Towards An Institution for Graph Transformation

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institutions vs. DPO rewr.

- Institutions: abstract model theory
  - sentences, models (and satisfiability)
- DPO rewriting: abstract rewriting formalism
  - rules, rule applications (and replacement)
    - lacking both sentences and models (hence, lacking an entailment system)

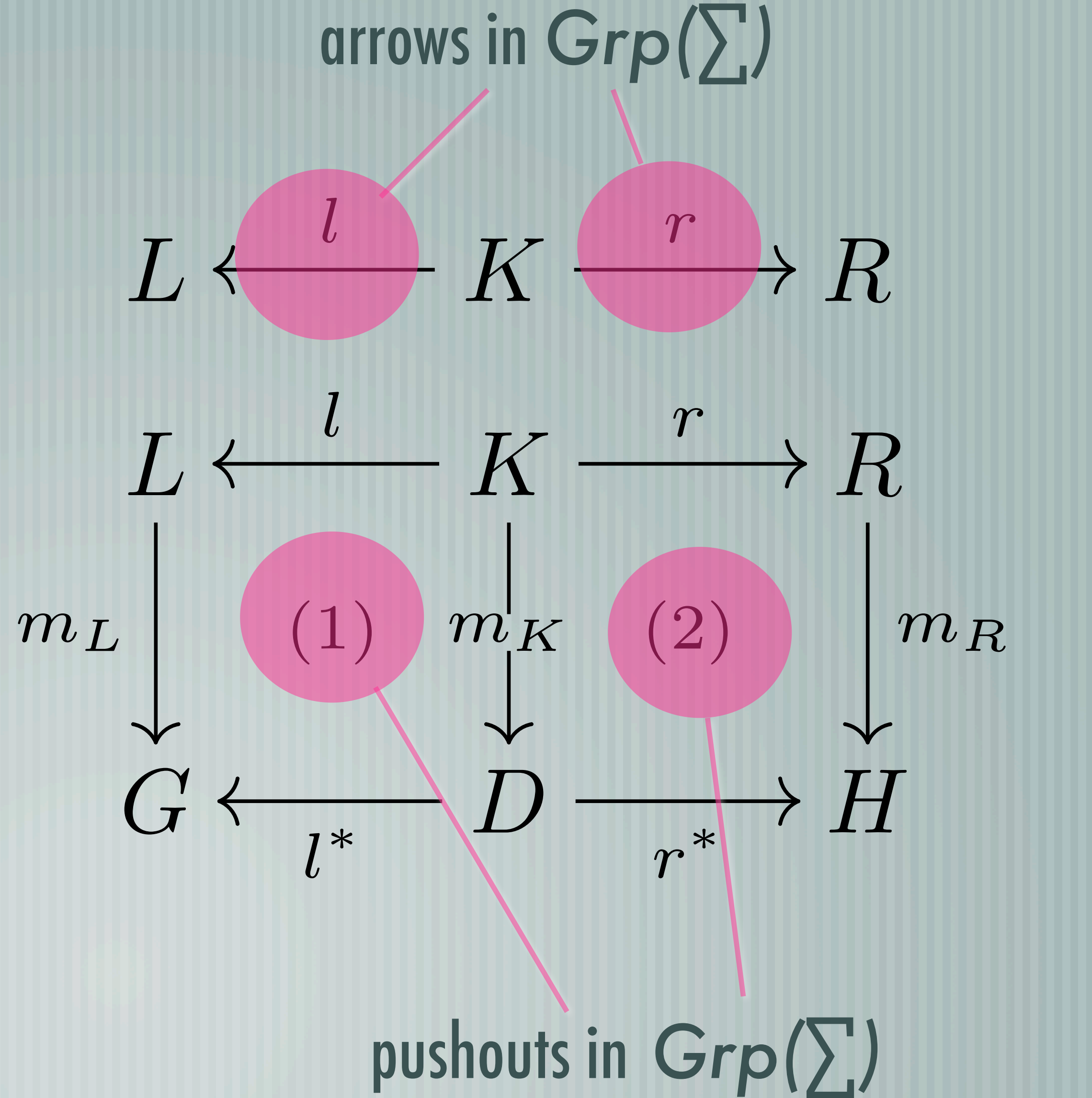
shortly, the DPO approach

Grp(∑) = Grp↓∑

a rule

a derivation step

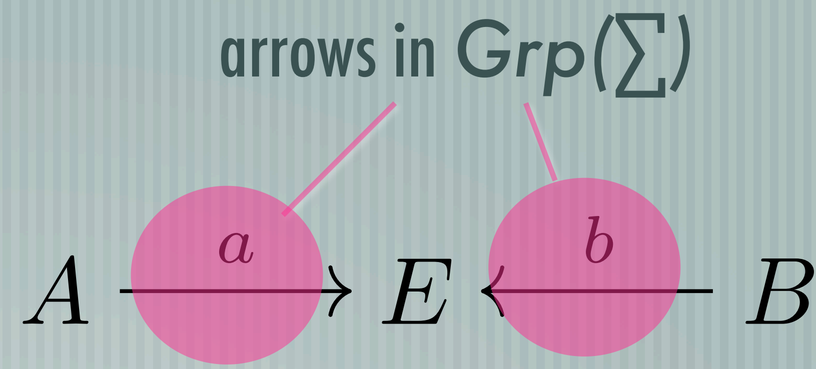
set of theoretical tools (concurrency, mostly) [holding for adhesive cats]



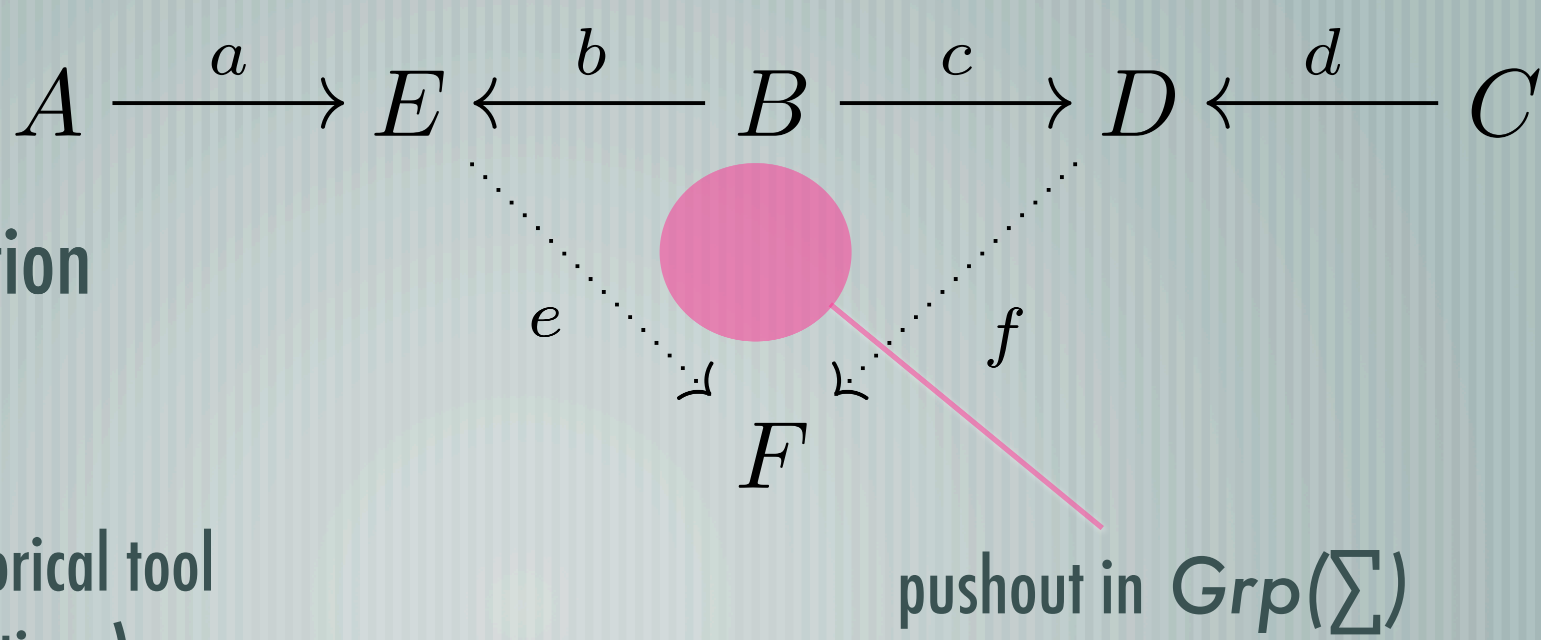
the CoSpan (bi-)category

[holding for any cocomplete cat]

an arrow

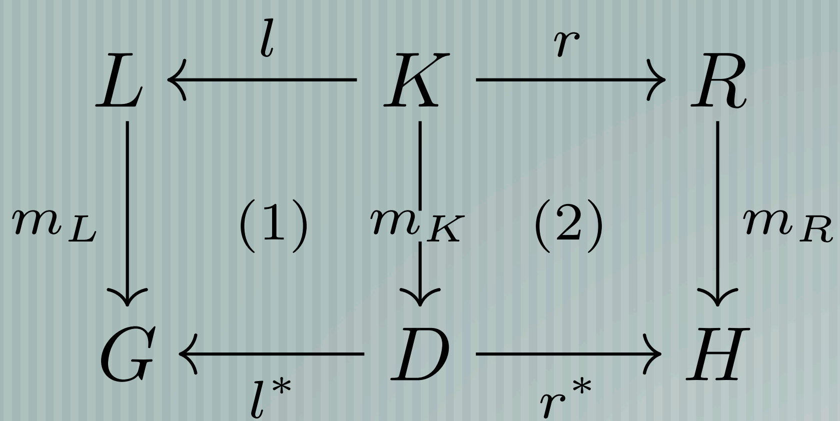


composition



powerful categorical tool
(cats of relations)

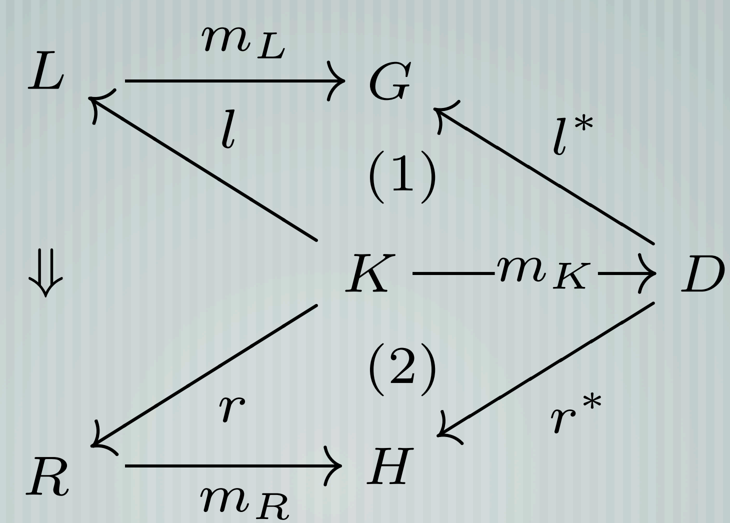
connecting with the DPO



a rule

a derivation step

operational vs.
induction-based

a “cell” $\emptyset$ $\Downarrow$  $\emptyset$

“whiskering”

DPO vs. CoSpans

- (A factorized sub-category of) Cospans over graphs (typed over $\sum$) form the free compact-closed category built from $\sum$ (with operators as basic arrows)
- The DPO approach is operational: search for the match, build the PO complement...
- The free construction (concretely, via cospans) is algebraic: inductive closure of a set of basic rules

first step: inductive sentences

- *DGSTh*(∑) is the self-dual, free symmetric (strict) monoidal category equipped with symmetric monoidal transformations

$$\nabla_a : a \to a \otimes a \qquad\qquad !_a : a \to e$$

- (intuitively representing pairing tuple <*x*,*x*> and empty tuple)
- plus two additional laws (relating transfs. and their dual)

main correspondence result

- I) (Isomorphic classes of) Cospans over graphs (typed on ∑) and sets of nodes as objects *1-1 correspond to* arrows in *DGSTh*(∑) (indeed, a categorical equivalence)
  - You abstract the identity of nodes not in the interface
  - ...but this way graphs get a “standard” notion of sentence
- II) The preorder on arrows obtained by replacing each DPO rule with an order on graphs *1-1 corresponds to* DPO rewrites
  - This way DPO rewriting gets an entailment system

very shortly, institutions

- A category *Sign* of specifications
- a functor $Sen\colon Sign \to Set$ for sentences
- a functor $Mod\colon Sign \to Cat^{op}$ for models
- a (satisfiability) relation $\models_\Sigma$ on $|Mod(\Sigma)| \times Sen(\Sigma)$
- a coherence axiom $\forall \phi : \Sigma \to \Sigma', e \in Sen(\Sigma), M' \in Mod(\Sigma')$

$$M' \models_{\Sigma'} Sen(\phi)(e) \Leftrightarrow Mod(\phi)(M') \models_\Sigma e$$

institution for algebraic specs.

$$Sen(\phi):(s,t)\rightarrow(\overline{\phi}(s),\overline{\phi}(t))$$

Sen

$$\phi:(\Sigma,E)\rightarrow(\Sigma',E')$$

$$\phi : \Sigma \hookrightarrow \Sigma'$$

$$\phi(E) \subseteq E'$$

Mod

$$Mod(\phi):\langle A,\iota_{\Sigma'}\rangle\rightarrow\langle A,\iota_{\Sigma'}\circ\phi\rangle$$

institution for algebraic specs.

$$Sen(\phi){:}(s,t)\rightarrow(\overline{\phi}(s),\overline{\phi}(t))$$

$$[s,t] \rightarrow [\overline{\phi}(s),\overline{\phi}(t)]$$

Sen

$$\phi{:}(\Sigma,E,R)\rightarrow(\Sigma',E',R')$$

$$\phi : \Sigma \hookrightarrow \Sigma'$$

$$\phi(E) \subseteq E'$$

$$\phi(R) \subseteq R'$$

Mod

$$Mod(\phi){:}\langle A,\iota_{\Sigma'},\leq_A\rangle\rightarrow\langle A,\iota_{\Sigma'}\circ\phi,\leq_A\rangle$$

the easy way out...

- exploit the categorical laws...
  - sentences as pairs of arrows in $DGSTh(\Sigma)$ (same homs.)
  - models as dgs-monoidal categories
    - obvious satisfiability
    - reductions via order enrichment
- unsatisfactory: looking for a "concrete" model characterisation, in terms of "classical" algebraic models (algebras for specs.)

a functorial detour

- The algebraic theory *Th*(∑) is concretely defined as
  - lists of vars as objects, (tuples of) typed terms as arrows
  - term substitution as composition
- (the theory is also the free cartesian category over ∑)
- Algebras over ∑ and axioms in E as functors

$$M \in [Th(\Sigma) \to Set]^{\times}_{E}$$

- product and axioms preserving (homs as natural transfs.)

a functorial detour, II

- PreAlgebras as rule-preserving functors

$$M \in [Th(\Sigma) \to Pre]^{\times}_{E,R}$$

$$s \to t \in R \Leftrightarrow \forall X.\, M(s) \leq M(t)$$

  - (still homomorphisms as natural transformations)

- How to generalize? Note that functors

$$M \in [Th(\Sigma) \to Rel]^{\times}_{E}$$

- still define algebras!!

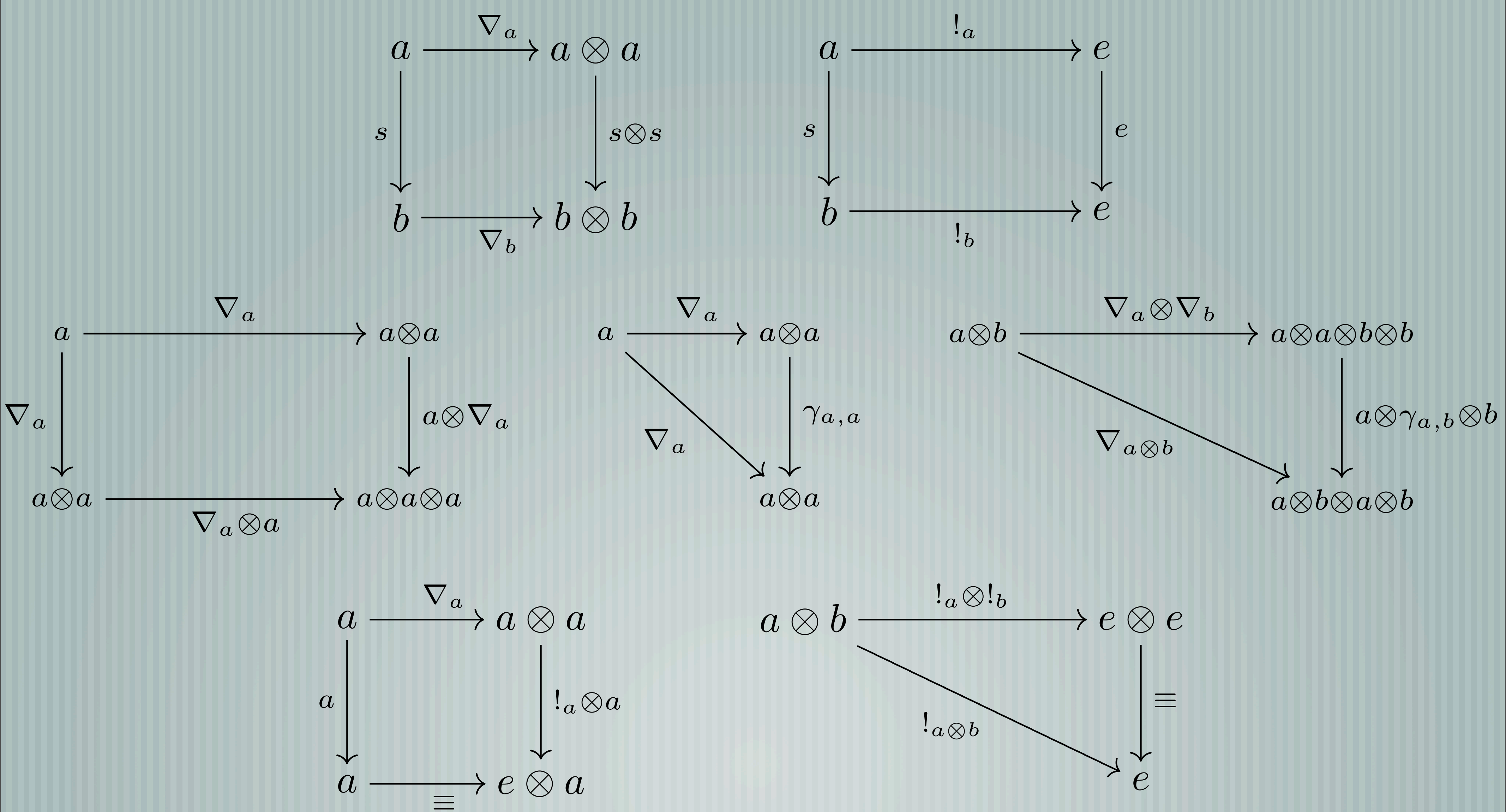
alternative take on *Th*(∑)

- *Th*(∑) is the free symmetric (strict) monoidal category equipped with symmetric monoidal natural transformations

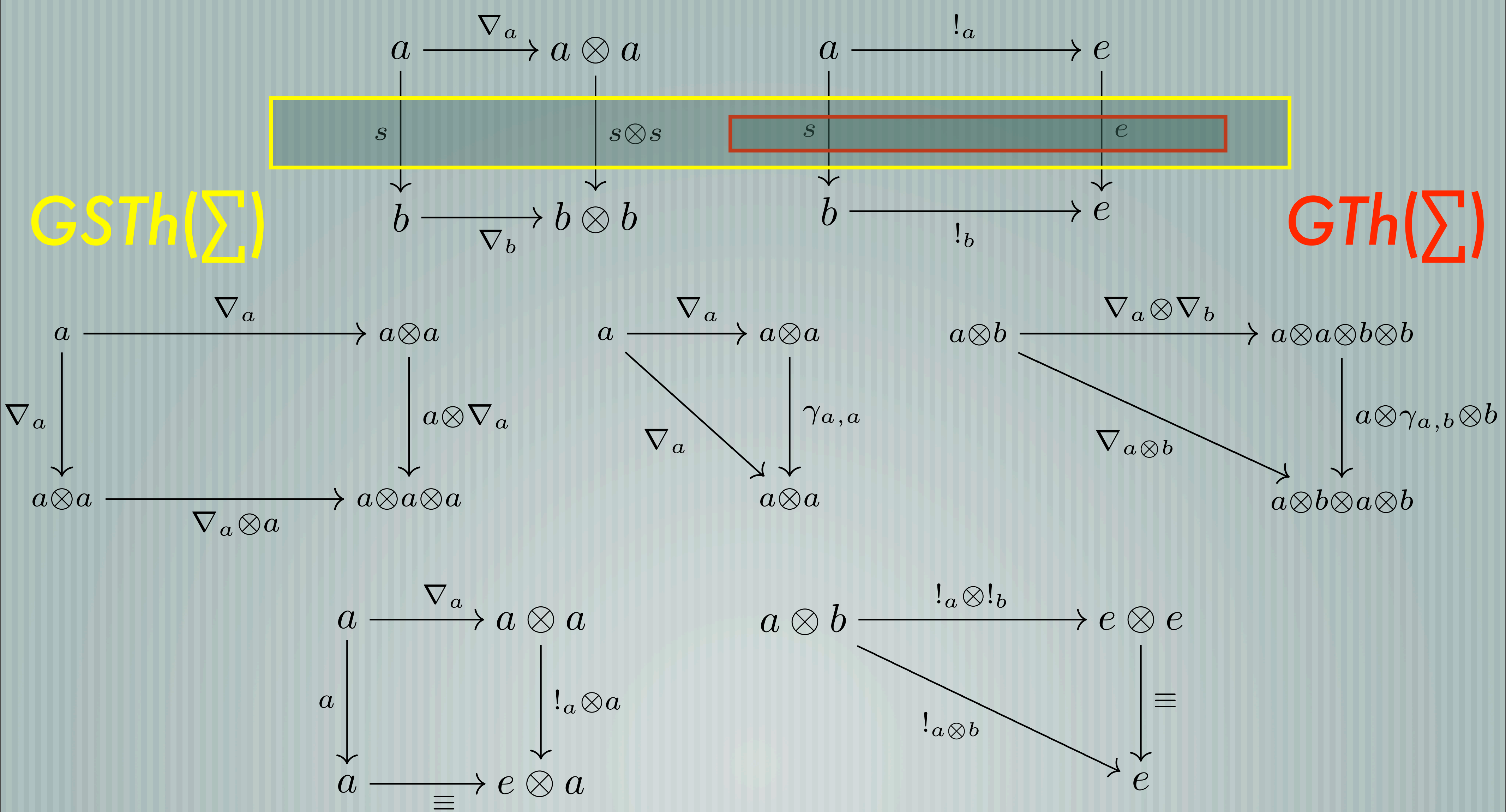
$$\nabla_a : a \to a \otimes a \qquad\qquad !_a : a \to e$$

- (intuitively representing pairing tuple <*x*,*x*> and empty tuple)

explicit definition of a theory



two alternative takes



another alternative take

- *DGSTh*(∑) as self-dual *GSTh*(∑) satisfying

$$
\begin{array}{ccc}
a \otimes a & \xrightarrow{\nabla_a^{op}} & a \\
{\scriptstyle \nabla_a \otimes a}\downarrow & & \downarrow{\scriptstyle \nabla_a} \\
a \otimes a \otimes a & \xrightarrow[a \otimes \nabla_a^{op}]{} & a \otimes a
\end{array}
\qquad\qquad
\begin{array}{ccc}
a & \xrightarrow{\nabla_a} & a \otimes a \\
 & {\scriptstyle a}\searrow & \downarrow{\scriptstyle \nabla_a^{op}} \\
 & & a
\end{array}
$$

some characterization results

- arrows in *DGSTh*(∑) are (isomorphic classes of) cospans of graphs (typed over ∑)
- arrows in *GSTh*(∑) are (isomorphic classes of) cospans of term graphs (typed over ∑)
- arrows in *GTh*(∑) are conditioned terms s | D (over ∑)
  - s a term (the functional)
  - D a sub-term closed set of terms (the domain restriction)

functorial characterizations

- Partial algebras with ⊥-preserving operators, tight homomorphisms and conditioned Kleene (in)equations
- Multialgebras with tight point-to-set operators, tight point-to-point homomorphisms and “term graph” (in)equations
- Multialgebras with tight point-to-set operators, tight point-to-point homomorphisms and “graph” (in)equations

$$[GTh(\Sigma) \to Set_\perp]^\times_E \qquad [DGSTh(\Sigma) \to 2^{Set}]^\times_E$$

$$[GSTh(\Sigma) \to 2^{Set}]^\times_E$$

back to institutions

$$Th(\phi)^2\colon Th(\Sigma)^2 \to Th(\Sigma')^2$$

Sen

$$\phi\colon(\Sigma,E)\to(\Sigma',E')$$

Mod

$$Th(\phi);\_\colon [Th(\Sigma')\to Set]^{\times}_{E'} \to [Th(\Sigma)\to Set]^{\times}_{E}$$

or

$$[GTh(\Sigma) \to Set_{\perp}]^{\times}_{E}$$

or

$$[(D)GSTh(\Sigma) \to 2^{Set}]^{\times}_{E}$$

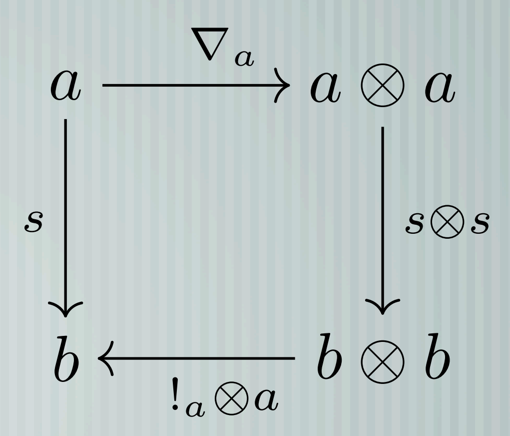
on entailment systems

- Claim: complete entailment system for partial algebras

$$\frac{s \mid D_s \equiv t \mid D_t}{s \mid D_s \cup D \equiv t \mid D_t \cup D} \qquad \frac{u_i \mid D_u \qquad (s \mid D_s, t \mid D_t) \in E}{s[\overline{u}/\overline{x}] \mid D_s[\overline{u}/\overline{x}] \cup D_u \equiv t[\overline{u}/\overline{x}] \mid D_t[\overline{u}/\overline{x}] \cup D_u}$$

- Conjecture: complete entailment system for multi-algebras

GSTh(∑) plus



back to insts. on preorders

$$\phi:(\Sigma,E,R)\to(\Sigma',E',R')$$

Sen: $2Th(\phi)^2: 2Th(\Sigma)^2 \to 2Th(\Sigma')^2$

Mod: $Th(\phi);\_: [Th(\Sigma')\to Set]^{\times}_{E',R'} \to [Th(\Sigma)\to Set]^{\times}_{E,R}$

or

$$[GTh(\Sigma) \to Pre_{\bot}]^{\times}_{E,R}$$

NOT! the bottom of the preorder

Smyth power-domain $X \leq Y \Leftrightarrow \forall y \in Y. \exists x \in X. x \leq y$

or $[(D)GSTh(\Sigma) \to 2^{Pre}]^{\times}_{E,R}$

preliminary conclusions

- uniform presentation of institutions for the DPO rewriting formalism over various graph-like structures
- sound and complete “abstract” entailment systems
- sound (possibly complete) “concrete” entailment systems

to be addressed...

- completeness for the entailment system
  - (rewriting) interpretation for up-to garbage law
- tackling hyper-graphs and hyper-signature
  - (singular vs plural) interpretation for hyper-operators
- considering cospans of adhesive categories
  - free construction for suitable algebraic varieties