_{A_j,A_i}$ and $f_{A_i,A_j} = f^{-1}_{A_j,A_i}$.
Notations :

- If $f : M_A \to M_B$, we will abusively denote f the functions obtained by extension of its domain or codomain such as :

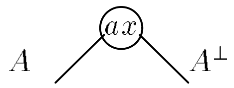
$$f : M_{A\otimes C} = M_A + M_C \to M_B$$
$$f : M_A \to M_{B\otimes C} = M_B + M_C$$

- $f \cup g$ denotes the function defined on the disjoint union $\mathrm{dom}(f) \cup \mathrm{dom}(g)$ by :

$$(f \cup g)(m) = \begin{cases} f(m) & m \in \mathrm{dom}(f) \\ g(m) & m \in \mathrm{dom}(g) \end{cases}$$

Definition:

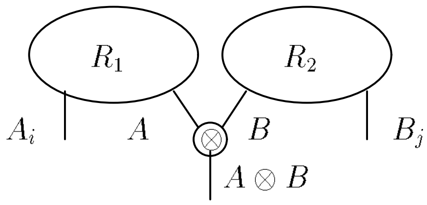
$R =$



$$\mathcal{M}_R = \begin{pmatrix} \emptyset & m \to m \\ m \to m & \emptyset \end{pmatrix}$$

The lines and columns are indexed by : $A, A^\perp$. $\emptyset$ denotes a nowhere defined function.

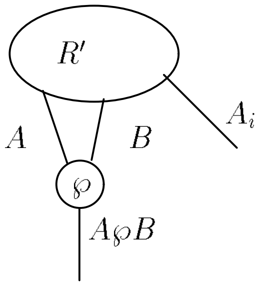
$R =$



$$\mathcal{M}_R = \begin{pmatrix} \mathcal{M}_{R_1}|_{A_1,..,A_n} & \emptyset & f^{R_1}_{A_i,A} \\ \emptyset & \mathcal{M}_{R_2}|_{B_1,..,B_m} & f^{R_2}_{B_i,B} \\ f^{R_1}_{A,A_j} & f^{R_2}_{B,B_j} & f^{R}_{A\otimes B,A\otimes B} \end{pmatrix}$$

where $f^R_{A\otimes B,A\otimes B} = f^{R_1}_{A,A} \cup f^{R_2}_{B,B}$
The lines and columns are indexed by: $A_1, ..., A_n, B_1, ..., B_m, A \otimes B$.

$R =$



$$\mathcal{M}_R = \begin{pmatrix} \mathcal{M}_{R'}|_{A_1,..,A_n} & f^{R'}_{A_i,A} \cup f^{R'}_{A_i,B} \\ f^{R'}_{A,A_j} \cup f^{R'}_{B,A_j} & f^{R}_{A\wp B,A\wp B} \end{pmatrix}$$

where :

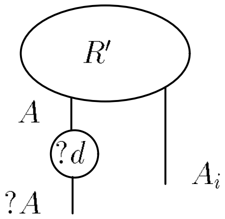
$$f^{R}_{A\wp B,A\wp B} = (f^{R'}_{A,A} \cup\ f^{R'}_{A,B}) \cup (f^{R'}_{B,A} \cup f^{R'}_{B,B})$$

(notice that $f^{R'}_{A,A}$ and $f^{R'}_{A,B}$ as well as $f^{R'}_{B,A}$ and $f^{R'}_{B,B}$ have disjoint domains).

In the following cases the matrix can be defined in a similar way, but it should be enough to give the main coefficients to reconstitute it.

– dereliction

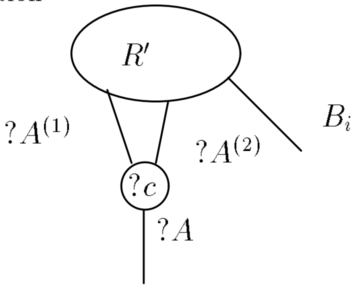
$R =$



$f^{R}_{A_k,A_{k'}} = f^{R'}_{A_k,A_{k'}}$
$f^{R}_{A_k,?A}(m) = (\alpha, 0.\vec{\imath}, c)$ if $f^{R'}_{A_k,A}(m) = (\alpha, \vec{\imath}, c)$

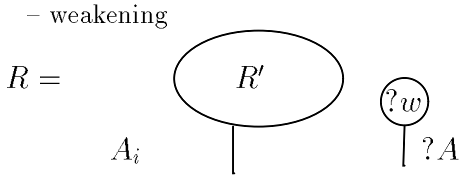
– contraction

$R =$



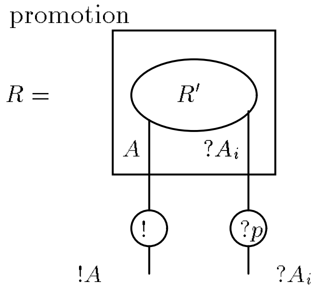
$$f^R_{B_k,B_{k'}} = f^{R'}_{B_k,B_{k'}}$$
$$f^R_{?A,B_k}(\alpha,(2i+1).\vec{\jmath},c) = f^{R'}_{?A^{(1)},B_k}(\alpha,i.\vec{\jmath},c)$$
$$f^R_{?A,B_k}(\alpha,(2i).\vec{\jmath},c) = f^{R'}_{?A^{(2)},B_k}(\alpha,i.\vec{\jmath},c)$$

– weakening



$f^R_{A_k,A_{k'}} = f^{R'}_{A_k,A_{k'}}$
$f^R_{?A,A_k}$ is nowhere defined.
$f^R_{?A,?A}$ is nowhere defined.

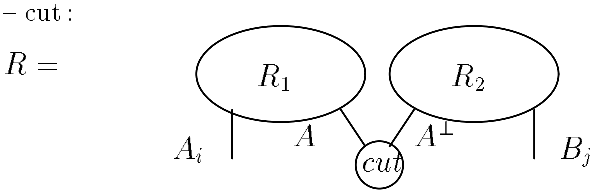
– promotion



Let $<.,.>$ denote a bijection $\mathcal{N}^2 \to \mathcal{N}$.

$f^R_{!A,!A}(\alpha,i.\vec{\jmath},c) = (\beta,i.\vec{\jmath'},c')$ if $f^{R'}_{A,A}(\alpha,\vec{\jmath},c) = (\beta,\vec{\jmath'},c')$
$f^R_{!A,?B_k}(\alpha,i.\vec{\jmath},c) = (\beta,<i,l>.\vec{\jmath'},c')$ if $f^{R'}_{A,?B_k}(\alpha,\vec{\jmath},c) = (\beta,l.\vec{\jmath'},c')$
$f^R_{?B_k,?B_{k'}}(\alpha,<i,l>.\vec{\jmath},c) = (\beta,<i,l'>.\vec{\jmath'},c')$ if $f^{R'}_{?B_k,?B_{k'}}(\alpha,l.\vec{\jmath},c) = (\beta,l'.\vec{\jmath'},c')$

– cut :



We apply the execution formula (cf [Gir89]).

$f^R_{A_k,B_l}(m) = f^{R_2}_{A^\perp,B_l} \circ [f^{R_1}_{A,A} \circ f^{R_2}_{A^\perp,A^\perp}]^{k_0} \circ f^{R_1}_{A_k,A}(m)$

where k_0 is the smallest integer such that this image is defined, if it exists (otherwise the function is not defined on m).
similarly

$f^R_{A_k,A_{k'}}(m) = f^{R_1}_{A_k,A_{k'}}(m)$ if $m \in \mathrm{dom}\, f^{R_1}_{A_k,A_{k'}}$,
otherwise
$f^R_{A_k,A_{k'}}(m) = f^{R_1}_{A_k,A_{k'}} \circ f^{R_2}_{A^\perp,A^\perp} \circ [f^{R_1}_{A,A} \circ f^{R_2}_{A^\perp,A^\perp}]^{k_0} \circ f^{R_1}_{A_k,A}(m)$
where k_0 is defined as previously.

It can easily be proved that these matrices map a move of atom α onto a move of atom $\alpha^\perp$ and preserve the content of the move (t.i.t.s. a question is mapped onto a question and an answer onto the same answer).

3.2 Proof-nets extended to the PCF-constants $T, F, n, Z, succ, pre, if_B, if_N$:

- T, F, n :

  We define $\mathcal{M}_T = (f_{B,B})$ where $f_{B,B} : (B, \epsilon, *) \to (B, \epsilon, tt)$.

  The matrices $\mathcal{M}_F$ and $\mathcal{M}_n$ are defined in a similar way by replacing tt respectively by ff and n.

  Notice that contrary to the "logical" proof-nets, these nodes don't change the atom of the move but turn a question into an answer.

- $succ$, pre, Z :

$$\mathcal{M}_{succ} = \begin{pmatrix} \emptyset & f_{N^\perp,N} \\ f_{N,N^\perp} & \emptyset \end{pmatrix}$$

  where

$$f_{N,N^\perp} : (N, \epsilon, *) \to (N^\perp, \epsilon, *)$$
$$f_{N^\perp,N} : (N^\perp, \epsilon, n) \to (N, \epsilon, n+1) \quad n \in \mathcal{N}$$

  The matrix $\mathcal{M}_{pre}$ is defined in a similar way ,by changing the coefficient $f_{N^\perp,N}$:

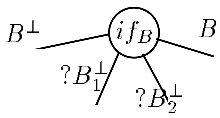
$$f_{N^\perp,N} : (N, \epsilon, n) \to (N^\perp, \epsilon, n-1) \quad if\ n \geq 1$$

$$\mathcal{M}_Z = \begin{pmatrix} \emptyset & f_{N^\perp,B} \\ f_{B,N^\perp} & \emptyset \end{pmatrix}$$

  where

$$\begin{array}{lcl} f_{B,N^\perp} : & (B, \epsilon, *) & \to (N^\perp, \epsilon, *) \\ f_{N^\perp,N} : & (N^\perp, \epsilon, 0) & \to (B, \epsilon, tt) \\ & (N^\perp, \epsilon, n+1) & \to (B, \epsilon, ff) \end{array}$$

- if_B, if_N :



  The matrix is defined by the following coefficients (the coefficients not mentioned are undefined functions):

$$\begin{array}{lcl} f_{B,B^\perp} : & (B, \epsilon, *) \to (B^\perp, \epsilon, *) & \\ f_{B^\perp,?B_1^\perp} : & (B^\perp, \epsilon, tt) \to (B_1^\perp, 0, *) & \\ f_{B^\perp,?B_2^\perp} : & (B^\perp, \epsilon, ff) \to (B_2^\perp, 0, *) & \\ f_{?B_1^\perp,B} : & (B_1^\perp, 0, r) \to (B, \epsilon, r) & r \in \{tt, ff\} \\ f_{?B_2^\perp,B} : & (B_2^\perp, 0, r) \to (B, \epsilon, r) & r \in \{tt, ff\} \end{array}$$

  The matrix for if_N is defined in a similar way.

The determinism property (1) is still satisfied but if we add Z or if we can't keep the reversibility property (2) ($\mathcal{M}_Z$ for example has a non-injective coefficient).

We have the following property:

Lemma 2 *The matrix associated to a proof-net is invariant by elimination of multiplicatives or axioma cuts.*

Yet this interpretation by matrices is not invariant by elimination of exponential cuts or by the new constant reductions.

4 Function of strategy induced by a proof-net :

R being a proof-net of conclusions $A_1, ..., A_n$ the determinism property (1) allows us to define a partial function $f_R : M_F^O \to M_F^P$, where $F = A_1 \wp ... \wp A_n$, in the following way:

$f_R(m) = f^R_{A_i,A_j}(m)$ if $m \in \mathrm{dom}\, f^R_{A_i,A_j}$.

Proposition 3 *The function f_R associated to a proof-net R induces a valid strategy σ_R on F, respectively to RP_F.*

Proof:

We prove by induction on a construction of R that if $s = m_1...m_{2k+1}$ is a position on F in which m_{2i+1} is a move by Opponent and $m_{2i+2} = f_R(m_{2i+1})$ (we say that *Player follows f_R in s*)then $s' = sf_R(m_{2k+1})$ is a position on F.

- R is an axioma . $F = A^\perp \wp A$

  The play is of the form
  $$\begin{array}{ccccc} m_1 & m_1 & m_3 & m_3 & m_5 ... \\ A & A^\perp & A^\perp & A & A \\ O & P & O & P & O \end{array}$$

  Suppose for instance $m_{2i+1} \in M_A$. The condition (1) is satisfied since $s'|A^\perp = s|A$. The switching condition (2) is satisfied. Condition (3) : suppose $f_R(m_{2k+1})$ is an answer, then it is easy to see that if it is matched with m_{k_0} we have $2k + 2 \equiv k_0 + 3[4]$. This implies that $f_R(m_{2k+1})$ is in the same component as m_{k_0}.

- It is easily checked that the constant nodes $V, F, n, Z, succ, pre, if_B, if_N$ induce a valid strategy on their conclusion.

- Suppose R is obtained by a $\wp$ on two conclusions of R'.

  Then $f_{R'} = f_R$ and the result follows from the induction hypothesis on R'.

- Suppose R is obtained from R_1 and R_2 satisfying the i.h. by a $\otimes$ rule . $F = A_1 \wp ... \wp A_n \wp (A \otimes B) \wp B_1 \wp ... \wp B_m$

  Let s_1(resp. s_2) be the restriction of s to its moves in the conclusions of R_1 (resp. R_2). By definition of f_R, Player follows f_{R_1} (resp.f_{R_2}) in s_1

(resp. s_2). Suppose for instance that m_{2k+1} is played in a conclusion of $R_1 : s_1 = t_1 m_{2k+1}$.

Let's show that s_1 (resp. s_2) is a position on $F_1 = A_1 \wp ... \wp A_n \wp A$ (resp. $F_2 = B \wp B_1 \wp ... \wp B_m$) using Proposition 1.

As s belongs to RP_F it satisfies conditions a, b, c. It implies that s_1 and s_2 satisfy condition a.

We need now to prove that the associations question-answer in s_1 (resp. s_2) match with the associations in s : we mean by that that if in s an answer m' belonging to M_{F_1} is associated to a question m , then m belongs to M_{F_1} and in s_1 m' is associated to m.

It suffices to prove that there is in s no question-answer couple m, m' such that m belongs to M_{F_1} and m' to M_{F_2} or *vice-versa*. This follows from the fact that s satisfies condition b.

Therefore the question-answer associations in s_1, s_2 are the same as in s and as s satisfies b, s_1 and s_2 satisfy b. Finally s_1 and s_2 satisfy c since s satisfies c. So applying proposition 1 we have: s_1 belongs to RP_{F_1} and s_2 to RP_{F_2}.

Now by i.h. on R_1, $s'_1 = s_1 f_{R_1}(m_{2k+1}) \in RP_{F_1}$ so it satisfies conditions a, b, c. If $f_{R_1}(m_{2k+1})$ is a question then the associations in s'_1 match with those in s' and we can conclude that s' satisfies a and b. If $f_{R_1}(m_{2k+1})$ is an answer we only need to prove that it is not associated in s' with a move in a conclusion of R_2 so that we may conclude in the same way as before.

Ab absurdo : suppose it is the case and let m denote the question associated to $f_R(m_{2k+1})$. Let u denote the sequence of the moves in F_2 played in s' between m and $f_R(m_{2k+1})$ (these two moves excluded). It follows from the definition of question-answer associations that moves between m and $f_R(m_{2k+1})$ are associated two by two and a move in F_2 is associated to a move in F_2 (we proved that before); so u is of even length $|u|$.

But on the other hand as s satisfies the switching condition it is of the following form:

$$s = \dots \quad \underbrace{\begin{matrix} \dots & m & \dots & B \\ & O & & P \end{matrix}}_{F_2} \underbrace{\begin{matrix} A & \dots & A \\ O & & P \end{matrix}}_{F_1} \underbrace{\begin{matrix} B & \dots & B \\ O & & P \end{matrix}}_{F_2} \quad \dots \quad \underbrace{\begin{matrix} A & \dots & f_R(m_{2k+1}) \\ O & & P \end{matrix}}_{F_1}$$

where A and B stand here for moves in A and B.

Since s satisfies alternacy we induce from that that $|u|$ is odd, which gives a contradiction.

So $f_R(m_{2k+1})$ is associated to a move in F_1 and the question-answer associations in s' match with those in s'_1 and s_2. So s' satisfies a and b.

As its last move is by Player, s' could only contradict the condition c with respect to a subformula $G\vec{1}$ where $G = G_1 \otimes G_2$ or $G = !G_1$. Such a subformula would have to be :

- either a subformula of F_1, but this would contradict the fact that s'_1 satisfies condition c.

- or $G = A \otimes B$: but as Player follows f_R and by definition of f_R, Player cannot switch component in $s'|A \otimes B$.

So s' is a position.

- Suppose R is obtained by a dereliction on a conclusion A of R'.

  In s' each move in $?A$ is of the form $(\alpha, 0.\vec{\imath}, c)$. Let $\overline{s'}$ be the sequence obtained from s' by replacing each $(\alpha, 0.\vec{\imath}, c)$ belonging to $M_{?A}$ by $(\alpha, \vec{\imath}, c)$ belonging to M_A. Then Player plays in $\overline{s'}$ according to $f_{R'}$. The associations question-answer in $\overline{s'}$ match with those in s', so since s satisfies the conditions a, b, c we can deduce that $\overline{s}$ does the same and therefore is a position on the conclusion F' of R'. The i.h. on R' then gives that $\overline{s'}$ is a position on F'; this implies immediately that s' satisfies conditions a and b. As to condition c as said before it is sufficient to consider the case of a subformula $G\vec{\imath}$ where $G = G_1 \otimes G_2$ or $G = !G_1$: such a subformula is a subformula of F' and s' satisfies c respectively to $G\vec{\imath}$ since $\overline{s'}$ does so.

  So s' belongs to RP_F.

- Suppose R is obtained from R' by contracting two conclusions $?A^{(1)}$,$?A^{(2)}$ into $?A$.

  Let $\overline{s'}$ be the sequence of $M_{F'}$ obtained from s' by replacing each $(\alpha, (2k+1).\vec{\imath}, c)$ element of M_A in s' by the move $(\alpha, k.\vec{\imath}, c)$ in $M_{?A^{(1)}}$ and each $(\alpha, (2k).\vec{\imath}, c)$ by the move $(\alpha, k.\vec{\imath}, c)$ in $M_{?A^{(2)}}$. Then Player plays in $\overline{s'}$ according to $f_{R'}$. The question-answer associations in $\overline{s'}$ match with those in s'. As previously we infer from that that since s is a position on F, $\overline{s}$ is a position on F'. Then by i.h. we have: $\overline{s'} \in RP_{F'}$.

  This implies that s' satisfies a and b. As to c we consider a subformula $G\vec{\imath}$ of F with $G = G_1 \otimes G_2$ or $G = !G'$:

  - if it is a subformula of an other R-conclusion than $?A$ the result follows directly from the fact that s' satisfies c respectively to $G\vec{\imath}$.
  - if it is a subformula of $?A$: suppose for instance $\vec{\imath} = (2k+1).\vec{\jmath}$, then the result follows from the fact that $\overline{s'}$ satisfies c respectively to the subformula $Gk.\vec{\jmath}$ of $?A^{(1)}$.

  So s' belongs to $RP_{F'}$.

- Suppose R is obtained from R' by the mean of a promotion rule.

  $F = !A_1 \wp ?B_1 \wp ... \wp ?B_n$, $F' = A_1 \wp ?B_1 \wp ... \wp ?B_n$

  Let s'_i be the sequence defined from s' by replacing each $(\alpha, i.\vec{\jmath}, c)$ belonging to $M_{!A}$ by $(\alpha, \vec{\jmath}, c)$ and each $(\alpha, < i, j > .\vec{k}, c)$ belonging to $M_{?B_k}$ by $(\alpha, j.\vec{k}, c)$ (these moves will be called their projections and i their copy index) and erasing the other moves. Then Player follows $f_{R'}$ in the alternate sequence s'_i for any i.

  Moreover as s is a position on F, we can induce that s_i satisfies conditions a,b,c and therefore is a position on F'.Let i_0 be the copy index of m_{2k+1}. Then by definition of f_R the copy index of $f_R(m_{2k+1})$ is i_0 and if $\overline{m_{2k+1}}$ denotes m_{2k+1}'s projection, the projection of $f_R(m_{2k+1})$ is $f_{R'}(\overline{m_{2k+1}})$.

  So $s'_{i_0} = s_{i_0} f_{R'}(\overline{m_{2k+1}})$, where the last move of s_{i_0} is $\overline{m_{2k+1}}$.

By i.h. on R' we can infer from this that s'_{i_0} belongs to $RP_{F'}$.

Now we need to show that the question-answer associations in s' match with those in s'_{i_0} and the plays s_i. In this purpose we suppose that $f_R(m_{2k+1})$ is an answer and we show that it is associated in s' to a question which has the same copy index i_0, using the same kind of argument as in the case of $\otimes$. *Ab absurdo* : suppose $f_R(m_{2k+1})$ is associated in s' to a question m of copy index $j \neq i_0$.

Let u denote the sequence of the moves of copy index i_0 played in s' between m and $f_R(m_{2k+1})$ (these two moves excluded). It follows from the definition of question-answer associations that moves between m and $f_R(m_{2k+1})$ are associated two by two and a move of copy index i_0 is associated to a move of the same copy index; so $|u|$ is even.

But as s' satisfies the switching condition it is of the following form:

$$s = \ldots \underbrace{\begin{matrix} !A & \ldots & m & \ldots & !A \\ O & & O & & P \end{matrix}}_{j} \underbrace{\begin{matrix} !A & \ldots & !A \\ O & & P \end{matrix}}_{k_1} \underbrace{\begin{matrix} !A & \ldots & !A \\ O & & P \end{matrix}}_{k_2} \ldots \underbrace{\begin{matrix} !A & \ldots & !A \\ O & & P \end{matrix}}_{k_l} \underbrace{\begin{matrix} !A & \ldots & f_R(m_{2k+1}) \\ O & & P \end{matrix}}_{i_0}$$

where the integers below stand for the copy indexes of the subsequences.

Since s satisfies alternacy we infer from this that $|u|$ is odd, which gives a contradiction.

So $f_R(m_{2k+1})$ is associated to a move of copy index i_0 and the question-answer associations in s' match with those in s'_{i_0} and the positions s_i.This implies in particular that s' satisfies conditions a and b. To check that s' satisfies condition c we only have to consider the cases of subformulas $G\vec{\imath}$ such that $G = G_1 \otimes G_2$ or $G = !G'$. Such a subformula would have to be:

- either a subformula of A or of one of the $?B_k$. The fact that s' satisfies c with respect to $G\vec{\imath}$ follows from the fact that a certain s'_i satisfies c.
- or $G = !A$. If $f_R(m_{2k+1})$ is a move in $!A$ notice that it follows from the definition of f_R that it satisfies the switching condition. It remains to show that in $s'|!A$ Player plays the same way in the different indexes; it follows from the definition of f_R and from the fact that Opponent plays the same way in the different indexes of $s'|?B_j$ for $1 \leq j \leq n$.

So s' is a position on F.

- Suppose R is obtained from R_1, R_2 by the mean of a cut rule on conclusions A, $A^\perp$.

  Let $\overline{s'}$ be the sequence obtained by completing s' with the intermediary moves in A, $A^\perp$ which were used to define Opponent's moves image by f_R with the execution formula.$\overline{s'}$ is an alternate sequence of moves on $F_{compl} = A_1 \wp \ldots \wp A_n \wp (A \otimes A^\perp) \wp B_1 \wp \ldots \wp B_m$. It follows from the definition of f_R that Player plays in $\overline{s'}$ according to $f_{R'}$, where R' is the proof-net obtained from R_1, R_2 by applying a tensor rule on A, $A^\perp$. Let t denote $\overline{s'}$ deprived of its last move $f_R(m_{2k+1})$; notice that t is not necesseraly equal to $\overline{s}$ as there might be additional moves in A,$A^\perp$. It can easily be proved that t is a position on F_{compl} using the fact that Opponent plays according to a “copy-cat” strategy in $t|A \otimes A^{\perp}$. Then , using the i.h. on R' we can infer from this that $\overline{s'}$ is a position on F_{compl}. This implies directly that s' satisfies condition a. As $\overline{s'}$ satisfies condition b the question-answer associations in $\overline{s'}$ match with those in s' and therefore s' satisfies condition b. Finally as $\overline{s'}$ satisfies condition c and as any subformula of F is a subformula of F_{compl} we know that s' satisfies condition c. So s' is a position on F.

5 Application to the PCF^{-} terms:

To a term $t : A$ of free variables $x_1 : A_1, ..., x_n : A_n$ we can associate the function of strategy given by its proof-net R on the formula:

$$F = (?A_1^{\perp})\wp...(?A_n^{\perp})\wp B \equiv (!A_1) \otimes ... \otimes (!A_n) \multimap B.$$

Is this interpretation of t the same as the AJM one? Let's recall that the AJM interpretation $|t|$ of t is an equivalence class of strategies on the game F with respect to the relation $\simeq_F$ (defined in [AJM95]).Intuitively, as said in [AJM94], $\sigma \simeq_F \tau$ means that σ and τ are the same modulo coding conventions. We want to show that the interpretation of t we described (via the matrix $\mathcal{M}_R$) and which relied on arbitrary choices in the matrix constructions corresponding to exponentials (dereliction, contraction, promotion) is an instantiation of the AJM one. Let $\mathcal{M}^t$ denote the matrix associated to the proof-net obtained by translation of t.

Proposition 4 *t being a PCF^{-} term, f_R induces a valid strategy respectively to P_F (and not only to RP_F). Moreover this strategy belongs to the AJM interpretation of t (which is an equivalence class of strategies).*

Proof: Let $\mathcal{M}_1$, $\mathcal{M}_2$ be matrices inducing strategies σ_1, σ_2 respectively to P_{Γ_1}, P_{Γ_2} where:

$$\Gamma_1 = A_1, ..., A_n \vdash A$$

$$\Gamma_2 = A, A'_1, ..., A'_n \vdash A'$$

(P_{Γ_1} is $P_{A1,...,A_n \multimap A}$).

Let $\mathcal{M}'$ denote the matrix obtained from $\mathcal{M}_1$, $\mathcal{M}_2$ by a cut construction (on A). Then $\mathcal{M}'$ induces a strategy σ' respectively to $P_{\Gamma'}$ ($\Gamma' = A_1, ..., A_n, A'_1, ..., A'_n \vdash A'$) and $\sigma' = (\sigma_1 \otimes Id_{A_1 \otimes ... \otimes A_n}); \sigma_2$ (which we will in the sequel denote abusively by $\sigma_1; \sigma_2$).

Let $\mathcal{M}$ be a matrix inducing a strategy σ respectively to P_Γ where $\Gamma_1 = A_1, ..., A_n \vdash A$. Let $\mathcal{M}'$ be the matrix obtained from $\mathcal{M}$ by a dereliction on A_{i_0}; then $\mathcal{M}'$ induces a strategy σ' respectively to $P_{\Gamma'}$ ($\Gamma' = A_1, .., ?A_{i_0}, ..., A_n \vdash A$) and $\sigma' = der^0_{A_{i_0}}; \sigma$. This follows directly from the definition of the composition of strategies in [AJM95].

Similarly the constructions tensor, contraction, promotion on matrices coincide with the corresponding interpretation by means of composition of strategies in [AJM95].

We can now prove the property for all PCF^{-} term t by induction on t.

- $t = x^A$: the associated matrix induces on $P_{!A \multimap A}$ the strategy der^0_A and the AJM interpretation of t is $[der^0_A]$

- $t = T, F, n, succ, pre, if_B, if_N$: the property is satisfied by definition of the matrices.
- $t = \lambda x^{A_1} u^{A_2} : A_1 \to A_2$

  Let $x_1, ..., x_n$ be the free variables of u and let $C_1, ..., C_n$ be their types. Then by i.h. $\mathcal{M}^u$ induces a strategy σ_u respectively to $P_{!C_1 \otimes ... \otimes !C_n \multimap A_2}$ which belongs to $|u|$.

  - first case: x is free in t

    Suppose for instance $x = x_n$, then $A_1 = C_n$. The following strategy is an element of $|\lambda x u|$:

    $\Lambda(\sigma_u) = \{\varphi(s), s \in \sigma_t\}$ where φ denotes the evident bijection

    $$\varphi : M_{!C_1 \otimes ... \otimes !C_{n-1} \otimes !A_1 \multimap A_2} \to M_{!C_1 \otimes ... \otimes !C_{n-1} \multimap (!A_1 \multimap A_2)}$$

    . On the other hand, by definition of the matrices associated to u and t, if f_u and f_t respectively denote the corresponding functions of strategy we have: $f_t = \varphi \circ f_u \circ \varphi^{-1}$. So f_t does induce the strategy $\Lambda(\sigma_u)$.
  - second case : x is not free in t

    In that case $|\lambda x.u|$ is given by the strategy:

    $$\Lambda(\sigma_u \otimes weak_{A_1}) : !C_1 \otimes ... \otimes !C_n \multimap (!A_1 \multimap A_2)$$

    where $weak_A : !A \multimap I$ is defined by $weak_A = \{\epsilon\}$. Again, the matrix interpreting $\lambda x.u$ induces this strategy.
- $t = (u)^{A_1 \to A_2} v_{A_1}$

  By i.h. on u and v, $\mathcal{M}^u$ and $\mathcal{M}^v$ induce strategies σ_u and σ_v belonging respectively to the AJM interpretation of u and v.

  $$\sigma_u : (!C_1) \otimes ... \otimes (!C_n) \multimap (!A_1 \multimap A_2)$$

  $$\sigma_v : (!C_1') \otimes ... \otimes (!C_m') \multimap A_1$$

  then $\sigma_u \otimes \sigma_v^\dagger$ is a strategy on :

  $$(!C_1) \otimes ... \otimes (!C_n) \otimes (!C_1') \otimes ... \otimes (!C_m') \multimap ((!A_1 \multimap A_2) \otimes !A_1)$$

  We consider the strategy $app_{!A_1, A_2} : (!A_1 \multimap A_2) \otimes !A_1 \multimap A_2$ defined by the function on $((M_{!A_1} + M_{A_2}) + M_{!A_1}) + M_{A_2}$ mapping a move in an occurence of $!A_1$ (resp. A_2) onto the same move in the other occurence of $!A_1$(resp. A_2) (see [AJM95] section 2.5).

  We claim that :

  Lemma 3 *An element of $|(u)v|$ is given by the strategy:*

  $$\sigma = cont_{i_1}; ...; cont_{i_k}; \sigma_u \otimes \sigma_v^\dagger; app_{!A_1, A_2}$$

  where the i_j are the indices of the common free variables x_{i_j} of u and v. $cont_{i_j} : (!C_{i_j}) \multimap (!C_{i_j}) \otimes (!C_{i_j})$

Proof of the lemma: Let's recall that for any couple of games A,B there is a isomorphism $e_{A,B}$ between $!(A\&B)$ and $!A\otimes!B$.

According to [AJM95] an element of $|(u)v|$ is given by:

$$cont_{i_1};...;cont_{i_k};e^{-1}_{C_1,\dots,C'_m};<\sigma_u,\sigma_v>^{\dagger};app'$$

where:

- $e_{C_1,\dots,C'_m}$ is an isomorphism between $!C=!(C_1\&...\&C_n\&C'_1\&...\&C'_m)$ and $!C_1\otimes...\otimes C'_m$.
- $<\sigma_u,\sigma_v>:!C\multimap(!A\multimap B)\&A$ is the pairing of σ_u and σ_v defined by: $<\sigma_u,\sigma_v>=e_{C_1,\dots,C'_m};\sigma_u^{\dagger}\otimes\sigma_v^{\dagger};e^{-1}_{!A_1\multimap A_2,A_1};der$
- $app':!(!A_1\multimap A_2)\otimes!A_1\multimap A_2$ is defined by: $app'=e_{(!A_1\multimap A_2),A_1};(der_{!A_1\multimap A_2}\otimes Id_{!A_1});app_{!A_1,A_2}$

Now by the Bang Lemma (see [AJM95]) we have:

$$\begin{aligned}<\sigma_u,\sigma_v>^{\dagger}&=(e_{C_1,\dots,C'_m};\sigma_u^{\dagger}\otimes\sigma_v^{\dagger};e^{-1}_{!A_1\multimap A_2,A_1};der)^{\dagger}\\&\simeq e_{C_1,\dots,C'_m};\sigma_u^{\dagger}\otimes\sigma_v^{\dagger};e^{-1}_{!A_1\multimap A_2,A_1}\end{aligned}$$

So:

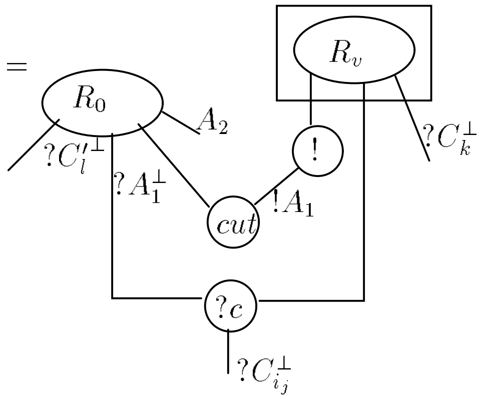
$$\begin{aligned}<\sigma_u,\sigma_v>^{\dagger};app'&\simeq e_{C_1,\dots,C'_m};\sigma_u^{\dagger}\otimes\sigma_v^{\dagger};e^{-1}_{!A_1\multimap A_2,A_1};e_{(!A_1\multimap A_2),A_1};(der_{!A_1\multimap A_2}\otimes Id_{!A_1});app_{!A_1,A_2}\\&\simeq e_{C_1,\dots,C'_m};\sigma_u^{\dagger}\otimes\sigma_v^{\dagger};(der_{!A_1\multimap A_2}\otimes Id_{!A_1});app_{!A_1,A_2}\\&\simeq e_{C_1,\dots,C'_m};(\sigma_u^{\dagger};der_{!A_1\multimap A_2})\otimes(\sigma_v^{\dagger};Id_{!A_1});app_{!A_1,A_2}\\&\simeq e_{C_1,\dots,C'_m};\sigma_u\otimes\sigma_v^{\dagger};app_{!A_1,A_2}\end{aligned}$$

And finally:

$cont_{i_1};...;cont_{i_k};e^{-1}_{C_1,\dots,C'_m};<\sigma_u,\sigma_v>^{\dagger};app'\simeq cont_{i_1};...;cont_{i_k};\sigma_u\otimes\sigma_v^{\dagger};app_{!A_1,A_2}$.
This ends the proof of the lemma.

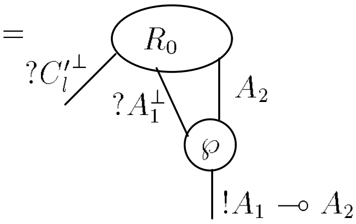
We want to show that the matrix $\mathcal{M}^t$ induces the strategy σ. The proof-net R_t can be reduced through the elimination of a multiplicative cut and a axioma cut into the following proof-net:

$R'=$



where R_0 is defined by:

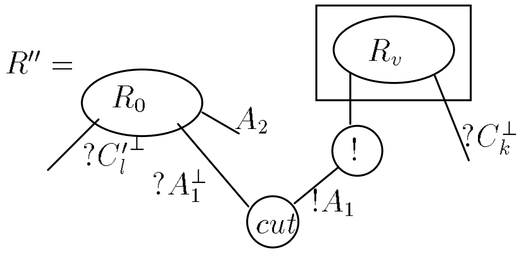
$R_u=$



So we have $\mathcal{M}_{R'}=\mathcal{M}^t$.

We also know that $\mathcal{M}_{R_0}$ induces a strategy σ_0 respectively to $P_{(!C_1)\otimes...\otimes(!C_n)\otimes(!A_1)\multimap A_2}$ and that $\sigma_t=\Lambda(\sigma_0)$.

Leaving aside the problem of the contractions we consider the following proof-net:



$\mathcal{M}_{R''}$ induces the strategy :

$$\tau = (Id_{\bigotimes_{l=1}^{n}(!C_l)} \otimes \sigma_v^{\dagger}); \sigma_0 : \bigotimes_{l=1}^{n}(!C_l) \otimes \bigotimes_{k=1}^{m}(!C_k') \multimap A_2$$

But $app_{!A_1,A_2}$ satisfies the following equality :

$$(\Lambda(\sigma_0) \otimes Id_{!A_1}); app_{!A_1,A_2} = \sigma_0$$

So, $(\sigma_u \otimes Id_{!A_1}); app_{!A_1,A_2} = \sigma_0$

Therefore we have:

$$\begin{aligned} \tau &= (Id_{\bigotimes_{l=1}^{n}(!C_l)} \otimes \sigma_v^{\dagger}); ((\sigma_u \otimes Id_{!A_1}); app_{!A_1,A_2}) \\ &= ((Id_{\bigotimes_{l=1}^{n}(!C_l)} \otimes \sigma_v^{\dagger}); (\sigma_u \otimes Id_{!A_1})); app_{!A_1,A_2} \\ &= ((Id_{\bigotimes_{l=1}^{n}(!C_l)}; \sigma_u) \otimes (\sigma_v^{\dagger}; Id_{!A_1})); app_{!A_1,A_2} \\ &= (\sigma_u \otimes \sigma_v^{\dagger}); app_{!A_1,A_2} \end{aligned}$$

We used successively the associativity of composition and the "compatibility" of $\otimes$ and composition.

Since $\mathcal{M}_{R''}$ induces the strategy τ, $\mathcal{M}_{R'}$ induces the strategy $cont_{i_1}; ...; cont_{i_k}; \tau$, which is precisely σ.

So $\mathcal{M}^t$ does induce a strategy that belongs to $|t|$.

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