Star-Height of an $\mathbb{N}$-Rational Series

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Abstract. We prove a new result on $\mathbb{N}$-rational series in one variable. This result gives, under an appropriate hypothesis, a necessary and sufficient condition for an $\mathbb{N}$-rational series to be of star-height 1. The proof uses a theorem of Handelman on integral companion matrices.

1 Introduction

The notion of star-height was introduced by Eggan ([EGG63]), and was later extended to rational formal power series in several commutative variables. In the case of formal languages, we know that the star-height is not bounded ([EGG63]), but is decidable : Hashiguchi has shown the existence of an algorithm for determining it in this case (the proof is given in [HAS89]). On the other hand, we have no similar result for rational formal power series.

The $\mathbb{N}$-rational series in one variable are particular formal power series, which we can define in several equivalent ways. One of these definitions gives the nth coefficient as the number of paths of length n in a graph G. Let $h(G)$ be the maximal number of loops necessarily fitted into each other in the graph G. The star-height is then equal to the minimum among the integers $h(G)$ when G runs over the set of graphs, which represent the same series. In this way, the series of star-height 0 are associated with acyclic graphs and are limited to polynomials. In this sense, the star-height can be interpreted as a measure of the loop complexity of series.

We know, from Soittola's theorem (see [SS78] p.60, [PER92]), that an $\mathbb{N}$-rational series in one variable is at most of star-height 2. This result has been obtained independently by Katayama, Okamoto and Enomoto ([KOE78]) by different methods.

In this paper, we establish a property of series of star-height 1 which is sufficient for deciding the star-height of a particular class of series. Unlike Soittola's theorem, the condition that we give not only concerns the greatest real positive root of the series, but the set of all its real positive roots.

The star-height of the series in one variable is linked with various notions, in particular with that of the representation of the series by a matrix with polynomial entries. Each series of the class which particulary interests us in the following (the $\mathbb{N}$-rational series having a dominating root) can be represented by a matrix over $z\mathbb{N}[z]$ of size 2 (for a general overview of matrices with polynomial entries, one can refer to the survey written by Boyle [BOY94]).

This paper is divided into three sections. In the first one, we recall some definitions about $\mathbb{N}$-rational series and nonnegative integral matrices, which we use for the linear representation of series studied in the following. In the second section, using a result established by Handelman ([HAN92]), we characterize spectral radii of irreducible matrices of size 1 having as only entry a polynomial of $z\mathbb{N}[z]$. They are strictly the algebraic integers greater than 1, of which some integral power satisfies the two following conditions : its modulus strictly exceeds moduli of all of its algebraic conjugates and none of these conjugates is a real positive number. We call these algebraic integers Handelman numbers.

The last section is devoted to the proof of the main result : to have exclusively Handelman numbers as real positive roots is a necessary condition for an $\mathbb{N}$-rational series to be of star-height 1 and a sufficient one for deciding the star-height of an $\mathbb{N}$-rational series having moreover a dominating root. The proof by induction uses the linear representation of a rational series and the characterization of its real positive roots.

2 Preliminaries

To begin with, we mention some definitions and results about rational series (see [BR88], [SS78]). A sequence $r = (r_n)_{n\geq 0}$ of elements of a semiring K is called *K-rational* if there exists a triple (l, M, c) with $l \in K^{1\times n}, M \in K^{n\times n}$ and $c \in K^{n\times 1}$ (n being an integer greater than 1), such that

$$\forall k \geq 0 \qquad r_k = lM^k c.$$

When K *is a field*, a sequence is K-rational if and only if it satisfies a relation of linear recurrence :

$$r_n = \sum_{i=1}^{k} q_i r_{n-i} \quad (n \geq k).$$

The only polynomial

$$Q(z) = z^k - \sum_{i=1}^{k} q_i z^{k-i},$$

where k is minimal, is called the *minimal polynomial* of the sequence r over K and its roots are called the *roots* of r. A polynomial Q has a *dominating root* if it has a real positive root α such that $\alpha > |\beta|$ for any other root β ($\beta \neq \alpha$) of Q. We say that a sequence r has a dominating root if its minimal polynomial does.

A sequence r is equivalently K-rational if and only if its generating function

$$f_r(z) = \sum_{n\geq 0} r_n z^n$$

can be written as :

$$f_r(z) = \frac{P(z)}{Q(z)}$$

where $P, Q \in K[z]$ and $Q(0) = 1$. If P and Q are relatively prime, then the minimal polynomial of r is the *reciprocal polynomial* $\overline{Q}$ of Q,

$$\overline{Q}(z) = z^{degQ} Q(1/z).$$

If K is a subfield of $\mathbb{C}$ and r a K-rational series which is not a polynomial, then for all sufficiently large n

$$r_n = \sum_{i=1}^{k} P_i(n)\alpha_i^n$$

where the numbers α_i are the inverses of the distinct poles of the generating series $f_r(z)$ and each P_i is a polynomial whose coefficients are algebraic over K and whose degree is equal to one less than the multiplicity of the pole $1/\alpha_i$.

When K *is not necessarily a field*, a sequence r is K-rational if and only if its generating function can be obtained from polynomials with coefficients in K, by making use of the three operations of sum, (convolution) product and the unary operation of *star*, defined by, when $P(0) = 0$,

$$P^\star = 1/(1-P).$$

The two definitions of a K-rational sequence coincide when K is a field.

A sequence r is a *merge* of sequences $(s_i)_{0\leq i\leq p-1}$ if, for $0 \leq i \leq p-1$ and for all positive integers n, $r_{pn+i} = s_{i,n}$. In terms of generating functions :

$$f_r(z) = \sum_{i=0}^{p-1} z^i f_{s_i}(z^p).$$

The following result gives a characterization of K_+-rational series, the proof of which is given in [SOI76], [BR88] and [PER92].

Theorem 1 (Soittola). *Let K be a subfield of $\mathbb{R}$. A sequence of positive numbers $r = (r_n)_{n\geq 0}$ is K_+-rational if and only if it is a merge of K-rational sequences having a dominating root.*

Since $\mathbb{Q}_+$ is additionally a Fatou extension of $\mathbb{N}$, then a $\mathbb{Q}_+$-rational series with coefficients in $\mathbb{N}$ is indeed $\mathbb{N}$-rational (see for example [BR88] p.90). Thus, a $\mathbb{Z}$-rational series with positive coefficients is $\mathbb{N}$-rational if and only if it is a merge of $\mathbb{Z}$-rational series having a dominating root.

In what follows, we'll exploit some properties of non-negative matrices (see [GAN59] volume 2 and [MIN88]) to establish results from the linear representation of series.

A square matrix is said to be *reducible* if, by a permutation, it can be reduced to a triangular block form ; it is otherwise said to be *irreducible*.

According to the Perron-Frobenius theorem, such a matrix (irreducible) has a positive simple eigenvalue λ which is its spectral radius. If it has h eigenvalues of maximal modulus, the matrix is said to be *primitive* if $h = 1$, or is otherwise said to be *cyclic* of index h, and h is then called *the index of cyclicity* of the matrix. Moreover, a real number λ is the spectral radius of a primitive integral matrix if and only if it is a Perron number (a *Perron number* is a real algebraic integer exceeding 1 and strictly greater than the modulus of all of its algebraic conjugates). This property was established by Lind in [LIN84].

The following result characterizes the spectral radius of a primitive integral companion matrix. For the proof one can refer to [HAN92].

Theorem 2 (Handelman). *Let P be a monic polynomial of degree n with integral coefficients such that $P(0) \neq 0$. Then the polynomial P has only one real positive root λ, which is a simple root and strictly exceeds the moduli of all other roots of P if and only if, for all sufficiently large integers M and N, there exist positive integers T_i such that the companion matrix of the polynomial C belonging to $P\mathbb{Z}[z]$*

$$C = z^{N+M+n} - \sum_{i=0}^{N+n-1} T_i z^i$$

is primitive and has λ as its spectral radius.

3 Star-height

We define the *star-height* of a rational series S as follows : let $(R_i)_{0 \leq i \leq n}$ be an increasing sequence (with respect to inclusion) of sets, whose union is the set of all rational series. The set of the polynomials is R_0 and each R_i is stable under sum and product ; additionally, if $S \in R_i$ is proper, that is, $S(0) = 0$, then $S^\star \in R_{i+1}$. The least integer n such that $S \in R_n$ is the star-height of the series S.

A consequence of Soittola's theorem is that any $\mathbb{N}$-rational series is at most of star-height 2 ([PER92]). This result has been independently established by Katayama, Okamoto and Enomoto ([KOE78]).

The following result gives a necessary condition for an $\mathbb{N}$-rational series to be of star-height 1 and allows us to decide its star height when it also has a dominating root.

Theorem 3. *Let r be an $\mathbb{N}$-rational series. If r is of star-height 1 then its real positive roots λ_i satisfy the following conditions*

$$\forall i, \exists k_i \text{ such that} \begin{cases} \lambda_i^{k_i} \text{ is a Perron number} \\ \lambda_i^{k_i} \text{ has no positive real algebraic conjugate.} \end{cases}$$

If the series r also has a dominating root, these conditions are sufficient to characterize the series of star-height 1.

In the following, we call real positive numbers which satisfy the conditions of the theorem 3 *Handelman numbers.*

In order to determine the star-height of an $\mathbb{N}$-rational series, we are only interested in the real positive roots of its minimal polynomial. In the proof of the theorem 3, we use the following lemma, which gives a characterization of real positive roots of polynomials of the form : $z^n - \sum_{i=0}^{n-1} a_i z^i$ where the numbers a_i are nonnegative integers and $a_0 \neq 0$.

Lemma 4. *Let λ be a real positive number. Then λ is a root of a polynomial of the form : $z^n - \sum_{i=0}^{n-1} a_i z^i$, where the numbers a_i are nonnegative integers, and $a_0 \neq 0$ if and only if λ is a Handelman number.*

Proof. Let $P(z)$ be the polynomial $z^n - \sum_{i=0}^{n-1} a_i z^i$, where the numbers a_i are nonnegative integers and $a_0 \neq 0$. The polynomial P has only one real positive root λ which, hence, has no real positive number as algebraic conjugate. Additionally, the conjugates λ_j of λ are roots of the polynomial P and thus have a modulus smaller than λ and, in case of equality, λ_j can be written as $\lambda\theta$ where θ is a root of unity. Let k be the common order of all these roots of unity, then λ^k is a Perron number and k is the index of cyclicity of the companion matrix of the polynomial P. Moreover if

$$P(z) = z^n - \sum_{n_1 > \cdots > n_j} a_i z^i - a_0,$$

where $a_{n_1}, a_{n_2}, \ldots, a_{n_j}, a_0$ are the strictly positive numbers of the sequence $(a_i)_{0 \leq i \leq n-1}$, the index k is the greatest common divisor of the differences

$$n - n_1, n_1 - n_2, \ldots, n_k - 0,$$

(the proof of this result, obtained by Frobenius, is given in [MIN88] p.34). Thus all the exponents, that appear in the polynomial P are multiples of k, and we can make the change of variable $y = z^k$. We then obtain the polynomial $y^{n/k} - \sum_{i=0}^{n-1} a_i y^{i/k}$ and we deduce that λ^k has no real positive algebraic conjugate. Hence the only real positive root of the polynomial P is a Handelman number.

Conversely, let λ be a Handelman number. Let P be the minimal polynomial of λ^k ; P is a monic polynomial of degree n with integral coefficients such that $P(0) \neq 0$ since $\lambda^k \neq 0$. Additionally, the polynomial P has only one real positive root λ^k and λ^k is a simple root that strictly exceeds the moduli of the other roots of P. As λ^k is a Perron number, by Handelman's theorem, there exists a polynomial

$$C_0(z) = z^m - \sum_{i=0}^{m-1} t_i z^i$$

where the numbers t_i are nonnegative integers, such that the companion matrix of the polynomial C_0 is primitive. Thus, the polynomial C defined by

$$C(z) = C_0(z^k)$$

is the polynomial that we look for, concluding the proof of the lemma.

Corollary 5. *Let M be a nonzero and irreducible integral matrix, whose spectral radius λ is a Handelman number, and let P be its characteristic polynomial. Then, in the ideal $P\mathbb{Z}[z]$, there exists a polynomial*

$$C(z) = z^n - \sum T_i z^i, \qquad T_i \in \mathbb{N}, \quad C(0) \neq 0$$

whose companion matrix is irreducible and has λ as spectral radius.

Proof. Let P be the characteristic polynomial of the matrix M. Let k be the index of cyclicity of the matrix M, k is the greatest common divisor of the differences between two exponents that successively appear in the polynomial P, thus all these exponents are multiples of k and we can make the change of variable $y = z^k$ in the polynomial P. The polynomial obtained in this way satisfies the conditions of Handelman's theorem the application of which leads to the announced result. The reciprocal change of variable is sufficient to obtain the matrix that we need.

Now we can prove the theorem 3.

4 Proof of the theorem

To begin, we show that an $\mathbb{N}$-rational series is of star-height 1 if and only if it can be written as

$$\sum_{i\in I} \frac{P_i}{\prod_{j\in J}(1-Q_{i_j})} \qquad \forall i,j \quad P_i, Q_{i_j} \in \mathbb{N}[z], \tag{1}$$

where I, J are finite sets and at least one of the polynomials Q_{i_j} is not zero.

Let R_0 be the set of polynomials with coefficients in $\mathbb{N}$, and R_1' the set of $\mathbb{N}$-rational series of star-height 1. The set of series of form (1) is included in R_1', and contains each element $P^\star$ where $P \in R_0$ and $P(0) = 0$.

Additionally, this set is stable for the sum and the (convolution) product, and we thus conclude that the expression (1) characterizes the $\mathbb{N}$-rational series of star-height 1.

Now, the real positive roots of a series of the form (1) are, each, a root of one of the reciprocal polynomials $\overline{(1-Q_{i_j})}$. By the lemma 4, they are algebraic integers, of which an integral power is a Perron number and has no real positive algebraic conjugate, and are hence Handelman numbers.

Remark. We can not use the characterization of $\mathbb{N}$-rational series by a merge of $\mathbb{Z}$-rational series having a dominating root for the converse proof. Actually, the star-height of series that appear in the merge do not give information on the star-height of the initial one. This leads to suppose the existence of a dominating root, an assumption that we use to prove that the auxiliary series introduced in the proof are $\mathbb{N}$-rational.

Example 1. The series defined by the function $f_r(z) = \frac{1}{(1-9z^2)(1-z-z^2)}$ is of star-height 1, and the square matrix of its minimal linear representation is cyclic of index 2. On the other hand, the series of terms of even index given by the function $h(z) = \frac{1-z}{(1-9z)(1-3z+z^2)}$ is of star height 2, because it has two real positive conjugated roots ($\frac{3+\sqrt{5}}{2}$ and $\frac{3-\sqrt{5}}{2}$).

Conversely, let r be an $\mathbb{N}$-rational series having a dominating root. We suppose that its real positive roots are Handelman numbers, that is that they verify the following conditions

$$\forall i, \exists k_i \text{ such that } \begin{cases} \lambda_i^{k_i} \text{ is a Perron number} \\ \lambda_i^{k_i} \text{ has no positive real algebraic conjugate.} \end{cases}$$

The aim is to establish that such a series is of star-height 1.

As r is an $\mathbb{N}$-rational series, it has a representation (l, M, c), where

$$l \in \mathbb{N}^{1\times p}, \quad M \in \mathbb{N}^{p\times p} \text{ and } \quad c \in \mathbb{N}^{p\times 1},$$

such that :

$$\forall k \geq 0 \qquad r_k = lM^k c.$$

Up to a possible permutation, we may suppose that the matrix M is of the form

$$\begin{pmatrix} M_{11} & 0 & \dots & 0 \\ M_{21} & M_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ M_{m1} & \dots & M_{mm-1} & M_{mm} \end{pmatrix}$$

where the square matrices M_{ii} are nonzero and irreducible. Then the characteristic polynomial of the matrix M can be written as the product of M_i, the characteristic polynomials of the matrices M_{ii}. From the corollary 5, we can construct an irreducible integral companion matrix having the same spectral radius as M_{ii} and whose characteristic polynomial C_i belongs to the ideal $M_i\mathbb{Z}[z]$. If the matrix M_{ii} is primitive, the matrix constructed is also primitive.

In these conditions, the generating series $f_r(z)$ of the series r can be written as :

$$f_r(z) = \frac{P(z)}{\prod_{i=1}^{m}(1-Q_i)(z)} \quad ,$$

where $P \in \mathbb{Z}[z]$ and $\forall i$, $1 - Q_i$ is the reciprocal polynomial of C_i and thus $Q_i \in \mathbb{N}[z]$.

We reason by induction on the number m of real positive roots of the series r. Let $\lambda_1, \lambda_2, \dots, \lambda_m$ be these roots arranged in increasing order (a multiple root occurs in this sequence as many time as its multiplicity). We set

$$(1 - Q_1)(z) = 1 - \sum_{j=1}^{q} q_j z^j,$$

and we define the sequence $(s_n)_{n\geq q}$ as :

$$s_n = r_n - \sum_{j=1}^{q} q_j r_{n-j} \qquad (\forall n \geq q).$$

If $m=1$, then $s_n = 0$ for $n \geq h = \max(\deg P + 1, q)$ and

$$r_{n+h} = lM^n c \qquad \forall n \geq 0,$$

where

$$l = \begin{pmatrix} r_{h+q} \\ \vdots \\ r_{h+1} \\ r_h \end{pmatrix}^t, \quad M = \begin{pmatrix} q_1 & 1 & 0 \ldots 0 \\ \vdots & 0 & 1 \ddots \vdots \\ \vdots & \vdots & 0 \ddots 0 \\ q_{q-1} & \vdots & \vdots \ddots 1 \\ q_q & 0 & 0 \ldots 0 \end{pmatrix} \quad \text{and} \quad c = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

We then obtain the following equality

$$\sum_{n \geq h} r_n z^n = \frac{1}{1-Q}(P - (1-Q)\sum_{n=0}^{h-1} r_n z^n),$$

where $1-Q$ is the reciprocal polynomial of the characteristic polynomial of the matrix $M$and thus $Q \in \mathbb{N}[z]$. The expression $(P-(1-Q)\sum_{n=0}^{h-1} r_n z^n)$ is of degree greater than h, so the only remaining terms result from the product $Q\sum_{n=0}^{h-1} r_n z^n$, and the demonitator of the fraction is a polynomial with nonnegative coefficients. Hence, the series r is of star height 1.

We now suppose that the result is true for an $\mathbb{N}$-rational series having a dominating root and exactly $(m-1)$ Handelman numbers as real positive roots.

We shall show that the series s verifies the hypothesis of induction stated above, and we shall then able to prove the annouced result.

First, we must show that there exists a nonnegative integer h such that

$$\forall n \geq h \qquad s_n \geq 0.$$

Since the series r is $\mathbb{N}$-rational, for all sufficiently large n

$$r_n = \sum_{0 \leq i \leq p} A_i(n)\alpha_i^n \quad \forall n \geq n_0,$$

where the numbers α_i are the inverses of the distinct poles of the generating function $f_r(z)$ of r, and each A_i is a polynomial whose degree is equal to one less than the multiplicity of the pole $1/\alpha_i$.

We then use the formula :

$$\begin{aligned} (*) \qquad & n^j\alpha_i^n - q_1(n-1)^j\alpha_i^{n-1} - \cdots - q_q(n-q)^j\alpha_i^{n-q} \\ &= n^j\alpha_i^n(1-\textstyle\sum_{k=1}^q \frac{q_k}{\alpha_i^k}) + jn^{j-1}\alpha_i^n \sum_{k=1}^q \frac{kq_k}{\alpha_i^k} + B(n)\alpha_i^n, \end{aligned}$$

where B is a polynomial of degree $j-2$.

Since $(r_n)_{n \geq n_0}$ is a linear combination of series of the form

$$\sum_{n \geq n_0} n^j \alpha_i^n z^n,$$

the series $\sum_{n\geq n_0+q} s_n z^n$ is a linear combination of series of the form

$$\sum_{n\geq n_0+q} \left[n^j\alpha_i^n - q_1(n-1)^j\alpha_i^{n-1} - \cdots - q_q(n-q)^j\alpha_i^{n-q}\right] z^n.$$

We deduce that

$$\forall n \geq n_0+q, \quad s_n = \sum_{0\leq i\leq p} B_i(n)\alpha_i^n.$$

We set $\alpha_0 = \lambda_m$. Since λ_m is the dominating root of the series r,

$$\forall i \in \{1,2,\ldots,p\}, \quad \lambda_m > |\alpha_i|.$$

Let β be the order of multiplicity of λ_m in the polynomial $\overline{1-Q}$ and let a_0 be the leading coefficient of the polynomial A_0,

$$r_n \sim_{n\to\infty} a_0 n^{\beta-1}\lambda_m^n.$$

since $r_n \geq 0$ and r is not a polynomial, $a_0 > 0$. We can now compute the equivalent of s_n when n tends toward infinity.

If $\lambda_1 \neq \lambda_m$, the leading coefficient of B_0 is, from $(*)$,

$$a_0\lambda_m^n(1 - \sum_{i=1}^{q} \frac{q_i}{\lambda_m^i}) = a_0\lambda_m^n(1-Q_1)(1/\lambda_m).$$

Furthermore, $(1-Q_1)(1/\lambda_m) > 0$ because this polynomial has only one real positive root, $1/\lambda_1$, and $1/\lambda_1 > 1/\lambda_m$. Since

$$s_n \sim_{n\to\infty} a_0\lambda_m^n(1-Q_1)(1/\lambda_m)n^{\beta-1},$$

we can deduce that for all sufficiently large n, $n \geq h$, $s_n \geq 0$.

If $\lambda_1 = \lambda_m$, $\beta = m$, since $1/\lambda_1$ is a root of the polynomial $(1-Q_1)$, the polynomial B_0 is of degree $m-2$ and its leading coefficient then is

$$a_0(m-1)\lambda_m^n \sum_{i=1}^{q} \frac{iq_i}{\lambda_m^i} > 0,$$

by $(*)$. Since

$$s_n \sim_{n\to\infty} a_0(m-1)\lambda_m^n \sum_{i=1}^{q} \frac{iq_i}{\lambda_m^i} n^{m-2},$$

we can deduce that for all sufficiently large n, $n \geq h$, $s_n \geq 0$.

We have shown that the sequence $(s_n)_{n\geq h}$ has nonnegative coefficients. Additionally, the generating function of the series s can be written as

$$f_s(z) = f_r(z)(1-Q_1) - W(z),$$

where W is a polynomial of degree $(q-1)$, so the series s is $\mathbb{Z}$-rational and has λ_m as dominating root. Since the series s has $(m-1)$ Handelman numbers as only real positive roots, by the hypothesis of induction, the series s is $\mathbb{N}$-rational of star-height 1. So, there exists a linear representation (l_s, M_s, c_s) in which the entries of matrices l_s, M_s et c_s are nonnegative integers, such that

$$s_{n+h} = l_s {M_s}^n c_s \quad \forall n \geq 0$$

and matrix M_s is associated with an automaton of cycle rank 1 (that is to say we can obtain from this automaton an acyclic automaton by eliminating a state and all edges incident to this state).

Finally, we consider the triple (L, N, C) consisting of

$$L = \begin{pmatrix} l_s \\ \hline \\ r_{h+q} \\ \vdots \\ r_{h+1} \\ r_h \end{pmatrix}^t , \quad N = \left(\begin{array}{c|cccccc} M_s & c_s & 0 & \dots & \dots & & 0 \\ \hline 0 & q_1 & 1 & 0 & \dots & & 0 \\ \vdots & \vdots & 0 & 1 & \ddots & & \vdots \\ \vdots & \vdots & \vdots & 0 & \ddots & & 0 \\ \vdots & q_{q-1} & \vdots & \vdots & \ddots & & 1 \\ 0 & q_q & 0 & 0 & \dots & & 0 \end{array}\right) \quad \text{and} \quad C = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

We verify by induction on n that, for all $n \geq 0$,

$$LN^n = (l_s {M_s}^n | r_{n+h+q}, \dots, r_{n+q})$$

and we deduce that

$$r_{n+h} = LN^n C \quad \forall n \geq 0.$$

concluding the proof the series r is of star-height 1.

Remark. The decidability of the star-height in the case of $\mathbb{N}$-rational series having a dominating root comes from the possibility of calculating, for each positive real root λ, the maximal integer k such that λ^k verifies the conditions of the theorem 3. The method is similar to that used to decide the $\mathbb{N}$-rationality of a $\mathbb{Z}$-rational series (see [SS78] p.74-75). We construct the polynomial with integral coefficients

$$T(x) = \prod_{i,j} (\beta_i x - \beta_j)$$

where β_i runs over the set of roots having λ as modulus. Next, we determine the integer n_T such that

$$\forall n > n_T, \qquad \phi(n) > \deg(T)$$

where ϕ is Euler's function and we factor the polynomial T as a product of cyclotomic polynomials of order smaller than n_T. The integer k is then the l.c.m. of the orders of the cyclotomic polynomials which appear in the factorization of T.

Remark. The previous theorem 3 allows us to decide the star-height of some $\mathbb{N}$-rational series if no new positive root appears in their writing as a merge of $\mathbb{Z}$-rational series. This happens, for example, when a series has a multiple periodic but unique root ($f_r(z) = \frac{1}{(1-z^n)^q}$).

We may conjecture that to have nothing but Handelman numbers as real positive roots characterizes $\mathbb{N}$-rational series of star-height 1.

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