Stratified coherence spaces: a denotational semantics for Light Linear Logic

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Abstract

Light linear logic (LLL) was introduced by Girard as a logical system capturing the class of polytime functions within the *proofs-as-programs* approach. In the present paper we undertake a semantical analysis of LLL: a variant of coherence spaces is introduced and we prove that it is a sound model for this system, but not for usual linear logic. A simpler version of the model yields a sound semantics of Elementary linear logic, which is the analog of LLL for the class of Kalmar elementary functions. We illustrate our semantical method by showing how various principles fail in these models.

1 Introduction

Linear logic and time complexity. Linear logic can be seen as a typing system where one provides fine-grain information about the use by functions of their arguments. This is achieved with modalities called *exponentials*, distinguishing arguments that can be used an arbitrary number of times from arguments that have to be used exactly once. It follows that the exponentials typing rules are the key to the control of the size of the computation flow during execution of typed programs. Tame exponentials can cut down the set of typeable programs to moderate time complexity classes. Indeed in [Gir98] Girard introduced such a variant of LL called *Light Linear Logic* (LLL) which

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captures the class of polytime functions. A polynomial bound was given on the number of steps of the cut-elimination procedure for LLL proofs (the degree of the polynomial depending on a certain parameter of the proof called the *depth*). Conversely, any polytime function on integers was shown to be representable by an LLL proof. A simplification of this system in an affine setting (Light Affine Logic) was done by Asperti in [Asp98] and adaptations of the system to the frameworks of lambda-calculus and combinatory logic have been studied (see [Rov98] and [Abr99]).

Semantics. Still, though the motivation lying behind the restrictions on exponential rules shows quite clearly when one examines the normalisation procedure, it is not so obvious to get some *positive* intuition on these new exponentials. Light linear logic is not born with a denotational semantics as its elder sister... The supply of models could facilitate the understanding of the system, give clues to fix some points where several options are possible (e.g. auto-duality of the modality §, provability of $!\mathbf{1}$, adding of full weakening as in Asperti's LAL), possibly help to write representations of concrete functions.

A first step was done by Kanovich *et al* who gave in [KOS97] a semantics of *provability* through phase spaces, establishing a completeness result similar to that for LL ([Gir87]). But this approach does not provide information about the proofs themselves. An alternative idea was to look for geometry of interaction models with intrinsic complexity bound on their dynamics. Such a work was undertaken in [BP01] for Elementary Linear Logic (ELL), another variant of LL corresponding to Kalmar elementary functions (see [Gir98] and [DJ]).

Our approach. Here we address the problem of denotational semantics for LLL. But let us first make clear what we are looking for. Indeed LLL can be encoded in LL in a natural way (compatible with cut-elimination), which is no surprise since it is a refinement of LL. Therefore any model of LL yields a model of LLL. So in fact we would like a model *specific* of LLL in the sense that it should not satisfy the principles excluded by this system (for instance the dereliction principle).

Our starting point is inspired by a semantics introduced by Martin Hofmann in [Hof99] where a size is associated to values and functions are required not to increase the size. We will instead consider sizes of *computations* defined from the number of *threads* of subcomputation opened. Morphisms will be required to offer a fixed bound on the difference between their number of output and input requests. This idea can be illustrated informally through

a games semantics analogy: the AJM-like strategy ([AJM94]) associated to the canonical proof of $!A \otimes !B \multimap !(A \otimes B)$ splits each thread of computation on the right-hand-side (output) into two threads of computation on the left-hand-side (input), so here the previously mentioned difference is equal to the size of the output and therefore is not bounded.

We will develop this idea in the context of a time-free semantics derived from coherence spaces. Actually the bounding condition will have to hold for each level of nesting of subcomputations. It then turns out that to obtain a compositional model we need to require a stronger coherence condition on morphisms, which leads us to stratified spaces and stratified cliques. Surprisingly, the stratified model (without the bounding condition) is interesting by itself as it provides a specific model of Elementary Linear Logic (ELL). It bears several similarities with the fibred phase model of [KOS97].

Recently Murawski and Ong have also proposed a games semantics for Light Affine Logic yielding a full completeness result ([MO00]). It should be interesting to try to relate their approach to the present work.

Outline of the paper. In section 2 we recall background on LLL, ELL and coherence spaces, then we present the stratified coherence spaces in section 3 and show that it models ELL. In 4 we define the measured spaces and the subcategory of locally bounded stratified cliques which is our model of LLL. Section 5 is devoted to the syntax of proof-nets, to their semantic interpretation and to the proof of the soundness theorem.

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2 Preliminaries

2.1 Some notations

Let us start with fixing a few notations.

Given two sets E and F we denote by $E + F$ their *disjoint union*.

A *multiset* u on E is a function $u \to \mathbb{N}$. Its *support* is the set $\{x \in E / u(x) \neq 0\}$. A finite multiset is a multiset with finite support. We will handle multisets as sets with repetitions (the number of repetitions of x in u is $u(x)$) and write a finite multiset as $u = [x_1, \ldots, x_n]$. We denote by $n[x]$ the multiset u given by: $u(x) = n$, $u(y) = 0$ for $y \neq x$.

We denote by $\mathcal{P}(E)$ the powerset of E and by $\mathcal{M}(E)$ the set of multisets over E.

Given a function $f : A \to B$ we extend it in the usual way into a function $f^s : \mathcal{P}(A) \to \mathcal{P}(B)$ defined by: $f^s(u) = \{b \in B / \exists a \in A, b = f(a)\}$. Similarly, we define $f^m : \mathcal{M}(A) \to \mathcal{M}(B)$ by: $f^m(u) = v$ with $v(b) = \sum_{a \in A, f(a)=b} u(a)$.

2.2 Light linear logic

Light Linear Logic arises from restrictions on the exponential connectives of linear logic. They still enjoy the contraction $!A \vdash !A \otimes !A$ and weakening $!A \vdash 1$ principles, but:

- the *dereliction* $!A \vdash A$ and *digging* $!A \vdash !!A$ principles are not accepted,
- the modality ! is functorial, that is to say the rule (from $A \vdash B$ deduce $!A \vdash !B$) is valid; it is not multifunctorial though and the principle $!A \otimes !B \vdash !(A \otimes B)$ is not valid;
- an important point is that the equivalence between $!(A \& B)$ and $!A \otimes !B$ is maintained.

To compensate for the lack of dereliction a new modality § is introduced, with principles $!A \vdash \S A$ and $\S A \otimes \S B \vdash \S(A \otimes B)$. Here we will not consider § to be self-dual (as suggested in [KOS97]) and we will denote its dual by $\bar{\S}$.

Girard showed how these principles could be organised into a sequent calculus ([Gir98]) which we recall below. Proof-nets were also introduced; they offer a more convenient syntax to describe the cut-elimination procedure and we will come back to them in section 5.

The *depth* of a proof-net is the maximal nesting of its exponential boxes. In [Gir98] it was shown that given a fixed depth d, a polynomial P of degree depending on d could be given such that the normalisation of any proof-net R of depth d can be performed in less than $P(|R|)$ steps (where $|R|$ measures the size of R). Conversely, the representation theorem states that any polytime function on integers can be represented by an LLL proof of the sequent $1^k; bint \vdash \S^k bint$, where:

- the formula 1^k stands for $!\ldots!1$ with ! repeated k times,

- *bint* is a type for integers in binary representation,
- $\S^k A$ stands for $\S \ldots \S A$ with $\S$ repeated k times.

Let us now give the sequent calculus (for the fragment without quantifiers). Light Linear Logic formulas are defined as Linear Logic formulas, but for the introduction of the new modality $\S$ (and its dual $\bar{\S}$).

The negation is a defined one, by which we mean that $A^\perp$ is used as a notation for the De Morgan dual of A. We also consider *discharged formulas* (denoted as $[A]$) which are expressions in waiting of contraction (they are not proper formulas and will not appear in conclusions).

A *block* is either a discharged formula $[A]$ or a multiset $A_1, \ldots A_n$ of formulas (to be thought of as $A_1 \oplus \ldots \oplus A_n$). A *sequent* is a multiset of blocks, denoted as $\vdash \mathbf{B_1}; \ldots; \mathbf{B_n}$ (to be thought of as $B_1 \wp \ldots \wp B_n$, where B_i is the formula associated to the block $\mathbf{B_i}$).

Identity group

$$\frac{}{\vdash A^\perp; A}\ (\text{Axiom}) \qquad \frac{\vdash A; \Gamma \quad \vdash A^\perp; \Delta}{\vdash \Gamma; \Delta}\ (\text{Cut})$$

Structural group

$$\frac{\vdash \Gamma}{\vdash \Gamma; [A]}\ (\text{M-weakening}) \qquad \frac{\vdash \Gamma; \mathbf{A}}{\vdash \Gamma; \mathbf{A}, B}\ (\text{A-weakening})$$

$$\frac{\vdash \Gamma; [A]; [A]}{\vdash \Gamma; [A]}\ (\text{M-contraction}) \qquad \frac{\vdash \Gamma; \mathbf{A}, B, B}{\vdash \Gamma; \mathbf{A}, B}\ (\text{A-contraction})$$

$$\frac{\vdash \Gamma; [A]}{\vdash \Gamma; ?A}\ (\text{Why not})$$

Logical group

$$\frac{}{\vdash 1}\ (\text{One}) \qquad \frac{\vdash \Gamma}{\vdash \Gamma; \perp}\ (\text{False})$$

$$\frac{\vdash A; \Gamma \quad \vdash B; \Delta}{\vdash A \otimes B; \Gamma; \Delta}\ (\text{Times}) \qquad \frac{\vdash A; B; \Gamma}{\vdash A \wp B; \Gamma}\ (\text{Par})$$

$$\frac{}{\vdash \Gamma; T}\ (\text{True})$$

$$\frac{\vdash \Gamma; A}{\vdash \Gamma; A \oplus B}\ (\text{Left Plus}) \qquad \frac{\vdash \Gamma; A \quad \vdash \Gamma; B}{\vdash \Gamma; A \,\&\, B}\ (\text{With})$$

$$\frac{\vdash \Gamma; B}{\vdash \Gamma; A \oplus B}\ (\text{Right Plus})$$

$$\frac{\vdash B_1, \ldots, B_n; A}{\vdash [B_1]; \ldots; [B_n]; !A} \text{ (Of course)} \qquad \frac{\vdash B_1 | \ldots | B_n; A_1; \ldots; A_m; A}{\vdash [B_1]; \ldots; [B_n]; \bar{\S} A_1; \ldots; \bar{\S} A_m; \S A} \text{ (Neutral)}$$

In the (Neutral) rule, each | is either a "," or ";".

2.3 Elementary linear logic

The system ELL is obtained by considering the same restrictions as before but allowing the principle $!A \otimes !B \vdash !(A \otimes B)$. The ELL ! is multifunctorial. The modality § is not necessary in this system.

Danos and Joinet showed in [DJ] how these requirements are fulfilled by a subsystem of plain LL proofs defined by constraints on the proof-nets. They established the fact that the class of functions characterised is that of Kalmar elementary functions.

2.4 Coherence spaces

We briefly recall the main definitions on coherence spaces (see [Gir95, Gir87] for a complete exposition).

A *coherence space* is a pair $(|X|, \frown_X)$ where $|X|$ is a countable set (the *web* of X, its elements are *points*) and $\frown_X$ is a binary reflexive and symmetric relation on $|X|$. Two elements of $|X|$ that are in this relation are said to be *coherent*. We will then write $x \frown_X y$ or $x \frown y \bmod X$. We write $x \frown_X y$ if $x \frown_X y$ and $x \neq y$ (*strict coherence*). The complement of the relation $\frown_X$ is the *strict incoherence* $\smile_X$ and its reflexive closure is the *incoherence* relation $\asymp_X$.

A *clique* of X is a subset c of $|X|$ whose elements are pairwise coherent. A *multiclique* is a multiset whose support is a clique.

The constructions needed to interpret linear logic formulas are the following ones:

- *negation*: $X^\perp = (|X|, \asymp_X)$,
- *tensor*: $X \otimes Y = (|X| \times |Y|, \frown_{X \otimes Y})$ where:

$$(x, y) \frown (x', y') \bmod X \otimes Y \text{ if}$$

$$x \frown x' \bmod X \text{ and } y \frown y' \bmod Y.$$

- *with*: $X \& Y = (|X| + |Y|, \frown_{X\&Y})$. We denote by $inl : |X| \hookrightarrow |X| + |Y|$ and $inr : |Y| \hookrightarrow |X| + |Y|$ the two canonical injections. Then the relation $\frown_{X\&Y}$ is defined by: for $x, x' \in |X|$, $y, y' \in |Y|$ we have

$$\begin{aligned} inl(x) &\frown inr(y) \bmod X \& Y, \\ inl(x) &\frown inl(x') \bmod X \& Y \text{ if } x \frown x' \bmod X, \\ inr(y) &\frown inr(y') \bmod X \& Y \text{ if } y \frown y' \bmod Y. \end{aligned}$$

Then we set: $X \wp Y = (X^\perp \otimes Y^\perp)^\perp$, $X \oplus Y = (X^\perp \& Y^\perp)^\perp$, $X \multimap Y = X^\perp \wp Y$. It follows that:

$$(x, y) \smile (x', y') \bmod X \multimap Y \text{ iff } (x \frown_X x' \text{ implies } y \smile_Y y').$$

We consider the multiset version of coherence spaces (it will be important in section 4): $|!_m X|$ is the set of finite multicliques on $|X|$ and $u \frown_{!_m X} v$ if $u + v$ is a multiclique.

If f is a clique on $X \multimap Y$, then $!_m f : !_m X \multimap !_m Y$ is defined by:

$$\begin{aligned} !_m f \;=\; & \{([x_1, \ldots, x_n], [y_1, \ldots, y_n]) \text{ s.t.} \\ & \forall i,\ (x_i, y_i) \in f \text{ and } [x_1, \ldots, x_n] \in |!_m X|\} \end{aligned}$$

3 Stratified coherence spaces

3.1 Definitions

We now introduce a stratified version of the coherence semantics. Think of it as a way of managing *partial information* on the points of the web, *ie* on values/computations. From such a point x we want to be able to retrieve successive approximations x^0, x^1, ... that we shall call *appearances*, providing us with increasing information about the computation. For each level of approximation a coherence relation is specified, so that the former compatibility/incompatibility single judgement on two computations is refined into a sequence of judgements, each of them corresponding to a particular level of approximation.

Definition 1 *A* stratified coherence space *(s.c.s.) X is given by a sequence $(X^i, \phi_i)_{i \in \mathbb{N}}$ where:*

- *each X^i is a coherence space $(|X^i|, \frown_{X^i})$,*
- *each ϕ_i is an application from $|X^{i+1}|$ to $|X^i|$,*

- *the sequence is stationary: there exists an integer d such that:*

$$\forall i \geq d, X^i = X^d \text{ and } \phi_i = id_{|X^d|}.$$

Note that ϕ_i must be defined on $|X^{i+1}|$. The least d such that the last condition is satisfied will be called the *depth* of the s.c.s. and denoted by $depth(X)$. Then X^d is called the *main space* of X and X^i is its *appearance at depth* i.

Actually we could have added the following condition to the definition of s.c.s. (*strong surjectivity*):

$$\forall c \text{ finite clique of } X^i, \exists d \text{ clique of } X^{i+1} \text{ s.t. } \phi_i(d) = c.$$

This condition implies that ϕ_i is a surjection (simply take for c a singleton set). It will be satisfied by all the s.c.s. we consider and it is preserved by the constructions we will describe. However as we do not need it for our results, we leave it aside. But one can keep in mind that we can require the ϕ_is to be onto.

For $i \leq d$, the *i-th appearance map* is the application $\pi_i : |X^d| \to |X^i|$ given by:

$$\pi_i = \phi_i \circ \phi_{i+1} \circ \ldots \circ \phi_{d-1}.$$

Whenever there is no risk of confusion, given x in $|X^i|$ and $j \leq i$ we will write simply x^j for $\phi_j \circ \phi_{j-1} \circ \ldots \phi_i(x)$. In particular if $i = d$ we have $x^j = \pi_j(x)$; we call it the *appearance of x at depth j*.

We will also denote by π_i its extension π_i^s (resp. π_i^m) to $\mathcal{P}(|X^d|)$ (resp. $\mathcal{M}(|X^d|)$) as defined in section 2. We do the same for ϕ_i.

We will tend to identify the stratified coherence space X with its main coherence space X^d, the reason for that being that when we consider interpretations of formulas a stratified coherence space can be reconstructed from its main space. In particular, $|X|$ will stand for $|X^d|$.

A convenient way to represent a s.c.s. X is as a forest $\mathcal{F}_X$ displayed by levels:

- the trees have height $(d+1)$, where d is the depth of X; nodes at level $0 \leq i \leq d$ are elements of $|X^i|$;
- there is an edge from b to a if a belongs to a $|X^{i+1}|$, b belongs to $|X^i|$ and we have $\phi_i^X(a) = b$;
- the roots of the trees are the elements of $|X^0|$.

Therefore the leaves of the trees are the elements of $|X^d|$. With this representation, finding the i-th appearance of an element means going down the tree (from leaf to root) until reaching level i. We will see in the sequel that logical constructions include infinite branchings.

If a at level $i+1$ is linked to b at level i we will say that a is an *immediate ascendant* of b. The *ascendance* relation is the transitive closure of this relation.

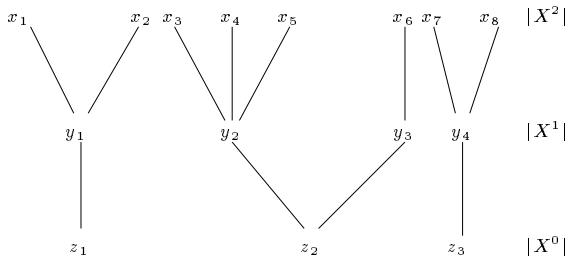

Figure 1: Example of a stratified coherence space as a forest.

The point x_3 has appearances at depth 0 and 1 $x_3^0 = z_2$ and $x_3^1 = y_2$.

Definition 2 *Let X be a stratified coherence space of depth d and c be a subset of $|X^d|$. Then c is a* stratified clique *of X if for all $i \leq d$, $\pi_i(c)$ is a clique of X^i.*

For convenience, we will now write c^i for $\pi_i(c)$. The previous definition can be rephrased in our tree terminology: a stratified clique is a subforest c of $\mathcal{F}_X$ such that for each level i, the nodes of c at level i form a clique of X^i.

Coming back to the example of figure 1: assume $c = \{x_3, x_4, x_6, x_7\}$ is a clique of X^2; then c is a stratified clique of X iff $\{y_2, y_3, y_4\}$ and $\{z_2, z_3\}$ are respectively cliques of X^1 and X^0.

3.2 Constructions on stratified coherence spaces

Let X, Y be two s.c.s.. For $\Box = \otimes, \&$ we define $X \Box Y$ by setting for all i's:

$$\begin{array}{rcl} (X \Box Y)^i & = & X^i \Box Y^i, \\ \phi_i^{X \otimes Y} & = & \phi_i^X \times \phi_i^Y \text{ if } \Box = \otimes, \\ \phi_i^{X \& Y} & = & \phi_i^X + \phi_i^Y \text{ if } \Box = \&. \end{array}$$

We define $X^\perp$ by setting for all i's:

$$(X^\perp)^i = (X^i)^\perp \text{ and } \phi_i^{X^\perp} = \phi_i^X.$$

Then $X \wp Y = (X^\perp \otimes Y^\perp)^\perp$ and $X \oplus Y = (X^\perp \,\&\, Y^\perp)^\perp$.

These constructions are therefore performed level by level. A proper action on levels is used for the interpretation of exponentials. The construction of $!X$ will be done in two steps:

(i) first we define a sequence $Y = (Y^i, \phi_i^Y)_{i \in \mathbb{N}}$, where the ϕ_i^Y are only *partial* maps, and which we call the pre-bang of X,

(ii) then we define the s.c.s. $!X$ by reducing the webs of the spaces.

The intermediary space Y is obtained by applying the usual multiset bang construction on coherent spaces (recalled in 2.4) level by level and then shifting the resulting sequence from one level:

- for $i \geq 0$, $Y^{i+1} = !_m(X^i)$,
- $Y^0 = 1$ (the coherence space with singleton web $\{*\}$).

The applications $\phi_i^Y : |Y^{i+1}| \rightharpoonup |Y^i|$ are defined by:

- if $i \geq 1$, $\phi_i^Y([x_1, \ldots, x_n])$ is defined if for all $1 \leq k \leq n$, $\phi_{i-1}^X(x_k) = x'_k$ is defined and if $[x'_1, \ldots, x'_k]$ belongs to $|!_m(X^{i-1})|$,
- for $i = 0$, ϕ_0^Y is the constant function equal to $*$.

Given i, we say that a point y of $|Y^{i+1}|$ is *visible* if it belongs to the domain of $\phi_0^Y \circ \phi_1^Y \circ \ldots \circ \phi_i^Y$. The only point of $|Y^0|$ is visible. We define $|(!X)^i|$ as the set of visible points of $|Y^i|$; the coherence relation on $(!X)^i$ is the restriction of that of Y^i and $\phi_i^{!X}$ is the restriction of ϕ_i^Y to $|(!X)^{i+1}|$. Then $!X$ is a s.c.s. (whereas Y is not as its maps are only partial).

Note that we have: $depth(!X) = depth(X) + 1$.

The s.c.s. $?X$ is defined as expected by: $?X = (!(X^\perp))^\perp$.

These constructions on s.c.s. naturally give a notion of appearance on formulas; we only need to add that for an atomic formula α we set $\alpha^i = \alpha$ for any i.

Let us now consider an example. We define N as the stratified coherent space of depth 1 given by: $N^0 = 1$; N^1 is the set $\mathbb{N}$ equipped with the discrete coherence relation. We denote the elements of N^1 by $\underline{n}$. The choice of N being of depth 1 is suggested by the representation of the integer type in second order ELL as $\forall\, \alpha(!(\alpha \multimap \alpha) \multimap !(\alpha \multimap \alpha))$.

It follows that $|(!N)^2| = \{k[\underline{n}],\, k, n \in \mathbb{N}\}$ where $k[\underline{n}]$ denotes $[\underline{n}, \ldots, \underline{n}]$ with $\underline{n}$ being repeated k times. We have:

$$k[\underline{n}] \supset l[\underline{m}] \bmod !N^1 \;\text{ iff } n = m \text{ or } k = 0 \text{ or } l = 0.$$

Let $F = !(!N \multimap N) \multimap N$. This s.c.s. has depth 3 and its appearances are given by:

$$\begin{array}{lcccccc} F^3 & = & !(!N \multimap N) & \multimap & N & = & F \\ F^2 & = & !(!1 \multimap N) & \multimap & N & & \\ F^1 & = & !(1 \multimap 1) & \multimap & N & & \\ F^0 & = & 1 & \multimap & 1 & & \end{array}$$

Consider at level 3 the point $x = ([([\underline{0}], \underline{0}), ([\underline{2}], \underline{0})], \underline{1})$. It is visible and we have:

$$\begin{array}{lcccccc} x^2 & = & (& [\,([*], \underline{0}), ([*], \underline{0})\,] & , & \underline{1} &) \\ x^1 & = & (& [(*,*), (*,*)] & , & \underline{1} &) \\ x^0 & = & (& * & , & * &) \end{array}$$

Let us comment on the intuitive meaning of these x^is. The type F is that of a functional taking as argument a numerical function and returning an integer.

- The point x witnesses a computation yielding 1 as result and where the argument function has been tested on values 0 and 2, using each time its own input exactly once and returning value 0 both times.
- At level of appearance 2 (on x^2) we only have the information that the argument function has been tested twice, using its argument only once each time, and returning 0 both times.
- At level 1 (on x^1) we only *see* that the argument function has been tested two times by the functional which gave 1, but we do not know neither on which values, nor what has been returned.

We now give an example of a non-visible point. Let $G = !N \multimap N$ and H be the pre-bang of G. Take on $|G^2|$ the points $y = ([\underline{0}], \underline{1})$ and $z = ([\underline{1}, \underline{1}], \underline{1})$. We have:

$$y \frown z \bmod G^2,$$

so $[y, z]$ belongs to $|H^3| = |!_m(G^2)|$. But besides:

$$\begin{array}{lccc} & ([*], \underline{1}) & \smile & ([*,*], \underline{1}) \quad \bmod G^1, \\ \text{so} & y^1 & \smile & z^1 \quad \bmod G^1. \end{array}$$

Therefore $[y^1, z^1]$ is not a multiclique of G^1 and hence does not belong to $|H^2|$. Hence, ϕ_2^H is not defined on $[y, z]$ and so $[y, z]$ does not belong to $|(!G)^3|$.

By the way this example also illustrates the fact that $x \supset y \bmod X^d$ does *not* imply $x^i \supset y^i \bmod X^i$. The coherence relation at level $i \leq d$ in a s.c.s. can not be directly deduced from that on the main space.

3.3 Composition

Let f and g be stratified cliques of s.c.s. $X^\perp \wp Y$ and $Y^\perp \wp Z$. Let d be the maximal depth of these two s.c.s. and define the composition of f and g as:

$$f;g = \{(x,z) \in |(X^\perp \wp Z)^d|, \exists y \in |Y^d| \text{ s.t.}$$

$$(x,y) \in f \text{ and } (y,z) \in g\}.$$

One easily checks that $h = f;g$ is a stratified clique. Indeed, for any (x,z) of h as f^i and g^i are cliques and $h^i \subseteq f^i;g^i$, we know that h^i is a clique.

This composition is naturally associative and it has the usual relational identities. So we have a category $\mathbb{SCOH}$ of stratified coherence spaces and stratified cliques.

Let us stress now an important property of this composition.

Lemma 1 *If $h = f;g$ is the composition of two elements respectively of $\mathbb{SCOH}[X,Y]$ and $\mathbb{SCOH}[Y,Z]$, and if (a,c) belongs to h^i (where i is inferior to the depth of $X^\perp \wp Z$) then there exists a unique b in $|Y^i|$ such that (a,b) and (b,c) belong respectively to f^i and g^i.*

Proof : The existence of b follows from the definition of the composition. The uniqueness is consequence of the fact that f^i and g^i are cliques. □

This property expresses a kind of independence of each layer: roughly speaking, the i-th appearance of the "interaction" of morphisms f and g with environment (x,z) only depends on the i-th appearances of x and z. It will be crucial when later we define a subcategory of $\mathbb{SCOH}$ modeling LLL.

3.4 A model of ELL

To the previous constructions on objects correspond constructions on morphisms. For the additive and multiplicative constructions they are done in the straightforward way, layer by layer using the constructions of $\mathbb{COH}$. For instance if f_1 and f_2 are respectively elements of $\mathbb{SCOH}[X_1,Y_1]$ and $\mathbb{SCOH}[X_2,Y_2]$ and d is the maximal depth of the s.c.s. then we have a clique $f_1 \otimes f_2$ of $((X_1 \otimes X_2) \multimap (Y_1 \otimes Y_2))^d$ and it gives a stratified clique of $(X_1 \otimes X_2) \multimap (Y_1 \otimes Y_2)$. Similarly for $\wp, \&$ and $\oplus$.

Let us consider now the exponential operations. Let f be an element of $\mathbb{SCOH}[A,B]$ and let d denote the depth. We define $!f$ as:

$$!f = \{(u,v) \in !_m f, \text{ s.t. } u \text{ is visible}\}$$

where $!_m$ is the usual multiset bang functor of coherent spaces (here used in the coherence space X^d). The following lemma ensures that $!f$ is included in $(!A \multimap !B)^{d+1}$ and is a stratified clique:

Lemma 2 *If (u,v) belongs to $!f$ then v is visible.*

It is easy to check that ! is a functor. We then have natural transformations given for objects A and B by $co_A \in \mathbb{SCOH}[!A, !A \otimes !A]$, $w_A \in \mathbb{SCOH}[!A, 1]$ and $m_{A,B} \in \mathbb{SCOH}[!A \otimes !B, !(A \otimes B)]$.

$$\begin{aligned} co_A &= \{(u+v, u, v) \in |!A \multimap !A \otimes !A|\} \\ w_A &= \{([\,], *)\}, \\ m_{A,B} &= \{([a_1, .., a_n], [b_1, \ldots, b_n], u) \in |(!A \otimes !B) \multimap !(A \otimes B)|, \\ &\quad \text{with } u = [(a_1, b_1), \ldots, (a_n, b_n)]\} \end{aligned}$$

Note that the definition of co_A requires that $u + v$ belongs to $|!A|$: there might be some u, v such that $u + v$ is not visible, in which case $(u + v, u, v)$ does not belong to co_A. The same remark holds for $m_{A,B}$.

If f belongs to $\mathbb{SCOH}[A_1 \otimes \ldots A_k, B]$ let us denote (abusively) by $!f$ the morphism of $\mathbb{SCOH}[!A_1 \otimes \ldots !A_k, !B]$ obtained in the obvious way from the functor ! and the natural transformation m.

Lemma 3 *Let $\Gamma = A_1 \otimes \ldots A_k$, f belong to $\mathbb{SCOH}[\Gamma, B]$, g belong to $\mathbb{SCOH}[B, (B_2 \otimes \ldots B_l) \multimap C]$ and let us denote by $!g$ the expected morphism in $\mathbb{SCOH}[!B, (!B_2 \otimes \ldots !B_l) \multimap !C]$, then we have:*

$$\begin{aligned} !f; co_B &= co_\Gamma; (!f \otimes !f) \\ !f; w_B &= w_\Gamma \\ !f; !g &= !(f; g) \end{aligned}$$

Proof :

These equations hold for the coherence spaces model, so here we only have to check that the constraint we added on visibility of points does not raise any problem. Let us just do it for the contraction case as an example, and assuming for simplicity that $\Gamma = A$.

Let us show first that $!f; co_B \subseteq co_A; (!f \otimes !f)$. Take an element of the left-hand side clique; it is of the form $(u, (v_1, v_2))$ where $(u, v_1 + v_2)$ belongs to $!f$. Then u is visible and there exist u_1, u_2 in $!A$ such that $u = u_1 + u_2$ and (u_1, v_1), (u_2, v_2) belong to $!f$. It follows that $(u, (u_1, u_2))$ belongs to co_A and $(u, (v_1, v_2))$ belongs to $co_A; (!f \otimes !f)$.

Conversely, if $(u, (v_1, v_2))$ belongs to $co_A; (!f \otimes !f)$, then there exists (u_1, u_2) in $|!A^d \otimes !A^d|$ such that $(u, (u_1, u_2))$ and $((u_1, u_2), (v_1, v_2))$ are respectively elements of co_A and $(!f \otimes !f)$. Now, to deduce that $(u_1 + u_2, v_1 + v_2)$ belongs to $!f$ we need to know that $u_1 + u_2$ is visible. This is ensured by the

fact that $u = u_1 + u_2$ and $(u, (u_1, u_2))$ belongs to co_A. Hence $(v_1 + v_2)$ is also visible by lemma 2, so $(v_1 + v_2, (v_1, v_2))$ belongs to co_B, $(u, (v_1, v_2))$ belongs to $!f; co_B$ and we are done. □

Lemma 4 *There is in* $\mathbb{SCOH}$ *an isomorphism between* $!(A \& B)$ *and* $!A \otimes !B$.

Proof : The isomorphism holds in $\mathbb{COH}$ and we consider the corresponding cliques in the coherent spaces $!(A \,\&\, B) \multimap !A \otimes !B$ and $!A \otimes !B \multimap !(A \,\&\, B)$. One only has to check that these cliques, once restricted to the main spaces of the s.c.s. associated to the same formulas give stratified cliques and that they yield an isomorphism. It is the case. □

Let us denote respectively by $is_{A,B}$ and $is'_{A,B}$ these morphisms of $\mathbb{SCOH}[!(A\& B), !A \otimes !B]$ and $\mathbb{SCOH}[!A \otimes !B, !(A \,\&\, B)]$.

It follows now that:

Proposition 5 *The category* $\mathbb{SCOH}$ *is a model of Elementary Linear Logic.*

The structure described on the category should already give convincing hints in favour of this statement. We shall not give a complete proof however as in section 5.4 we will prove an analogous result for Light Linear Logic, which is more interesting.

Note that the dereliction $(!A \vdash A)$ and digging $(!A \vdash !!A)$ principles are not valid in this semantics. So $\mathbb{SCOH}$ does not give a model of Linear Logic.

Indeed, considering the equations required and looking at the resulting cliques at depth high enough, we note that the only possible candidates would be:

$$\begin{aligned} der_A &= \{([a], a) \in |!A \multimap A|\}, \\ dig_A &= \{(u_1 + \ldots + u_n, [u_1, \ldots, u_n]) \in |!A \multimap !!A|\} \end{aligned}$$

Let us check that these are not in general stratified cliques. In fact for dig_A it is never the case if $|A|$ has at least one element. Indeed, given a in $|A|$ take $x = ([a], [[a]])$ and $y = ([a, a], [[a, a]])$. Then x and y belong to dig_A and their appearances at depth 1 are: $x^1 = ([a_0], [*])$ and $y^1 = ([a_0, a_0], [*])$ in $|!A^0 \multimap !1|$.

But $([a_0], [*]) \smile ([a_0, a_0], [*]) \bmod (!A^0 \multimap !1)$, and so ${dig_A}^1$ is not a clique.

As to der_A let us assume that : $\exists a, a' \in |A|$ s.t. $a^0 \smile a'^0 \bmod A^0$. Then we take $x = ([a], a)$ and $y = ([a'], a')$. These points belong to der_A and their appearances at depth 0 are: $x^0 = (*, a^0)$, $y^0 = (*, a'^0)$ in $|1 \multimap A^0|$. We have $x \smile y \bmod (1 \multimap A^0)$ and so ${der_A}^0$ is not a clique. This argument can be extended to the case where: $\exists i, \exists a, a' \in |A|$ s.t. $a^i \smile a'^i \bmod A^i$. Consequently der_A is not in general a stratified clique.

3.5 Example of refutation: the iterator

As an example, we will show that the iteration principle is excluded in $\mathbb{SCOH}$. Indeed this principle is not provable in ELL, which is normal since the class of elementary functions is not closed under iteration. Actually, strictly speaking we are not *refuting* the iteration principle as we assume a particular interpretation of integers, choose the candidate clique to interpret the iteration and show that it does not satisfy our stratification condition. But as this candidate clique is the natural one, we consider that this is already a good test for our model.

To interpret the (tally) integers take as before the stratified coherence space N of depth 1 given by: $N^0 = 1$; N^1 is the set $\mathbb{N}$ equipped with the discrete coherence relation.

The clique $Iter_A$ would be defined on:

$$F = A \otimes !(A \multimap A) \otimes N \multimap A,$$

by:

$$Iter_A = \{(a_0, [(a_0, a_1), \ldots, (a_{n-1}, a_n)], \underline{n}, a_n) \in |F|\}.$$

Let us take the example of $A = N$. Take two elements of $Iter_A$ obtained with the same integer n:

$$\begin{aligned} x &= (a_0, [(a_0, a_1), \ldots, (a_{n-1}, a_n)], \underline{n}, a_n) \\ x' &= (a'_0, [(a'_0, a'_1), \ldots, (a'_{n-1}, a'_n)], \underline{n}, a'_n) \end{aligned}$$

Say we chose $a'_0 = a_0$, but $[(a_0, a_1), \ldots, (a_{n-1}, a_n)]$ and $[(a'_0, a'_1), \ldots, (a'_{n-1}, a'_n)]$ (i.e. the iterated functions) such that $a_n \neq a'_n$. For a concrete example one can consider $x = (\underline{0}, [(\underline{0}, \underline{0})], \underline{1}, \underline{0})$ and $x' = (\underline{0}, [(\underline{0}, \underline{1})], \underline{1}, \underline{1})$. Then at depth 1 we have:

$$\begin{aligned} x^1 &= (a_0, n[(*, *)], \underline{n}, a_n) \\ x'^1 &= (a_0, n[(*, *)], \underline{n}, a'_n) \end{aligned}$$

Then as $a_n \smile a'_n \bmod N$, we have: $x^1 \smile x'^1 \bmod F^1$. Hence $Iter_A$ is not a stratified clique.

The following modified iteration principle is valid in $\mathbb{SCOH}$ though and is provable in (second-order) ELL:

$$\begin{aligned} ItEL_A &: !A \otimes !(A \multimap A) \otimes N \multimap !A \\ ItEL_A &= \{(\sum_{j=1}^{m} [a_0^j], \sum_{j=1}^{m} u_j, \underline{n}, \sum_{j=1}^{m} [a_n^j]) \in |F|, \\ &\qquad \text{where } m \in \mathbb{N} \text{ and for } 1 \leq m, \\ &\qquad u_j = [(a_0^j, a_1^j), \ldots, (a_{n-1}^j, a_n^j)]\}. \end{aligned}$$

4 Locally bounded stratified cliques

4.1 Measured coherence spaces

We want to enrich our semantical structure with a quantitative feature, a measuring function. As said in the introduction our goal here is to keep track of the I/O balance within a computation of the program. We mean by I/O balance the difference between the number of times the program has been called and the number of times it has requested an input. This balance will be evaluated at each level.

A *measured* coherence space X is a coherence space given together with a measuring function: $s^X : |X| \to \mathbb{Z}$. We will consider *stratified measured coherence spaces* (s.m.c.s.) (X^i, s_i, ϕ_i) adapted from definition 1. Actually spaces will be measured only starting from level 1 and s_i will denote the measure of X^{i+1}: $s_i : |X^{i+1}| \to \mathbb{Z}$ (one can think of X^0 as having constant measure s_{-1} equal to 0). The third condition is strengthened to:

there exists an integer d such that:

$$\forall i \geq d, X^i = X^d, s_{i-1} = s_{d-1} \text{ and } \phi_i = id_{|X^d|}.$$

The reason for this mismatch on indexes is that what we really care about is the size of the multisets corresponding to exponentials, and in the interpretation of formulas these only appear from level 1. We will use upper-scripts to indicate to which s.m.c.s. the measure function is relative: s_i^X is defined on $|X^{i+1}|$ where X^{i+1} is the level $i+1$ of X. We will omit the upper-script X when there is no risk of confusion and we will sometimes write $s_i(x)$ for $s_i(x^{i+1})$, when $x \in |X^d|$. We introduce the following measure functions for the various constructions:

$$\begin{aligned} s_i^{X^\perp} &= -s_i^X, \\ s_i^{X \,\square\, Y}(x,y) &= s_i^X(x) + s_i^Y(y) \text{ for } \square = \otimes, \wp, \\ s_i^{X \,\square\, Y} &= [s_i^X, s_i^Y] \text{ for } \square = \&, \oplus, \\ s_i^{\mathbf{1}}(*) &= s_i^{\perp}(*) = 0 \end{aligned}$$

The main case is that of the ! construction, for which we set: $s_0^{!X}([x_1, \ldots, x_n]) = n$ and $s_{i+1}^{!X}([x_1, \ldots, x_n]) = \sum_{k=1}^n s_i^X(x_k)$.

Note that by duality we then have

$$s_0^{?X}([x_1, \ldots, x_n]) = -n$$

and

$$s_{i+1}^{?X}([x_1, \ldots, x_n]) = \sum_{k=1}^n s_i^X(x_k).$$

As to the new modality § we interpret its action as a shifting operation: the s.m.c.s. $\S X$ is given by

- for $i \geq 0$, $(\S X)^{i+1} = X^i$,
- $(\S X)^0 = 1$.

Actually, it will be more convenient to denote elements of $|(\S X)^{i+1}|$ as singleton elements over $|X^i|$: $|(\S X)^{i+1}| = \{[x], x \in |X^i|\}$.

The measure functions are defined by: $s_0^{\S X}([x]) = 1$, and $s_{i+1}^{\S X}([x]) = s_i^X(x)$.

Note that § is not self-dual, contrarily to the definition in [Gir98]. We shall denote its dual by $\bar{\S}$. A possibility to have a self-dual § would be to set $s_0^{\S X}$ equal to the constant zero function.

Observe that the previous bang construction can be decomposed using the § operation : $!X = \S !_l X$, where $!_l X$ is obtained by applying $!_m$ level by level and restricting to visible elements.

4.2 Locally bounded stratified cliques

Definition 3 *Let f be a stratified clique on a m.s.c.s. X. We say that f is* locally bounded *if for any i, for any x in f^i the following integers set is bounded:*

$$s_i(\phi_i^{-1}(\{x\}) \cap f^{i+1}).$$

Recall the forest definition of a stratified clique c given in section 3.1. Now, this forest c is a locally bounded stratified clique if to *each node* (at level i) we can associate an integer M such that the size s_i of its immediate ascendants is bounded by M (in absolute value).

Notice that if all branchings of the forest c are finite, then the condition is trivially satisfied and so c is a locally bounded stratified clique. But this will not often be the case for cliques interpreting proofs (in the case of a cut-free proof it would mean that it does not make any use of !-promotion).

Another particular case of locally bounded clique is that where the measures are always zero: $\forall x \in f, \forall i \in \mathbb{N},\ s_i(y) = 0$. This condition is satisfied for instance by the identity maps, so they are bounded. It seems to be a fairly degenerate case, but surprisingly enough it turns out that it would suffice to account for all constructions of LLL ... but the §. Yet, as the § is essential to get a significative expressivity we definitely cannot stick to this particular case.

Lemma 6 *Identities, contraction and weakening morphisms and the morphisms $is_{A,B}$ and $is'_{A,B}$ are locally bounded cliques.*

Proof : One can check that they satisfy the property we just pointed out: the size of all their elements at any depth is always zero. □

Proposition 7 *Locally bounded stratified cliques are preserved by composition.*

Proof : Let f and g belong respectively to $\mathbb{SCOH}[X,Y]$ and $\mathbb{SCOH}[Y,Z]$ and assume they are locally bounded. Let i be an integer, (x,z) be an element of $(f;g)^i$ and let us show that $f;g$ satisfies the property w.r.t. (x,z) at depth i.

There exists a y in $|Y^i|$ such that (x,y) and (y,z) belong respectively to f^i and g^i. Let us denote:

$$\begin{aligned} M_1 &= sup\{\, |s_i(x',y')|,\ (x',y') \in f^{i+1} \text{ s.t.} \\ &\quad \phi_i(x',y') = (x,y)\}, \\ M_2 &= sup\{\, |s_i(y',z')|,\ (y',z') \in g^{i+1} \text{ s.t.} \\ &\quad \phi_i(y',z') = (y,z)\}. \end{aligned}$$

We claim that $(M_1 + M_2)$ provides a suitable bound for $(f;g)$ and (x,z) at depth i. Indeed, let (x',z') be an element of $(f;g)^{i+1}$ such that $\phi_i(x',z') = (x,z)$. There exists a y' such that (x',y') and (y',z') belong respectively to f^{i+1} and g^{i+1}. As $\phi_i(x',z') = (x,z)$ it follows from lemma 1 that $\phi_i(y') = y$ and therefore we have:

$$\phi_i(x',y') = (x,y) \text{ and } \phi_i(y',z') = (y,z).$$

Hence $|s_i(x',y')| \leq M_1$, $|s_i(y',z')| \leq M_2$, and as:

$$s_i(x',z') = s_i(x',y') + s_i(y',z'),$$

we conclude that: $|s_i(x',z')| \leq (M_1 + M_2)$. □

Note that lemma 1 is used in a crucial way in the proof. If we were to define a stratified relational model in the lines of the stratified coherence model and then a notion of locally bounded stratified relations, these morphisms would not be preserved by composition.

4.3 The model of LLL

We denote by $\mathbb{BSCOH}$ the category of s.m.c.s. and locally bounded stratified cliques. We will consider it as a subcategory of $\mathbb{SCOH}$ even if in fact the objects are not the same (we added the measure functions). Let us illustrate now how the quantitative condition rules out monoidality of !.

Lemma 8 *The stratified clique $m_{A,B}$ of $\mathbb{SCOH}[!A \otimes !B, !(A \otimes B)]$ is not locally bounded.*

Indeed for any n we have in $m_{A,B}$ an element of the shape:

$$x = (([a_1, \ldots a_n], [b_1, \ldots b_n]), [(a_1, b_1), \ldots, (a_n, b_n)])$$

(take for instance the a_i's equal to a single element a of $|A|$, and similarly for the b_i's).

We have: $s_0(x) = (-n) + (-n) + n = -n$. So the set$\{s_0(y), y \in m_{A,B}\}$ is not bounded and therefore $m_{A,B}$ is not a locally bounded stratified clique. Consequently, the category $\mathbb{BSCOH}$ is not a model of ELL.

Proposition 9 *The subcategory $\mathbb{BSCOH}$ is stable by $\otimes$, $\wp$, $\oplus$ and $\&$.*

For instance if f, f' belong respectively to $\mathbb{BSCOH}[X, Y]$ and $\mathbb{BSCOH}[X', Y']$ it is easy to check that the stratified clique $f \otimes f'$ of $\mathbb{SCOH}[X \otimes X', Y \otimes Y']$ is also locally bounded .

Proposition 10 *The subcategory $\mathbb{BSCOH}$ is stable by the functor $!$.*

In this proof we once again use the coherence relations.

Proof : Assume f belongs to $\mathbb{BSCOH}[X, Y]$ and let us show that then $!f : !X \multimap !Y$ is locally bounded. Take (u, v) an element of $(!f)^{i+1}$. Then u and v are of the form $u = [x_1, \ldots, x_n]$, $v = [y_1, \ldots, y_n]$, where for $1 \leq k \leq n$, (x_k, y_k) are elements of f^i. As f is locally bounded we know that for each k the set:

$$\{|s_i(x', y')|, (x', y') \in f^{i+1} \text{ and } \phi_i(x', y') = (x_k, y_k)\}$$

is bounded. Denote by M_k its upper bound. Now we claim that $\sum_{i=1}^{n} M_k$ provides a suitable bound at depth $(i + 1)$ for $!f$ and (u, v). Indeed, take (u', v') in $(!f)^{i+2}$ such that $\phi_{i+1}(u', v') = (u, v)$. Then it means that u' and v' are of the form $u' = [x'_1, \ldots, x'_n]$, $v = [y'_1, \ldots, y'_n]$ with:

$$\forall 1 \leq k \leq n,\ \phi_i(x'_k) = x_k, \phi_i(y'_k) = y_k.$$

Besides, as (u', v') belongs to $(!f)^{i+2}$ there is a permutation $\sigma : \{1, \ldots n\} \to \{1, \ldots n\}$ such that:

$$\forall 1 \leq k \leq n,\ (x'_{\sigma(k)}, y'_k) \in f^{i+1}.$$

Note that here we cannot assume that σ is the identity as the indexing of the multiset elements has already been chosen so that $\phi_i(x'_k) = x_k, \phi_i(y'_k) = y_k$.

Let us fix one k. As f^i is a clique and contains (x_k, y_k) and $(x'^i_{\sigma(k)}, y'^i_k) = (x_{\sigma(k)}, y_k)$ we have:

$$(x_k, y_k) \frown (x_{\sigma(k)}, y_k) \bmod (X^i \multimap Y^i),$$

Now, as u is visible we know that $[x_1, \ldots, x_n]$ is a multiclique of X^i, and so in particular:

$$x_k \frown x_{\sigma(k)} \bmod X^i.$$

Therefore $x_k = x_{\sigma(k)}$. So finally $(x'^i_k, y'^i_k) = (x_k, y_k)$ and consequently the bound applies:

$$|s_i(x'_k, y'_k)| \leq M_k.$$

Hence we have:

$$\begin{aligned} |s_{i+1}(u', v')| &= |s^{?X^\perp}_{i+1}(u') + s^{!Y}_{i+1}(v')| \\ &= |\sum_{k=1}^{n} s^{?X^\perp}_i(x'_k) + \sum_{k=1}^{n} s^{Y}_i(y'_k)| \\ &= |\sum_{k=1}^{n} s^{X \multimap Y}_i(x'_k, y'_k)| \leq \sum_{k=1}^{n} M_k. \end{aligned}$$

As to the depth 0 we clearly have a bound equal to zero. □

Let us define the action of § on morphisms. Given f in $\mathbb{SCOH}[X, Y]$ we set:

$$\S f = \{([x], [y]) \in |\S X \multimap \S Y|, (x, y) \in f\}.$$

It is clear that if f is locally bounded , then $\S f$ is locally bounded . This gives us a functor on $\mathbb{BSCOH}$ and we have natural transformations mp and nd defined by:

$$\begin{aligned} mp_{A,B} &= \{(([a], [b]), [(a, b)]) \in \\ &\quad |\S A \otimes \S B \multimap \S(A \otimes B)|\}, \\ nd_A &= \{([a], [a]) \in |\S A \multimap !A|\}. \end{aligned}$$

Theorem 11 (Soundness) *The category $\mathbb{BSCOH}$ is a model of Light Linear Logic.*

Now, to prove this theorem we need first . . . to agree on a syntax for proofs with a suitable normalisation process. Unfortunately the sequent calculus recalled in 2.2 is not satisfactory for this purpose; cut-elimination in it is too complicated to handle. That is why we will establish the result for LLL proof-nets, for which Girard proved the polytime complexity bound in

[Gir98]. Hence we need to postpone the proof to section 5.4, after we have given the definition of proof-nets and their semantic interpretation through *experiments*.

However it is worth noting already that certain extra principles are validated in this semantics, like $\S(A \otimes B) \multimap \S A \otimes \S B$.

4.4 Examples

The ELL iterator. We want to show that the ELL iterator $ItEL$ recalled in section 3.5 is excluded in $\mathbb{BSCOH}$. We need first to set a measure for the space N. As in second-order LLL, tally integers can be given the type $\forall\alpha(!(\alpha \multimap \alpha) \multimap \S(\alpha \multimap \alpha))$, we set:

$$\forall n \in |N|,\ s_0(\underline{n}) \ = \ 1 - n.$$

To simplify we consider elements of $ItEL_A$ of the form:

$$x = (m[a_0], mu, \underline{n}, m[a_n]),$$

(this is the particular case when all the a_0^j's are equal and so are the u_j's). We then have:

$$\begin{aligned} x^0 &= (*,*,*,*), \\ x^1 &= (m[*], mn[*], \underline{n}, m[*]). \end{aligned}$$

Hence:

$$\begin{aligned} s_0(x^1) &= -m - mn - (1-n) + m \\ &= n(1-m) - 1 \end{aligned}$$

So we might fix for instance $m = 2$ and choose elements x with arbitrary large n's, which shows that $ItEL$ is not a locally bounded stratified clique.

Notice that if m is constrained to be 1, this argument does not apply. This is what happens for the LLL iterator, as the right-hand side ! is replaced by a §.

The exponential function. The exponential function $n \to 2^n$ is representable in ELL. Recall that the type of tally integers in ELL is $N_X = !(X \multimap X) \multimap !(X \multimap X)$. Here we are not considering second order quantifiers, so our integer type needs to be instantiated on a particular formula X. Then the exponential function is computed by an ELL proof corresponding to the lambda-term[1]:

$$exp = \lambda x(x)\lambda f\lambda y(f)(f)y.$$

[1] We use the Krivine notation: application of term u to v is denoted by $(u)v$.

The conclusion of the proof (a type for the term) is $N_{A\multimap A} \multimap !N_A$. Danos and Joinet showed in [DJ] how the syntax of LL proof-nets can be used to denote ELL proofs. This is the representation we use to give the proof on figure 2, using as abbreviation $B = !(A \multimap A)$. Note that this syntax should note be confused with that of the LLL proof-nets ([Gir98]) that we will recall in section 5.

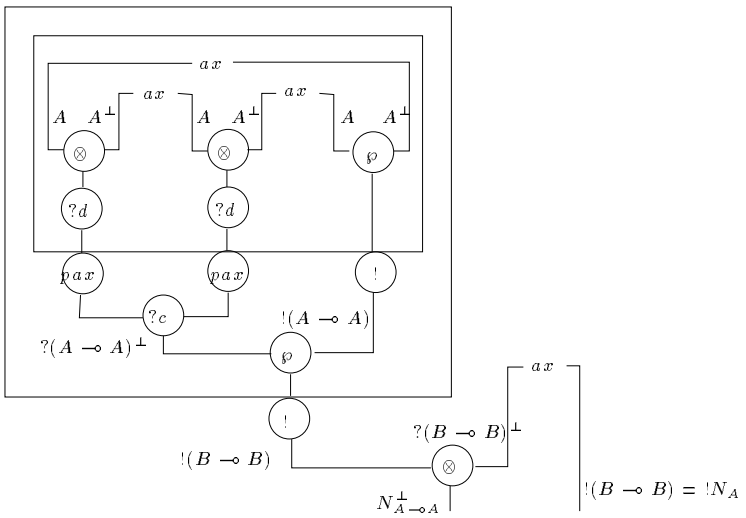

Figure 2: Proof-net R for the term exp.

The interpretation R^* is a stratified clique which is not locally bounded. Indeed, its elements are of the form:

$$x = (([\alpha_1, \ldots, \alpha_n], \beta), \beta) \in |(!(B \multimap B) \multimap !(B \multimap B)) \multimap !(B \multimap B)|,$$

where $[\alpha_1, \ldots, \alpha_n]$ belongs to the interpretation of the external box. We have $s_0(x) = n$, and as R^* has elements with arbitrary large n's the condition is not satisfied at level 0.

Note however that we could remedy to this problem by typing the algorithm as $(!1 \otimes N_{A\multimap A}) \multimap !N_A$. This amounts to add to the proof-net a $\perp$ node (followed by dereliction and promotion) at depth 1 with an adequate jump. Call R' this new proof-net. Now, the element of R'^* are of the form:

$$x = (n[*], ([\alpha_1, \ldots, \alpha_n], \beta), \beta),$$

and the conditions at level 0 are satisfied. Nevertheless they are contradicted at level 1. To see that, let us take arbitrary elements a and β respectively in $|A|$ and $|!(B \multimap B)|$. Then for any m the following point belongs to R'^*:

$$x_m = ([*], ([(2m[(a,a)], m[(a,a)])], \beta), \beta)$$

(using the previous notations, we have fixed $n = 1$ and are now making α_1 vary). For any m we have:

$$x^1_m = ([*], ([(*,*)], \beta^1), \beta^1),$$

which is independent of m, but:

$$x^2_m = ([*], [(2m[(a^0, a^0)], m[(a^0, a^0)])], \beta^2, \beta^2)$$

and $s_1(x^2_m) = 2m - m = m$, hence the contradiction.

5 Proof-nets

5.1 Definition

In this section we recall the syntax of proof-nets for LLL introduced in [Gir98] and their normalisation. For background on additive proof-nets (i.e. proof-nets *without* additive boxes) see [Gir96] or the account in [Lau99].

A proof-structure is a labelled graph built from a certain class of nodes that we will give below. For additive proof-nets we need to handle *weights*: elementary weights are boolean variables (denoted as p, q) and weights (denoted as w, w') are products of elementary weights p and negations of elementary weights $\bar{p}$. If such a product contains a variable p and its negation we replace it by its value 0. A weight w *depends* on p if p or $\bar{p}$ appears in it. Given a valuation ψ of the variables we denote by $\psi(w)$ (or simply by w if there is no ambiguity) the value of w for this valuation. We write $w \leq w'$ if w is a subproduct of w'.

Now, the nodes and boxes are the following ones:

- node with two conclusions (resp. two premises): ax (resp. cut),
- nodes with two premises and one conclusion: $\otimes$, $\&$, $\wp$,
- nodes with one premise and one conclusion: $\oplus_1$ and $\oplus_2$,
- node with one conclusion and no premise: $\bot$, 1,

  (for the edges of all these nodes the typing is the natural one)
- *why not* node $?$: an arbitrary number of premises (possibly zero, which is the case of weakening) labelled by the same discharged formula $[A]$ and one conclusion $?A$,
- additive contraction (denoted as C): at least two premises labelled by the same formula A and one conclusion A,

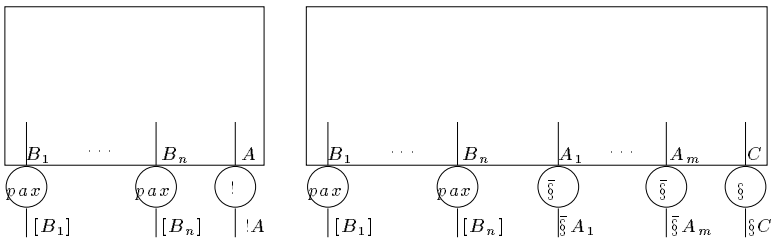

Figure 3: The bang box and paragraph box.

- boxes: see figure 3.

To each & node of the graph we associate a distinct elementary weight, called its *eigen weight*. To each $\perp$ and ? node we associate a non empty set of nodes (not *cut* nodes), called its *jumps*.

A weight $w(A)$ is then associated to each edge A and the following conditions must hold:

- if $A \,\&\, B$ is conclusion of a & node of weight p, then its weight $w = w(A \,\&\, B)$ does not depend on p, and the premises of & have weights $w.p$ and $w.\bar{p}$,
- for an additive contraction node of premises $A^1, \ldots A^n$ and conclusion A: $\sum_{k=1}^{n} w(A^k) = w(A)$ and for $i \neq j$ $w(A^i).w(A^j) = 0$,
- for a *why not* node (or $\perp$ node) of conclusion $?A$, if F is one of its premises or the conclusion of one of the jumps, then $w(F) \leq w(?A)$,
- for all other nodes, all adjacent edges must have same weight,
- if w is the weight of the conclusion of a & node with eigen weight p, and if w' is the weight of an edge depending on p, then $w' \leq w$.
- conditions on boxes: using the notations of figure 3 we should have:
  - for a bang box: $\sum_{i=1}^{n} w(B_k) = w(A)$,
  - for a paragraph box: there is a partition $I_1 \cup \ldots \cup I_k = \{1, \ldots n\}$ such that $\forall j \in \{1, \ldots, k\}$ $\sum_{i \in I_j} w(B_i) = w(C)$; furthermore, $\forall i \in \{1, \ldots m\}$ $w(A_i) = w(C)$.

Let $\Gamma = [\Delta]; A_1; \ldots; A_n$ be an LLL sequent: each A_i is a block $A_i = A_{i1}, \ldots, A_{ik_i}$. An LLL proof-structure of conclusion Γ is a graph satisfying

the previous conditions and such that:

$$\sum_{j=1}^{k_i} w(A_{ij}) = 1.$$

A proof-structure coming from the translation of an LLL proof is called a proof-net. Proof-nets can also be characterised by a correctness criterion ([Gir98]).

To illustrate these definitions we give on figure 4 the proof-net corresponding to the proof of the sequent $\vdash ?A^{\perp}; ?B^{\perp}; !(A \,\&\, B)$, i.e. of the principle $!A \otimes !B \multimap !(A \,\&\, B)$ (note that $?A^{\perp}$ and $?B^{\perp}$ could also have been given weight 1). Another example (a proof-net implementing the predecessor) is given in appendix A.

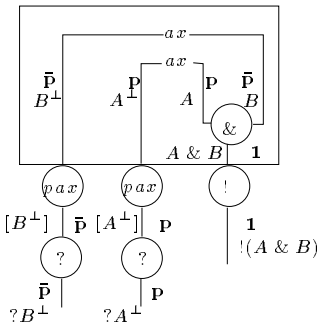

Figure 4: Example of an LLL proof-net.

5.2 Normalisation

Let us recall that the normalisation procedure from [Gir98] is a lazy one: we do not reduce commutative additive cuts and only certain exponential cuts (*special* cuts). A *special* cut[2] is an exponential cut such that if one of the premises is a ? node, then either it is a weakening (it has no premise) or at least one of the premises of this node has weight 1.

- **Contraction**: the elimination of a special contraction cut is carried out in the way described on figure 5 in the case where $w_1 = 1$. In the resulting proof-net R', the arity of each ? node below a discharged formula $[A]$ of $[\Gamma]$ has increased by one: the edge $[A]$ has been replaced

[2]This terminology clashes with that used in the thesis of Regnier, where special cut has another meaning.

by two edges $[A^{(1)}]$ and $[A^{(2)}]$ (this is what we represent by the dashed lines below $[\Gamma^{(1)}]$ and $[\Gamma^{(2)}]$).

- **Weakening**: if $!A$ is cut with a ? node with no premise then the box enclosing Θ is simply erased (the ? nodes which were conclusion of a $[A]$ in $[\Gamma]$ consequently lose one premise).

- **Paragraph**: on figure 6 we give the paragraph reduction step, which amounts to the merging of two boxes.

We do not recall here the other reduction steps, which are the same as for (multiplicative additive) LL proof-nets ([Gir96]).

5.3 Experiments

The theory of experiments was introduced in [Gir87] in order to give a direct interpretation of proof-nets in coherence spaces; it was studied in detail in [TdF00] for LL proof-nets with additive boxes. Here we simply give the modifications needed to handle the LLL proof-nets.

Assume we have chosen an interpretation of atomic formulas in s.m.c.s. Then we have an interpretation for each formula. An *experiment* of the proof-net R is an application e sending each edge A to a multiset over the web (of the main space) $|A|$ (called its *tag*) and satisfying certain conditions we detail below.

We define experiments by induction on the depth of the proof-structure. In the case of a proof-structure of depth 0 (i.e. without any box) we choose a valuation ψ and all edges of weight zero with this valuation are tagged with the empty multiset, the others are tagged with a singleton multiset. Furthermore, the following constraints must hold:

- for an axiom (resp. cut) node of conclusions (resp. premises) A, $A^\perp$ $e(A) = e(A^\perp)$,

- for a $\otimes$ (resp. $\wp$) node: if $e(A) = [a]$, $e(B) = [b]$ then $e(A \otimes B) = [(a, b)]$ (resp. $e(A \wp B) = [(a, b)]$),

- for a & node: $e(A \& B) = e(A) + e(B)$ (note that $e(A) = [\,]$ or $e(B) = [\,]$), and for a $\oplus$ node $e(A_1 \oplus A_2) = e(A_i)$ if A_i is the premise;

- for an additive contraction node: the tag associated to the conclusion is equal to the sum of the tags associated to the premises (at most one of the premises tags is not $[\,]$),

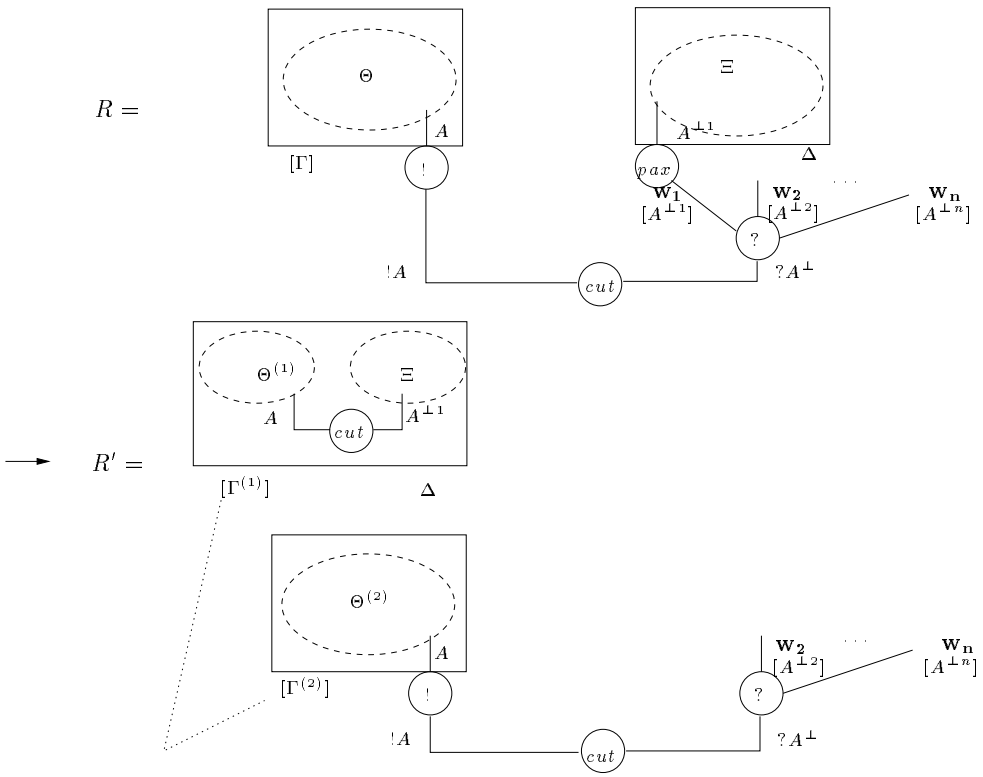

Figure 5: Contraction reduction step.

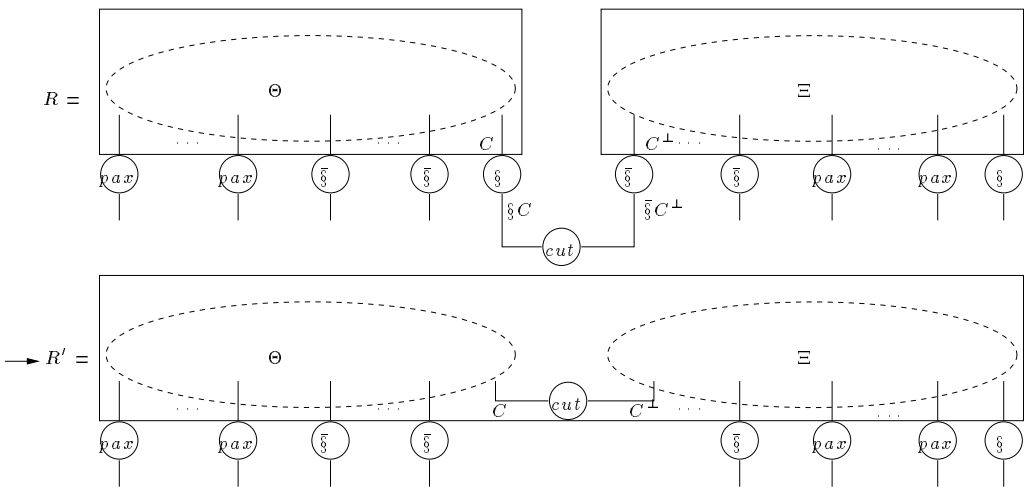

Figure 6: Paragraph reduction step.

- for a contraction node whose premises are tagged by $[u_1], \ldots, [u_n]$, the conclusion $?A$ is tagged by $[u_1 + \ldots + u_n]$, where $(u_1 + \ldots + u_n)$ must be an element of $|?A|$,

- for an edge labelled by 1 or $\bot$ and of weight non zero, the tag is $[*]$.

Now, for an arbitrary proof-structure R we define an experiment in the following way:

- for each !-box at depth 0 and of content S we take $e_1, \ldots e_n$ (possibly 0) experiments of S and tag each edge A of S with $e(A) = e_1(A) + \ldots + e_n(A)$ (empty multiset if $n = 0$); the conclusions of the box are tagged with $e(!C) = [e(C)]$ and $e([B]) = [e(B)]$; $e(C)$ and $e(B)$ should respectively belong to $|!C|$ and $|?B|$;

- for each §-box at depth 0 at depth 0 and of content S we take e' experiment of S, tag each edge A of S with $e(A) = e'(A)$ and the conclusions of the box with $e([B]) = [e(B)]$, $e(\bar{\S}C) = [e(C)]$, $e(\S D) = [e(D)]$.

For convenience when $e(A)$ is a singleton $[x]$ we denote by $\underline{e}(A)$ its content x.

To illustrate the notion of experiment we give an example in appendix B. If e is an experiment of the proof-net R with conclusion $\Gamma = A_1; \ldots; A_n$, then it gives a singleton $[a_i]$ for each block A_i. The *result* of the experiment is the element $(a_1, \ldots, a_n)$ of $|\Gamma|$.

We define the interpretation R^* of the proof-net R as the set of results of all its experiments. Of course this interpretation coincides with the one we could define directly for sequent calculus proofs: if Π is a sequent calculus proof, R is the proof-net associated and Π^* denotes the interpretation of Π (using the categorical constructions introduced in sections 3.4 and 4.3), then $R^* = \Pi^*$. It follows then from the results of sections 4.2 and 4.3 that:

Proposition 12 *If R is an LLL proof-net, then R^* is a locally bounded stratified clique.*

5.4 Soundness

We are now equipped to prove the soundness of $\mathbb{BSCOH}$ for LLL (theorem 11).

Proof : We claim that for any reduction step, if R reduces to R' then $R^* = R'^*$. For multiplicative, additive, axiom and weakening reduction steps

there is no novelty compared to the soundness result holding for linear logic and (usual) coherent spaces, so we do not need to go into more detail.

Now, the case of the § reduction step is simple, so we might as well go on with that. We want to show that if R reduces to R' through this step, then for any experiment of one of these proof-nets, we can give an experiment of the other proof-net with same result. Note that given the modularity of the definition of experiments, we can restrict our attention to the case where R is obtained by a cut between two §-boxes (the § rule is the "last rule" of the proof-net) as on figure 6. Take e an experiment of R and define for R' the experiment e' simply by taking for any edge D the tag $e'(D)$ equal to the tag $e(D)$ of the corresponding edge of R. There is no new condition required for e' and so it is indeed a valid experiment of R', with same result as e. Conversely, any experiment e' of R' yields a corresponding experiment e of R obtained by setting $e(\S C) = [e'(C)]$ and $e(\bar{\S} C^{\perp}) = [e'(C^{\perp})]$ (where $\S C$ and $\bar{\S} C^{\perp}$ are the premises of the cut).

Now, let us consider the contraction reduction step. We will use the notations of figure 5.

Let e be an experiment of R and γ be the result of e: $\gamma = (u_1, \ldots, u_k, \vec{y}, \vec{z})$ where $u_1, \ldots, u_k$ are in $[\Gamma]$, $\vec{y}$ is in Δ and $\vec{z}$ are in the other conclusions Δ' of the proof-net. Then e gives us experiments $e_1, \ldots e_n$ of the box containing Θ with:

$$e_k(A) = e(A^{\perp k}).$$

We define an experiment e' of R' in the following way:

- for C edge of $\Theta^{(1)}$: $e'(C) = e_1(C)$,
- for C edge of Ξ: $e'(C) = e(C)$,
- for C edge of $\Theta^{(2)}$: $e'(C) = \sum_{k=2}^{n} e_k(C)$,
- for other edges C: $e'(C) = e(C)$.

The validity conditions for e' are straightforward, but for the ones dealing with the new contractions on $[\Gamma]$, where there is something to check. By definition of e' we have:

$$\underline{e'}([\Gamma^{(1)}]) + \underline{e'}([\Gamma^{(2)}]) = \underline{e}([\Gamma]),$$

and we know that $\underline{e}([\Gamma])$ belongs to the web of the s.m.c.s. associated to Γ, so the condition for the contraction holds. So do the box conditions.

Conversely, if e' is an experiment of R' with result γ' we define an experiment e of R in the following way:

- for any edge C of Θ: $e(C) = e'(C^{(1)}) + e'(C^{(2)})$,
- for $!A$: $e(!A) = [e'(A^{(1)}) + e'(A^{(2)})]$,
- for $?A^{\perp}$: $e(?A^{\perp}) = e(!A)$,
- for $[A^{\perp 1}]$: $e([A^{\perp 1}]) = [e'(A^{\perp 1})]$,
- for other edges C: $e(C) = e'(C)$.

We need to check that the element of $e(!A)$ belongs to $|!A|$ and that the contraction condition on $e([A^{\perp 1}])$, ...,$e([A^{\perp n}])$ holds.

Let $\mathcal{A}$ denote the box enclosing Θ. First we know that $\underline{e}([\Gamma])$ belongs to $|?\Gamma|$ and as $\mathcal{A}$ is a proof-net we infer that $\underline{e}(!A)$ belongs to $|!A|$. Second, by definition of e we have:

$$e(!A) = e(?A^{\perp}) = [\sum_{k=1}^{n} e([A^{\perp k}])].$$

Therefore $\sum_{k=1}^{n} e([A^{\perp k}])$ belongs to $|?A^{\perp}|$ and so the condition is satisfied. □

6 Conclusion and open questions

In this work we provided a semantical account of the restricted exponentials of Elementary and Light Linear Logic, using the framework of coherence spaces. We showed as an example how the iteration principle failed in our model.

An important question now is whether this interpretation extends to second-order quantifiers. Indeed these are needed in the syntax to get enough expressivity, i.e. to be able to represent all polytime numerical functions.

Another interesting question arising is whether the model itself would offer some complexity bound property: is there a notion of computational complexity for stratified cliques and would then the locally stratified cliques fall in a kind of polynomial class? If this is not the case would there be a further constraint on stratified cliques that could ensure such a tractability property?

Going back to the syntax, we saw that our framework incorporates interpretation of ELL and LLL in a unified setting. What about some possible intermediate systems, capturing complexity classes between PTIME and the elementary class? Terui introduced a subsystem of Light Affine Logic corresponding to the PSPACE class ([Ter00]), and it would be interesting to find a suitable semantic interpretation of this system.

Finally, from a semantical point of view one can also wonder whether a class of functions corresponds to locally bounded stratified cliques in a similar way stable functions correspond to cliques in the set version of coherence spaces. Thomas Ehrhard and Nuno Barreiro characterised the class of functions arising from the multiset coherence model ([BE99]) and maybe their work could be adapted to our stratified setting.

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A Example of LLL proof-net: the predecessor

We give here the proof-net corresponding to the proof computing the predecessor in [Gir98]. The dashed line stands for a jump. For more readability we use as abbreviation $\alpha^\dagger$ for $\alpha \,\&\, \alpha$. The edges with no weight indicated have weight 1.

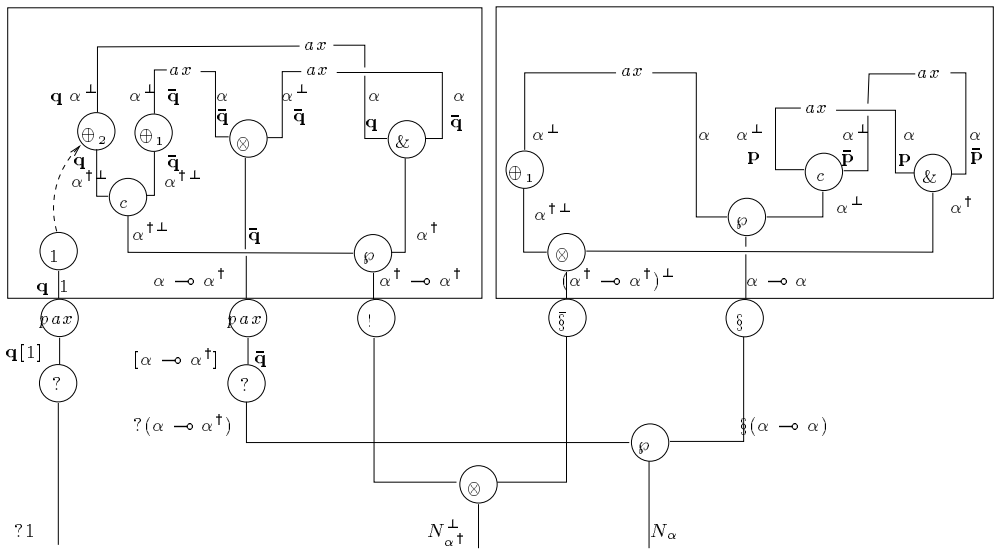

Figure 7: Predecessor.

B Example of experiment

We give in figure 8 an example of experiment for the predecessor proof-net (we omit the formulas and weights to improve the readability). We are considering an interpretation of α by a coherence space with a single point denoted as a (so this is in fact the coherence space 1). We denote the two points of $|1 \,\&\, 1|$ as l and r.

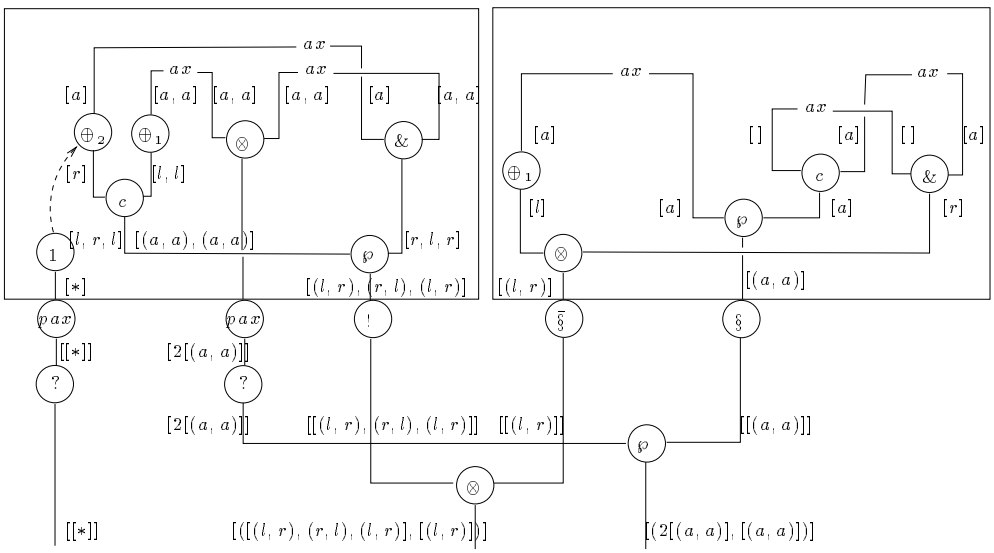

Figure 8: Experiment for the predecessor proof-net.