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Many-sorted Universal Algebra: Some Technical Nuances

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Algebraic Specifications

Some basic concepts and facts:

- algebras
- equations
- equationally definable classes
  - Birkhoff variety theorem
- equational calculus
  - soundness & completeness
- modularisation and compositionality
  - amalgamation
  - interpolation

classical single-sorted algebra
vs.
many-sorted algebra

Quickly through the basics

Algebraic signature:

$$\Sigma = (S, \Omega = \langle \Omega_{w,s} \rangle_{w \in S^*, s \in S})$$

the class of Σ-algebras: $\mathbf{Alg}(\Sigma)$

Σ-*algebra*:

$$A = (|A|, \langle f_A \rangle_{f \in \Omega})$$

$|A| = \langle |A|_s \rangle_{s \in S}$ and $f_A \colon |A|_{s_1} \times \ldots \times |A|_{s_n} \to |A|_s$, for $f \colon s_1 \times \ldots \times s_n \to s$.

And then:

- Σ-*subalgebra* $A_{sub} \subseteq A$ ...
- Σ-*homomorphism* $h \colon A \to B$ ...
- Σ-*congruence* $\equiv \subseteq |A| \times |A|$ ...
- *quotient algebra* $A/\!\equiv$ ...
- *product of* $\langle A_i \rangle_{i \in \mathcal{I}}$, $\prod_{i \in \mathcal{I}} A_i$ ...
- *terms* $t \in |T_\Sigma(X)|$ ...
- *term algebra* $T_\Sigma(X)$ ...
- *term evaluation*: $t_A(v) \in |A|_s$ for $t \in |T_\Sigma(X)|_s$, $v \colon X \to |A|$ ...
- ...

Equations

Equation:

$$\forall X . t = t'$$

where: X is a finite set of variables, and $t, t' \in |T_\Sigma(X)|_s$ are terms of a common sort.

Satisfaction relation:

$$A \models \forall X . t = t'$$

when for all $v\colon X \to |A|$, $t_A(v) = t'_A(v)$.

Models of a set of equations:

$$Mod(\Phi) = \{A \in \mathbf{Alg}(\Sigma) \mid A \models \Phi\}$$

Semantic entailment:

$$\Phi \models \varphi$$

φ is a *semantic consequence* of Σ-equations Φ if $A \models \varphi$ for all $A \in Mod(\Phi)$.

Birkhoff's Variety Theorem

$\mathcal{V} \subseteq \mathbf{Alg}(\Sigma)$is a *variety* if $\mathcal{V}$ is closed under products, subalgebras and homomorphic images:

$$\boxed{\mathcal{V} = \mathcal{HSP}(\mathcal{V})}$$

Fact: *A class $\mathcal{V} \subseteq \mathbf{Alg}(\Sigma)$ of Σ-algebras is equationally definable (that is, $\mathcal{V} = Mod(\Phi)$ for some set Φ of Σ-equations) if and only if $\mathcal{V}$ is a variety.*

$$\boxed{\mathcal{V} = \mathcal{HSP}(\mathcal{V}) \text{ iff } \mathcal{V} = Mod(EQ(\mathcal{V}))}$$

BTW: reachable initial/free models exist by "only if"

the equational theory of $\mathcal{V}$: $EQ(\mathcal{V})$

Birkhoff's Variety Theorem

Birkhoff's Variety Theorem essentially holds;

the standard proof essentially carries over

BUT:

One of the following additional assumptions is needed:

- only algebras with no carriers empty are considered;
- the set of sorts in the signature is finite;
- there may be infinitely many variables named in equations.

Counterexample: *Consider a signature with no operations and an infinite set of sorts. Let $\mathcal{V}$ be the class of algebras with finitely many sorts with non-empty carriers, or with all carriers containing at most one element. $\mathcal{V} = \mathcal{HSP}(\mathcal{V})$ but $\mathcal{V}$ is not equationally definable.*

Exercise: *Check that any of the assumptions above makes $\mathcal{V}$ equationally definable.*

(Finitary) Birkhoff's Variety Theorem

Fact: *A class $\mathcal{V} \subseteq \mathbf{Alg}(\Sigma)$ of Σ-algebras is equationally definable (that is, $\mathcal{V} = Mod(\Phi)$ for some set Φ of Σ-equations) if and only if $\mathcal{V}$ is a variety and is closed under directed sums (unions of directed families of algebras).*

Classical equational calculus

$$\frac{}{t = t} \qquad \frac{t = t'}{t' = t} \qquad \frac{t = t' \quad t' = t''}{t = t''}$$

$$\frac{t_1 = t'_1 \quad \ldots \quad t_n = t'_n}{f(t_1 \ldots t_n) = f(t'_1 \ldots t'_n)} \qquad \frac{t = t'}{t(\theta) = t'(\theta)} \text{ for } \theta \colon X \to |T_\Sigma(Y)|$$

where *naive equation* $t = t'$ stands for $\forall FV(t, t').t = t'$.

Naive equational calculus is essentially sound and complete

BUT: Mind the variables!

$a = b$ does *not* follow from $a = f(x)$ and $f(x) = b$, unless...

- We need to assume that only algebras with no carriers empty are considered.

Equational calculus

$$\frac{}{\forall X.t = t} \qquad \frac{\forall X.t = t'}{\forall X.t' = t} \qquad \frac{\forall X.t = t' \quad \forall X.t' = t''}{\forall X.t = t''}$$

$$\frac{\forall X.t_1 = t'_1 \quad \dots \quad \forall X.t_n = t'_n}{\forall X.f(t_1 \dots t_n) = f(t'_1 \dots t'_n)} \qquad \frac{\forall X.t = t'}{\forall Y.t(\theta) = t'(\theta)} \text{ for } \theta\colon X \to |T_\Sigma(Y)|$$

Fact: *The above calculus is sound and complete:*

$$\Phi \models \varphi \text{ iff } \Phi \vdash \varphi$$

Moving between signatures

Signature morphism:

$$\sigma : \Sigma \to \Sigma'$$

maps sorts to sorts and operation names to operation names preserving their profiles.

Translating syntax and semantics:

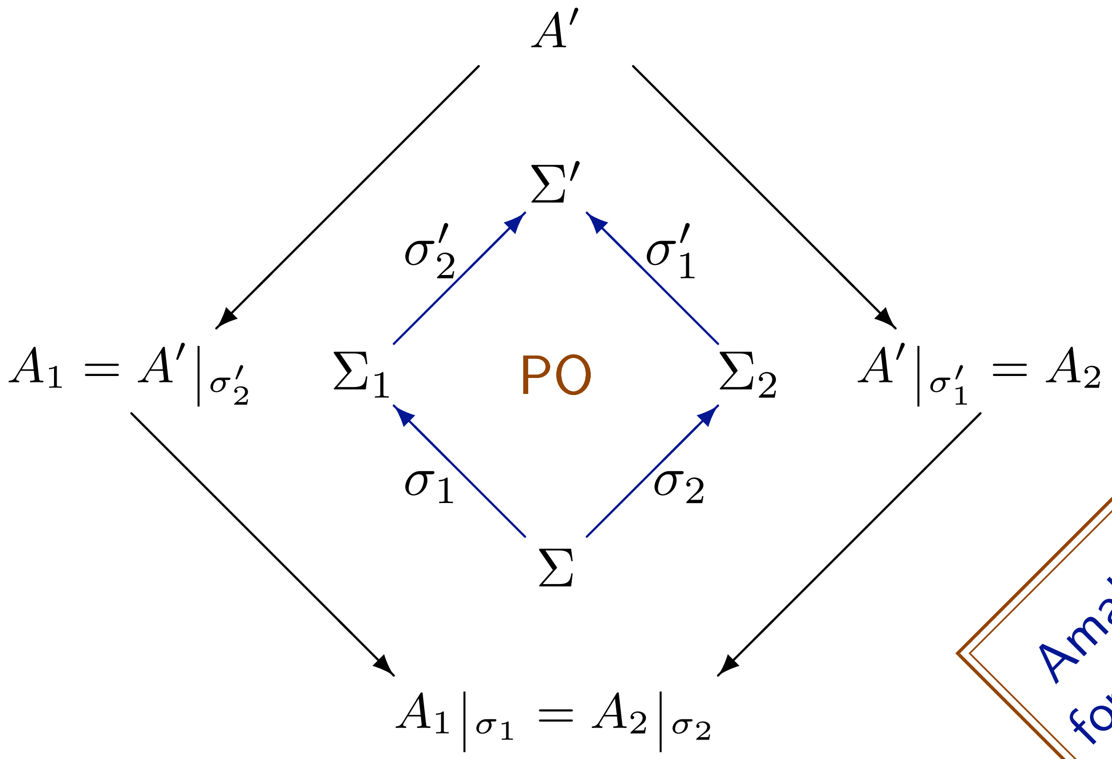
- *translation of equations*: $\sigma(\forall X.t_1 = t_2)$ yields $\forall X'.\sigma(t_1) = \sigma(t_2)$
- σ*-reduct*: $\_|_\sigma : \mathbf{Alg}(\Sigma') \to \mathbf{Alg}(\Sigma)$ where for $A' \in \mathbf{Alg}(\Sigma')$, $A'|_\sigma$ interprets sorts and operation names in Σ as A' interprets their image under σ.

Satisfaction condition:

Fact: *For all signature morphisms $\sigma : \Sigma \to \Sigma'$, Σ'-algebras A' and Σ-equations φ:*

$$A'|_\sigma \models_\Sigma \varphi \iff A' \models_{\Sigma'} \sigma(\varphi)$$

Amalgamation



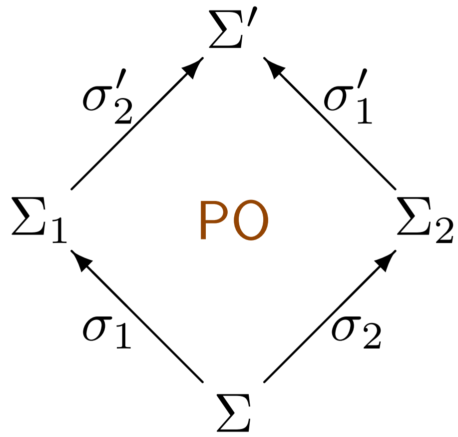
Amalgamation property may be stated for any commuting square of morphisms

Fact: *Amalgamation property holds for all pushouts of signature morphisms:* *for all* $A_1 \in \mathbf{Alg}(\Sigma_1)$ *and* $A_2 \in \mathbf{Alg}(\Sigma_2)$ *with* $A_1|_{\sigma_1} = A_2|_{\sigma_2}$, *there is a unique* $A' \in \mathbf{Alg}(\Sigma')$ *with* $A'|_{\sigma'_1} = A_2$ *and* $A'|_{\sigma'_2} = A_1$.

UFF!!!

Interpolation

A logic has the *interpolation property* for a pushout of signature morphisms



if for all $\varphi_1 \in \mathbf{Sen}(\Sigma_1)$ and $\varphi_2 \in \mathbf{Sen}(\Sigma_2)$ such that $\sigma'_2(\varphi_1) \models_{\Sigma'} \sigma'_1(\varphi_2)$ there is an *interpolant* $\theta \in \mathbf{Sen}(\Sigma)$ such that $\varphi_1 \models_{\Sigma_1} \sigma_1(\theta)$ and $\sigma_2(\theta) \models_{\Sigma_2} \varphi_2$.

Fact: $\mathbf{FOEQ}$ *has the interpolation property for all pushouts of pairs of morphisms, where at least one of the morphisms is injective on sorts.*

Equational interpolation

Equational interpolation essentially holds

when sets of interpolants are allowed

BUT:

Mind the nuances!

- Such equational interpolation holds when only algebras with no carriers empty are considered, and the signature morphisms are injective (on sorts).
- There may be no set of interpolants when algebras with some carriers empty are admitted, even if all signature morphisms are inclusions.
- In the general case we need to require surjectivity of reducts wrt signature morphisms involved (at least wrt σ_1).

Equational interpolation

Counterexample: $\Sigma = \mathtt{sorts}\ s, s_1, s_2\ \mathtt{opns}\ a, b \colon s$

$\Sigma_1 = \mathtt{enrich}\ \Sigma\ \mathtt{by}\ \mathtt{opn}\ c \colon s_1$

$\Sigma_2 = \mathtt{enrich}\ \Sigma\ \mathtt{by}\ \mathtt{opn}\ f \colon s_1 \rightarrow s_2$

Consider Σ_1*-equation* $\forall x{:}s_2.a = b$ *and* Σ_2*-equation* $a = b$*.*

Then $\forall x{:}s_2.a = b \models_{\Sigma_1 \cup \Sigma_2} a = b$*.*

BUT: there is no set Θ *of* Σ*-equations such that* $\forall x{:}s_2.a = b \models_{\Sigma_1} \Theta$ *and* $\Theta \models_{\Sigma_2} a = b$*.*

To show this, consider $A_1 \in \mathbf{Alg}(\Sigma_1)$ *with* $|A_1|_{s_2} = \emptyset$ *and* $a_{A_1} \neq b_{A_1}$*, a subalgebra of* $A_1|_\Sigma$ *with the carrier of sort* s_1 *empty, and its* Σ_2*-expansion* $A_2 \in \mathbf{Alg}(\Sigma_2)$*. Given a set of equational interpolants* Θ *as above, all these algebras satisfy* Θ*, and hence* $A_2 \models_{\Sigma_2} a = b$ *— contradiction.*

Conclusions

Many-sorted universal algebra

is essentially the same as

in the classical single-sorted case

But watch out: technical nuances may differ!