Errata

The following theorem gives the β-expansion of 1 for any cubic Pisot number.

Theorem 2. *Let β be a cubic Pisot number and let*

$$M_\beta(x) = X^3 - aX^2 - bX - c$$

be its minimal polynomial. Then the beta-expansion of 1 is

- *Case 1 : When $b \geq a$, then $d_\beta(1) = (a+1)(b-1-a)(a+c-b)(b-c)c$.*
- *Case 2: When $0 \leq b \leq a$, if $c > 0$, $d_\beta(1) = abc$, otherwise,*

$$d_\beta(1) = a[(b-1)(c+a)]^\omega.$$

- *Case 3: When $-a < b < 0$, if $b + c \geq 0$, then $d_\beta(1) = (a-1)(a+b)(b+c)c$, otherwise $d_\beta(1) = (a-1)(a+b-1)(a+b+c-1)^\omega$*
- *Case 4: When $b \leq -a$, let k be the integer of $\{2, 3, \ldots, a-2\}$ such that, denoting $e_k = 1 - a + (a-2)/k$, $e_k \leq b + c < e_{k-1}$.*
  - *If $b(k-1) + c(k-2) \leq (k-2) - (k-1)a$, $d_\beta(1) = d_1 \ldots d_{2k+2}$ with*

$$\begin{aligned}
&d_1 = a - 2,\\
&d_{k+2-i} = -(k+3-i) + a(k+2-i) + b(k+1-i) + c(k-i), 3 \leq i \leq k\\
&d_k = -k + ak + b(k-1) + c(k-2)\\
&d_{k+1} = -(k-1) + ak + bk + c(k-1)\\
&d_{k+2} = -(k-2) + a(k-1) + bk + ck\\
&d_{2k+2-i} = -(i-2) + a(i-1) + bi + c(i+1) \quad k \geq 3, 2 \leq i \leq (k-1)\\
&d_{2k+1} = b + 2c \quad \textit{and} \quad d_{2k+2} = c.
\end{aligned}$$

  - *If $b(k-1) + c(k-2) > (k-2) - (k-1)a$, let m be the integer defined by $m = \left\lfloor \frac{1-c}{1-a-b-c} \right\rfloor$.*

    When $m = 1$, $d_\beta(1) = (a-2)(2a+b-2)(2a+2b+c-2)(2a+2b+2c-2)^\omega$.

    When $m > 1$, $d_\beta(1) = d_1 d_2 \ldots d_{m+2} d_{m+3}^\omega$, with

$$\begin{aligned}
&d_1 = a - 2, \quad d_2 = 2a + b - 3,\\
&d_{m+3-i} = 2a + b - 3 + (m+1-i)(a+b+c-1) \quad m \geq 3, 3 \leq i \leq m,\\
&d_{m+1} = 2a + b - 2 + (m-1)(a+b+c-1),\\
&d_{m+2} = a + b - 1 + m(a+b+c-1),\\
&d_{m+3} = (m+1)(a+b+c-1).
\end{aligned}$$

Example 1. When $a \geq b \geq 0$ and $c > 0$, we obtain the only beta-expansion of 1 of length 3.

The smallest Pisot number has $M_\beta = X^3 - X - 1$ as minimal polynomial, it is a simple beta-number and $d_\beta(1) = 10001$.

The positive root β of $M_\beta = X^3 - 3X^2 + 2X - 2$ is a simple beta-number and $d_\beta(1) = 2102$.

The case where $b \leq -a$ shows that from a cubic simple beta-number, we can obtain an arbitrary long beta-expansion of 1. For any integer k greater than or equal to 2, the real root β of the irreducible polynomial $X^3-(k+2)X^2+2kX-k$, is a simple beta number whose integer part is equal to k, and the beta-expansion of 1 has length $2k+2$. For $k=2$, we get $d_\beta(1) = 221002$; for $k=3$, we get $d_\beta(1) = 31310203$.

Example 2. The greatest positive root β of $M_\beta = X^3-2X^2-X+1$ is a beta-number and $d_\beta(1) = 2(01)^\omega$.

If β is the positive root of X^3-5X^2+3X-2 , then $d_\beta(1) = 413^\omega$. When β is the greatest positive root of X^3-5X^2+X+2 , then $d_\beta(1) = 431^\omega$.

For any integer k greater than or equal to 3, the real root β of the irreducible polynomial $X^3-(k+2)X^2+(2k-1)X-(k-1)$, is a beta number whose integer part is equal to k, and the beta-expansion of 1 is eventually periodic of period 1, the length of its preperiod k. For $k=3$, we get $d_\beta(1) = 3302^\omega$; for $k=4$, we get $d_\beta(1) = 42403^\omega$.

Proof. It is known that Pisot numbers are beta-numbers, thus, for any cubic Pisot number β, the beta-expansion of 1 is finite or eventually periodic. In any case, we first compute the associated beta-polynomial P. Next we prove that the sequence $d=(d_i)_{i\geq 1}$ of nonnegative integers obtained from the beta-polynomial satisfy lexicographical order conditions: for all $p \geq 1$, $\sigma^p(d) < d$.

First of all, we recall that, from Theorem 1, a cubic number β, greater than 1 and having

$$M_\beta(X) = X^3 - aX^2 - bX - c$$

as minimal polynomial, is a cubic Pisot number if and only if it both

$$|b-1| < a+c \quad \text{and} \quad (c^2-b) < sgn(c)(1+ac)$$

hold.

Denote by Q the *complementary factor* of the beta-polynomial P defined by $P(X) = M_\beta(X)Q(X)$. As we shall see in what follows, the value of Q depends upon the value of the coefficients of M_β.

Case 1: When $b > a$, as β is a Pisot number, from Theorem **??**, c is a positive integer. In this case, the complementary factor is $Q(X) = X^2-X+1$ and $d_\beta(1) = (a+1)(b-1-a)(a+c-b)(b-c)c$.

Indeed, as $(c^2-b) < sgn(c)(1+ac)$ and $c > 0$, we get $c \leq a+1$. As $|b-1| < a+c$, we get $b-1-a \leq a$ and $0 \leq a-b+c$. From $b > a$, we get that $0 \leq b-a-1$ and, as $c \leq a+1$, that $a-b+c \leq a$. Finally as $0 \leq a-b+c \leq a$, we obtain $0 \leq b-c \leq a$.

Case 2: When $0 \leq b \leq a$, the complementary factor is then $Q(X) = 1$ and the associated beta-polynomial is equal to the minimal polynomial.

If $c > 0$, then $d_\beta(1) = abc$. Indeed, as $(c^2-b) < sgn(c)(1+ac)$, we get $c \leq a$.

If $c < 0$, then $d_\beta(1) = a[(b-1)(a+c)]^\omega$. As $|b-1| < a+c$, we get $b-1 \leq a-2$. As $(c^2-b) < sgn(c)(1+ac)$, we get that $c \geq -a$ and, consequently, $0 \leq c+a \leq a-1$.

Case 3: When $-a < b < 0$, if $b + c \geq 0$ then the complementary factor is $Q(X) = X + 1$ and $d_\beta(1) = (a-1)(a+b)(b+c)c$. Indeed, as $-a < b < 0$, we obtain $1 \leq a + b \leq a - 1$. Since $b + c \geq 0$, c is a positive integer. From $(c^2 - b) < sgn(c)(1 + ac)$, we get that $c \leq a - 1$ and $b + c \leq a - 2$.

If $b + c < 0$, then $Q(X) = 1$ and $d_\beta(1) = (a-1)(a+b-1)(a+b+c-1)^\omega$. As $-a < b < 0$, we get $0 \leq a + b - 1 \leq a - 2$. From $|b-1| < a + c$, we get that $1 \leq a + b + c - 1$ and as $b + c < 0$, we obtain $a + b + c - 1 \leq a - 2$.

Case 4: First of all, since $|b-1| < a+c$, we get $-a+2 \leq b+c$. Moreover as $b \leq -a$, we get $c \geq 2$ and as $(c^2 - b) < sgn(c)(1 + ac)$, we obtain $c \leq a - 2$, thus $b + c \leq -2$. So, there exists an integer k in $\{2, 3, \ldots, a-2\}$, such that, denoting $e_k = 1 - a + (a-2)/k$, $e_k \leq b + c < e_{k-1}$.

When $b(k-1) + c(k-2) \leq (k-2) - (k-1)a$, the complementary factor is

$$Q(X) = \frac{(X^k - 1)(X^{k+1} - 1)}{(X-1)^2}$$

and $d_\beta(1) = d_1 \ldots d_{2k+2}$ with

$$\begin{aligned}
&d_1 = a - 2, \\
&d_{k+2-i} = -(k+3-i) + a(k+2-i) + b(k+1-i) + c(k-i), k \geq 3, 3 \leq i \leq k \\
&d_k = -k + ak + b(k-1) + c(k-2) \\
&d_{k+1} = -(k-1) + ak + bk + c(k-1) \\
&d_{k+2} = -(k-2) + a(k-1) + bk + ck \\
&d_{2k+2-i} = -(i-2) + a(i-1) + bi + c(i+1) \quad k \geq 3, 2 \leq i \leq (k-1) \\
&d_{2k+1} = b + 2c \quad \text{and} \quad d_{2k+2} = c.
\end{aligned}$$

We now verify that the lexicographical order conditions on $d_\beta(1)$ are satisfied.

As $2 \leq c \leq a-2$ and $b + c \leq -2$, we get $d_{2k+1} \leq a - 4$. From $e_k \leq b + c$ and $b(k-1) + c(k-2) \leq (k-2) - (k-1)a$, we get $d_{2k+1} \geq 0$.

For $k \leq 3$ and $2 \leq i \leq k-1$, $d_{2k+2-i} = -(i-2) + a(i-1) + bi + c(i+1)$. As $b + c < e_i$, we get $d_{2k+2-i} < c$. As $-a + 2 \leq b + c$ and $b + 2c \geq 0$, we get $d_{2k+2-i} \geq i$.

As $e_k \leq b + c$, we obtain $d_{k+2} \geq 0$. Since $c \leq a - 2$, $d_{k+1} > d_{k+2}$ and since $b + c \leq -2$, $d_k > d_{k+1}$. Moreover from $b(k-1) + c(k-2) \leq (k-2) - (k-1)a$, we get $d_k \leq a - 2$.

For $k \leq 3$, as $|b-1| < a + c$, we obtain $d_2 < \cdots < d_{k-1}$. As $b + c < e_{k-1}$ and $b + 2c \leq 0$, we get $d_{k-1} < a - 2$. Moreover from $c \leq a - 2$ and $a + b + c - 1 > 0$, we get that $d_2 = 2a + b - 3$ is nonnegative.

All d_i's are smaller than d_1, only d_{2k+2} and d_k can be equal to d_1. Therefore we have to verify that $d_2 \geq d_{k+1}$ when $k \geq 3$ (otherwise $d_2 = d_k$ and $d_k > d_{k+1}$). If $d_k = a - 2$, then $b + c = e_k$, and $d_{k+1} = a - c - 1$. As $a + b + c - 1 > 0$, we obtain $d_{k+1} \leq d_2$. In case of equality, if $k = 3$, then $d_3 = d_k$ and $d_k > d_{k+2}$, otherwise $d_3 > d_2$ and $d_{k+1} > d_{k+2}$, therefore $d_3 > d_{k+2}$.

So lexicographical order conditions are satisfied and $d_1 \ldots d_{2k+2}$ is the beta-expansion of 1.

When $b(k-1) + c(k-2) > (k-2) - (k-1)a$, as $b \leq -a$, we get $k \geq 3$. Let m be the integer defined by $m = \lfloor \frac{1-c}{1-a-b-c} \rfloor$. Note that by definition of m,

$m \leq k-2$ and since $b \leq -a$, $m \geq 1$. In this case, the complementary factor is

$$Q(X) = \sum_{i=0}^{m} X^i.$$

The beta-expansion of 1 is then eventually periodic with period 1, the length of the preperiod is $m+2$.

When $m = 1$, $P(X) = X^4 - (a-1)X^3 - (a+b)X^2 - (b+c)X - c$ and

$$d_\beta(1) = (a-2)(2a+b-2)(2a+2b+c-2)(2a+2b+2c-2)^\omega.$$

Here $d_3 = d_{m+2} = a+b-1+m(a+b+c-1)$ and $d_4 = d_{m+3} = (m+1)(a+b+c-1)$.

When $m > 1$,

$$P(X) = X^{m+3} - (a-1)X^{m+2} - (a+b-1)X^{m+1} - \textstyle\sum_{i=3}^{m}(a+b+c-1)X^i$$
$$-(a+b+c)X^2 - (b+c)X - c$$

and $d_\beta(1) = d_1 d_2 \dots d_{m+2} d_{m+3}^\omega$, with

$$\begin{aligned}
&d_1 = a-2, \quad d_2 = 2a+b-3,\\
&d_{m+3-i} = 2a+b-3+(m+1-i)(a+b+c-1) \quad m \geq 3, 3 \leq i \leq m,\\
&d_{m+1} = 2a+b-2+(m-1)(a+b+c-1),\\
&d_{m+2} = a+b-1+m(a+b+c-1),\\
&d_{m+3} = (m+1)(a+b+c-1).
\end{aligned}$$

In both cases, $d_1 = a-2$. Since $b(k-1)+c(k-2) > (k-2)-(k-1)a$ and $c \leq a-2$, we get $-2a+3 \leq b$. Moreover as $b \leq -a$, $1 \leq d_2 \leq a-2$ when $m=1$, and $0 \leq d_2 \leq a-3$ otherwise. By definition of m, $(m+1)b+mc > m-(m+1)a$, thus $d_{m+2} \geq 0$ and $d_{m+3} \geq c$. Since $e_k \leq b+c < e_{k-1}$ and $m \leq k-2$, we obtain $d_{m+3} \leq a-3$ and $d_{m+2} \leq a-c-3$.

When $m > 1$, since $mb+(m-1)c \leq (m-1)-ma$, we get $d_{m+1} \leq a-2$. As $0 \leq 2a+b-2$ and $a+b+c-1 > 0$, $d_{m+1} > 0$. Moreover as $a+b+c-1 > 0$, one has $d_2 < d_3 < \ldots < d_{m+1}$. Note that, when $m \geq 3$, $d_2 \neq a-2$.

We now study the cases where d_i is not strictly smaller than d_1. When $m=1$, only d_2 may be equal to $a-2$, then $b=-a$ and $d_3 = c-2$, thus $d_3 < d_2$. When $m > 1$, only d_{m+1} may be equal to $a-2$, then $mb = -ma-(m-1)c+(m-1)$, and thus $d_2 - d_{m+2} = a-1-c$ is a positive integer.

We have proved that the lexicographical order conditions on $d_\beta(1)$:

$$d_1 d_2 \dots d_{m+3}^\omega >_{lex} d_i d_{i+1} \dots d_{m+3}^\omega \quad \text{for } 2 \leq i \leq m+3,$$

are satisfied, showing in this way that the announced beta-expansions of 1 are right.

References

[6] D. W. Boyd. On beta expansions for Pisot numbers. *Mathematics of Computation*, 65(214):841–860, 1996.