# Trader Multiflow and Box-TDI Systems in Series-Parallel Graphs

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**Abstract**

Series-parallel graphs are known to be precisely the graphs for which the standard linear systems describing the cut cone, the cycle cone, the $T$-join polytope, the cut polytope, the multicut polytope and the $T$-join dominant are TDI. We prove that these systems are actually box-TDI. As a byproduct, our result yields a min-max relation for a new problem: the trader multiflow problem. The latter generalizes both the maximum multiflow and min-cost multiflow problems and we show that it is polynomial-time solvable in series-parallel graphs.

*Keywords:* box-TDI system, series-parallel graph, multiflow
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## 1. Introduction

Throughout the paper, all the entries will be rational. A linear system $Ax \geq b, x \geq 0$ is *totally dual integral* (*TDI* for short) if the maximum in the LP-duality equation

$$\min\{c^\top x : Ax \geq b,\ x \geq \mathbf{0}\} = \max\{b^\top y : A^\top y \leq c,\ y \geq \mathbf{0}\}$$

has an integer optimal solution for all integer vectors $c$ for which the optimum is finite. This property is much sought-after since such systems describe

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integer polyhedra when $b$ is integer and yield min-max relations [1]. An even stronger property than TDIness is box-TDIness, where a *box-TDI system* is a TDI system $Ax \geq b, x \geq \mathbf{0}$ which remains TDI when adding box-constraints $\ell \leq x \leq u$, for all rational[1] vectors $\ell, u$. In other words, it is box-TDI if

$$\max\{b^\top y + \ell^\top z^1 - u^\top z^2 : A^\top y + z^1 - z^2 \leq c,\ y \geq \mathbf{0},\ z^1, z^2 \geq \mathbf{0}\}$$

has an integer solution for all integer vectors $c$ and all rational vectors $\ell, u$ for which the optimum is finite. General properties of such systems can be found in Cook [2] and Chapter 22.4 of Schrijver [3]. Note that, although every rational polyhedron $\{x : Ax \geq b, x \geq \mathbf{0}\}$ is described by a TDI system $\frac{1}{k}Ax \geq \frac{1}{k}b, x \geq \mathbf{0}$, for some integer $k$, not every polyhedron is described by a box-TDI system.

The book by Schrijver [4] contains numerous min-max relations of combinatorial optimization derived from TDI systems. When such systems are box-TDI, most of the time, the matrix $A$ is totally unimodular. The past few years, this topic has received a renewed interest [5, 6], and other box-TDI systems have been studied [7, 8, 9], with matrices that are not totally unimodular. A 0-1 matrix $A$ so that the linear system $Ax \geq \mathbf{1}$, $x \geq \mathbf{0}$ is (box-) TDI is called *(box-) Mengerian.* In 1977, Seymour [10] proved that a 0-1 matrix associated with a binary clutter is Mengerian if and only if it does not contain $Q_6$ as a minor. In 2008, Chen, Ding and Zang [8] proved that such matrices are box-Mengerian if and only if they contain neither $Q_6$ nor $Q_7$ as a minor. Recently, Ding, Tan and Zang [11] announced a characterization of the graphs for which a box-TDI system describes the matching polytope.

In 2009, Chen, Ding and Zang [9] proved that a graph is series-parallel if and only if the system $\frac{1}{2}Ax \geq \mathbf{1}$, $x \geq \mathbf{0}$ describing the 2-edge-connected spanning subgraph polytope is box-TDI, where $A$ is the cut-edge incidence matrix of the graph. Another set of characterizations of series-parallel graphs given by Schrijver asserts that they are precisely the graphs for which the standard linear systems describing the cut cone, the cycle cone [12], the cut polytope [13], the $T$-join polytope [14] and the $T$-join dominant [15] are TDI — see Corollary 29.9c of [4]. Moreover, it is proved in [16] that a graph is series-parallel if and only if the standard linear system describing its multicut polytope is TDI.

Multiflows are among the most famous NP-hard problems in combinato-

[1]allowed to take infinite values

rial optimization and have been considerably studied, see for instance [4]. We focus on integer multiflows in the present paper. Multiflow problems involve two simple undirected graphs, a *supply graph* $G = (V, E)$ and a *demand graph* $H = (V, R)$, and two vectors, a *capacity vector* $c \in \mathbb{Z}_+^E$ and a *demand vector* $d \in \mathbb{Z}_+^R$. An edge $e \in E$ is a *link* of *capacity* $c_e$ whereas an edge $r \in R$ is a *net* of *demand* $d_r$. From now on, $(G, H, c, d)$ will refer to such a quadruplet. For a net $r = st$, let $\mathcal{P}(r)$ be the set of all $st$-paths in $G$, and let $\mathcal{P}$ be the union of $\mathcal{P}(r)$ for all nets $r$. A *multiflow of* $(G, H, c, d)$ is an integer vector $y \in \mathbb{Z}^{\mathcal{P}}$ satisfying the following system of linear inequalities:

$$
(\textsc{mflow}) \begin{cases} \displaystyle\sum_{P \in \mathcal{P}(r)} y_P & \geq d_r \qquad \text{for each net } r \in R, \\ \displaystyle\sum_{P \in \mathcal{P}:\, e \in P} y_P & \leq c_e \qquad \text{for each link } e \in E, \\ \qquad y & \geq \mathbf{0}. \end{cases}
$$

Two famous NP-hard problems are related to multiflows. Given $G$, $H$ and $c$, the *maximum multiflow problem* asks for a demand vector $d$ such that there exists a multiflow for $(G, H, c, d)$ and $\sum_{r \in R} d_r$ is maximum.

Given $(G, H, c, d)$ and some cost vector $w \in \mathbb{Z}_+^E$ on the links, the *min-cost multiflow problem* asks for a multiflow minimizing the sum of $w_e y_e$ over all links $e \in E$, where $y_e := \sum_{P \in \mathcal{P}:\, e \in P} y_P$ is the amount of flow through link $e$.

A necessary condition for the existence of a multiflow in $(G, H, c, d)$ is the *cut condition* which requires that $d(D \cap R) \leq c(D \cap E)$ for all cuts $D$ of $G + H$, the latter being $G + H = (V, E \cup R)$ where $E$ and $R$ are considered as disjoint, that is, $G + H$ may contain parallel edges. Seymour [14] proved that a graph $(V, F)$ is series-parallel if and only if for all partitions $F$ into $E$ and $R$, and for all $c \in \mathbb{Z}_+^E$ and $d \in \mathbb{Z}_+^R$, the cut condition implies the existence of a multiflow.

*Contribution.* In this paper, we investigate some box-TDI systems related to multiflows. Our main result is to strengthen the TDI characterizations of series-parallel graphs mentioned earlier by proving that the standard linear systems describing the cut cone, the cycle cone, the $T$-join polytope, the cut polytope, the multicut polytope, and the $T$-join dominant are actually box-TDI systems for series-parallel graphs — see Theorem 1.

From the box-TDIness of the cut cone, we derive a min-max relation for series-parallel graphs that involves a new multiflow problem generalizing both the maximum multiflow and min-cost multiflow problems. Given $(G, H, c, d)$, a *profit* $\ell \in \mathbb{Z}_+^R$ and a *cost* $u \in \mathbb{Z}_+^E$, the *trader multiflow problem* asks to maximize $\ell^\top z^1 - u^\top z^2$ over all $(y, z^1, z^2) \in \mathbb{Z}_+^{\mathcal{P}} \times \mathbb{Z}_+^R \times \mathbb{Z}_+^E$ such that $y$ is a multiflow of $(G, H, \tilde{c}, \tilde{d})$ with $\tilde{c} = c + z^2$ and $\tilde{d} = d + z^1$. Therefore, in this new multiflow problem, we gain $\ell_r$ for each additional unit of demand on net $r \in R$ that we are able to satisfy, we pay $u_e$ to add a unit of capacity on link $e \in E$, and the goal is to maximize the total benefit. The min-max relation we derive connects the trader multiflow problem and box-multicuts, where box-multicuts are a generalization of multicuts. We also show that the trader multiflow problem is polynomial time solvable in series-parallel graphs.

*Outline.* In Section 2, we establish our characterization of series-parallel graphs in terms of box-TDI systems. Section 3 is devoted to the trader multiflow problem. We first show how it generalizes both the maximum multiflow and min-cost multiflow problems. Then, we provide our min-max relation for the trader multiflow problem in series-parallel graphs and explain why this problem is polynomial in these graphs. For the sake of clarity, the most technical part of the proof of Theorem 1 is postponed to the Appendix. The rest of this section is devoted to definitions.

*Definitions.* Throughout, $G = (V, E)$ will denote an undirected graph and $T \subseteq V$ a set of vertices of even cardinality. A graph is *series-parallel* if it is obtained from a forest by repeating the operations of replacing one edge by two edges in parallel, or by two edges in series. Equivalently, these are the graphs without $K_4$ minor [17]. Then, a series-parallel graph is planar and its planar dual is also series-parallel. Following [4], a *cycle* is a subset $C \subseteq E$ so that every vertex of $(V, C)$ has an even degree. A minimal nonempty cycle is a *circuit*. The *cut* defined by a subset of vertices $U$, denoted by $\delta(U)$, is the set of edges having one extremity in $U$ and the other one in $V \setminus U$. A minimal nonempty cut is a *bond*. Note that cycles (resp. cuts) are disjoint unions of circuits (resp. bonds). A *multicut* is the set of all the edges between different classes of some partition of the vertex set. A *$T$-join* is a subset of edges $F$ such that the odd degree vertices of $(V, F)$ are the ones in $T$. Note that a cycle is an $\emptyset$-join. A *$T$-cut* is a cut $\delta(U)$ with $|U \cap T|$ odd. For $x \in \mathbb{R}^E$ and $F \subseteq E$, we use the notation $x(F) = \sum_{e \in F} x_e$. We will make no difference between combinatorial objects and their characteristic vectors, that is, for instance, we will speak of nonnegative combinations of cycles instead of nonnegative combinations of characteristic vectors of cycles.

## 2. Box-TDI systems of series-parallel graphs

In this section, we first provide the systems involved in our main theorem. Then, we state and prove Theorem 1, which establishes that the standard linear systems describing the cut cone, the cycle cone, the $T$-join polytope, the cut polytope, the multicut polytope and the $T$-join dominant are box-TDI if and only if the graph is series-parallel. These systems were already known to be TDI [16, 4].

*2.1. TDI systems of series-parallel graphs...*

Let us write now the systems involved in Theorem 1. Let $G = (V, E)$ be an undirected graph and $T \subseteq V$ a set of vertices of even cardinality.

Seymour [12] proved that the *cycle cone* of $G$, that is, the set of non-negative combinations of cycles of $G$, is described by the following set of inequalities.

$$\text{(Cycle cone)} \begin{cases} x(\delta(U) \setminus \{e\}) - x_e \geq 0 & \text{for each } U \subseteq V \text{ and each } e \in \delta(U), \\ x \geq \mathbf{0}. \end{cases}$$

The *$T$-join polytope* of $G$ is the convex hull of its $T$-joins. Seymour [14] proved that it is described by the following set of inequalities.

$$(T\text{-join}) \begin{cases} x(F) - x(\delta(U) \setminus F) \leq |F| - 1 & \begin{array}{l}\text{for each } U \subseteq V,\ F \subseteq \delta(U) \\ \text{with } |U \cap T| + |F| \text{ odd,}\end{array} \\ \mathbf{0} \leq x \leq \mathbf{1}. \end{cases}$$

The $T$-join dominant of $G$ is the set of vectors greater than or equal to some $T$-join of $G$. This dominant is described by the following set of inequalities, see Corollary 29.2b in [4].

$$(T\text{-join dominant}) \begin{cases} x(C) \geq 1 & \text{for each } T\text{-cut } C, \\ x \geq \mathbf{0}. \end{cases}$$

Sebő [18] provided a minimal TDI system describing the $T$-join dominant of $G$.

Let us assume that $G$ is planar and let $G^*$ denote its dual graph. Recall that the cycles of $G$ are the cuts of $G^*$. Hence,

$$\text{(Cut cone)} \begin{cases} x(C \setminus \{e\}) - x_e \geq 0 & \text{for each circuit } C \text{ and each edge } e \in C, \\ x \geq \mathbf{0}, \end{cases}$$

describes the *cut cone* of $G$, that is, the set of nonnegative combinations of cuts of $G$. Moreover, by taking $T = \emptyset$ in system ($T$-join), and then writing the planar dual, we have the following description of the *cut polytope* of $G$, that is, the convex hull of its cuts.

$$\text{(Cut)} \begin{cases} x(F) - x(C \setminus F) \le |F| - 1 & \text{for each circuit } C \text{ and } F \subseteq C \text{ with } |F| \text{ odd,} \\ \mathbf{0} \le x \le \mathbf{1}. \end{cases}$$

Actually, the systems (Cut cone) and (Cut) describe the cut cone and the cut polytope for a larger class than planar graphs, namely graphs with no $K_5$-minor — see [14] and [13], respectively.

Schrijver showed that the systems (Cycle cone), ($T$-join) and ($T$-join dominant) are TDI if and only if the graph is series-parallel — see Corollary 29.9c of [4]. A graph being series-parallel if and only if its dual is, this result, combined with the fact that cycles are $\emptyset$-joins, implies that (Cut cone) and (Cut) are TDI if and only if the graph is series-parallel.

Multicuts can be equivalently defined as arbitrary unions of cuts, or as sets of edges $D \subseteq E$ such that $|D \cap C| \neq 1$ for all cycles $C$. The *multicut polytope* of a graph is the convex hull of its multicuts, and is therefore contained in the polyhedron defined by the inequalities of (Cut cone) and $x \le \mathbf{1}$. Chopra [19] showed that the following system, called (Multicut), describes the multicut polytope of a graph if and only if the graph is series-parallel.

$$\text{(Multicut)} \begin{cases} x(C \setminus \{e\}) - x_e \ge 0 & \text{for each circuit } C \text{ and each edge } e \in C, \\ \mathbf{0} \le x \le \mathbf{1}. \end{cases}$$

Corollary 4.1 of [16] strengthens the result of Chopra [19] by stating that system (Multicut) is TDI if and only if the graph is series-parallel.

### *2.2. ... are actually box-TDI*

We now strengthen the aforementioned TDIness results. More precisely, we show that each system mentioned in Section 2.1 which is TDI for series-parallel graphs is actually box-TDI for these graphs. Our theorem implies Corollary 4.1 of [16] and Corollary 29.9c of [4].

**Theorem 1.** *Let $G = (V, E)$ be a graph. The following statements are equivalent.*

*(i)* *G is series-parallel.*

*(ii)* *System (Cut cone) is box-TDI.*

*(iii)* *System (Cycle cone) is box-TDI.*

*(iv)* *System ($T$-join) is box-TDI, for all $T \subseteq V$ of even cardinality.*

*(v)* *System (Cut) is box-TDI.*

*(vi)* *System (Multicut) is box-TDI.*

*(vii)* *System ($T$-join dominant) is box-TDI, for all $T \subseteq V$ of even cardinality.*

*Proof.* Proof. Series-parallelness is already necessary for the systems of $(ii)$-$(vii)$ to be TDI — see [16] for $(vi)$ and Corollary 29.9c of [4] for the others. A box-TDI system being TDI, the necessity of $(i)$ follows. For the other directions, we will show that $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v)$ and $(ii) \Rightarrow (vi)$ and $(iv) \Rightarrow (vii)$.

$(i) \Rightarrow (ii)$: Let $G = (V, E)$ be series-parallel, $c \in \mathbb{Z}^E$ and $\ell, u \in \mathbb{Q}^E$ with $\ell \leq u$. The primal problem is to optimize over the system (Cut cone) intersected with the box $\{x : \ell \leq x \leq u\}$. Since we have $x \geq \mathbf{0}$, we may suppose that $\ell \geq \mathbf{0}$ and we get:

$$(P) \begin{cases} \min c^\top x \\ x(C \setminus \{e\}) - x_e \geq 0 \qquad \text{for each circuit } C \text{ of } G \text{ and each edge } e \in C, \\ \mathbf{0} \leq \ell \leq x \leq u. \end{cases}$$

To prove box-TDIness, one has to show that if the dual given below has an optimal solution, then it also has an integer one.

$$(D) \begin{cases} \max \ell^\top z^1 - u^\top z^2 \\ \sum\limits_{\text{circuit } C \ni e} \Big( \sum\limits_{f \in C \setminus \{e\}} y_{C,f} \quad - \quad y_{C,e} \Big) \leq c_e - z^1_e + z^2_e \qquad \text{for each } e \in E, \\ y \geq \mathbf{0}, \quad z^1, z^2 \geq \mathbf{0}. \end{cases}$$

The feasible set for $(D)$ has the form $Q = \{z^1, z^2 \geq \mathbf{0},\ y \geq \mathbf{0} : z^1 - z^2 + Ay \leq c\}$, and its projection onto the space of $z = (z^1, z^2) \in \mathbb{R}^{E \times E}$ is

$\text{proj}_z(Q) = \{z^1, z^2 \geq \mathbf{0} : v^\top z^1 - v^\top z^2 \leq v^\top c, \text{ for each } v \in K\}$ where $K$ is the projection cone $K = \{v \in \mathbb{R}^E : v^\top A \geq \mathbf{0}^\top, v \geq \mathbf{0}\}$. Observe that $K$ is the set of $v \in \mathbb{R}^E$ satisfying the inequalities of the system (Cut cone). Since $G$ is series-parallel, $K$ is the cut cone of $G$ [14]. Therefore

$$\text{proj}_z(Q) = \{(z^1, z^2) \in \mathbb{R}_+^{E \times E} : z^1(D) - z^2(D) \leq c(D), \text{ for each cut } D \text{ of } G\}.$$

The following claim states that $\text{proj}_z(Q)$ is an integer polyhedron. It is a direct corollary of a technical result whose statement and proof are postponed to the Appendix.

**Claim 2.** *$\text{proj}_z(Q)$ is integer whenever $c$ is integer.*

Suppose $D$ has an optimal solution. By Claim 2, there exists an integer optimal solution $(\bar{z}^1, \bar{z}^2)$ of $\max \ell^\top z^1 - u^\top z^2$ over $\text{proj}_z(Q)$. We now build an optimal solution $(\bar{y}, \bar{z}^1, \bar{z}^2)$ of $(D)$ as follows.

Let $b := c - \bar{z}^1 + \bar{z}^2$. Then $b$ is integer and satisfies $b(D) \geq 0$ for each cut $D$ of $G$. Define $R$ as the set of all $e \in E$ with $b_e \leq 0$ and $E' = E \setminus R$. Let $G' = (V, E')$ and $H = (V, R)$. Let $c' \in \mathbb{Z}_+^{E'}$ and $d \in \mathbb{Z}_+^R$ be defined by $c'_e = c_e$ for all $e \in E'$ and $d_r = -c_r$ for all $r \in R$. Then $d(D \cap R) \leq c'(D \cap E')$ for each cut $D$ of $G' + H$. In other words, the cut condition is satisfied in $(G', H, c', d)$. Hence, $G' + H = G$ being series-parallel, Theorem 8.1 of [14] implies that there exists a multiflow $\hat{y}$ of $(G', H, c', d)$. Define $\bar{y}$ as follows:

$$\bar{y}_{C,e} := \begin{cases} \hat{y}_P & \text{if } b_e \leq 0 \text{ and } P = C \setminus \{e\}, \\ 0 & \text{otherwise.} \end{cases}$$

By construction, $(\bar{y}, \bar{x}^1, \bar{x}^2)$ is an integer optimal solution of $(D)$, and we are done.

*(ii)* $\Rightarrow$ *(iii)*: The system (Cycle cone) of a series-parallel graph is the system (Cut cone) of its planar dual which is also a series-parallel graph. As the latter system is box-TDI precisely for such graphs, we get the desired implication.

*(iii)* $\Rightarrow$ *(iv)*: In the following, $Ax \leq b$ is a system whose underlying polyhedron $P = \{x : Ax \leq b\}$ is pointed. The *vertex system* associated with a vertex $z$ of $\{x : Ax \leq b\}$ is the system $A_z x \leq b_z$ composed of the inequalities of $Ax \leq b$ satisfied with equality by $z$.

**Claim 3.** *The system $Ax \leq b$ is box-TDI if and only if the vertex system associated with each vertex of $P = \{x : Ax \leq b\}$ is box-TDI.*

*Proof.* Cook proves that a system is box-TDI if and only if, for each face $F$ of the associated polyhedron, the set of active rows for $F$ forms a box Hilbert basis [2, Proposition 2.2].

Suppose that all the vertex systems of $P$ are box-TDI. Let $F$ be a proper face of $P$ and $z$ be a vertex of $F$. Then, the active rows in $A_z x \leq b_z$ for the minimal face of $\{x : A_z x \leq b_z\}$ containing $F$ are exactly the same as those in $Ax \leq b$ for $F$. Hence, by [2, Proposition 2.2], the set of active rows for $F$ forms a box Hilbert basis. Since this holds for every face of $P$, [2, Proposition 2.2] implies that $Ax \leq b$ is box-TDI. The converse can be proved in a similar way. ∎

Let $T \subseteq V$. Recall that vertices of the polytope defined by the system ($T$-join) correspond to $T$-joins of $G$, and conversely. Let $J$ be any $T$-join of $G$. By Claim 3, it suffices to show that the vertex system of ($T$-join) associated with vertex $J$ is box-TDI. Let $\phi_J : \mathbb{R}^E \to \mathbb{R}^E$ be defined by

$$[\phi_J(x)]_e := \begin{cases} 1 - x_e & \text{if } e \in J, \\ x_e & \text{if } e \in E \setminus J. \end{cases}$$

The next two claims exhibit properties of $\phi_J$.

**Claim 4.** *The system obtained from (Cycle cone) by replacing $x$ by $\phi_J(x)$ is the vertex system of ($T$-join) associated with $J$.*

*Proof.* Schrijver proves that replacing $x$ by $\phi_J(x)$ in the vertex system of ($T$-join) associated with $J$ gives the system (Cycle cone) — see (29.61) to (29.63) page 506 in [4] for the details. As $\phi_J(\phi_J(x)) = x$, the assertion follows. ∎

**Claim 5.** *Replacing $x$ by $\phi_J(x)$ preserves box-TDIness.*

*Proof.* From the definition of box-TDI systems, it follows that replacing some coordinates by their opposite preserves box-TDIness. So does translation, see Theorem 5.34 in [4]. ∎

The (Cycle cone) being box-TDI by (*iii*), Claims 4 and 5 imply the box-TDIness of the vertex system of ($T$-join) associated with $J$. Since this holds for any $T$-join $J$ of $G$, Claim 3 gives the box-TDIness of ($T$-join).

(*iv*) $\Rightarrow$ (*v*): We have already shown that ($T$-join) is box-TDI if and only if the graph is series-parallel. Recall that the cuts of a planar graph are the cycles of its planar dual, and that cycles are $\emptyset$-joins. Therefore, (Cut) is nothing but the system ($\emptyset$-join) for the planar dual of the graph, and since planar duality preserves series-parallelness, we get that $(iv)$ implies $(v)$.

$(ii) \Rightarrow (vi)$: This is immediate because (Multicut) is nothing but the box-TDI system (Cut cone) together with the box-constraints $x \leq \mathbf{1}$.

$(iv) \Rightarrow (vi)$: The system describing the $T$-join polytope being box-TDI, the TDI system ($T$-join dominant) describing its dominant is also box-TDI — by Theorem 22.11 of [3]. $\square$

Box-TDI systems have the remarkable property that any TDI system describing the same polyhedron is also box-TDI [2]. This gives the following consequence of Theorem 1. The minimal TDI system describing the $T$-join dominant given by Sebő [18] becomes box-TDI when the graph is series-parallel.

## 3. Trader multiflow vs box-multicut

In this section, we first explain how the trader multiflow problem generalizes both the min-cost multiflow and maximum multiflow problems. We then provide a min-max relation involving the trader multiflow problem and the so-called box-multicuts. Finally, we briefly explain why the trader multiflow problem is polynomial in series-parallel graphs.

### *3.1. Related multiflow problems*

Recall that an instance $(G, H, c, d, \ell, u)$ of the trader multiflow problem is composed of two simple undirected graphs $G = (V, E)$ and $H = (V, R)$, a capacity $c \in \mathbb{Z}_+^E$, a demand $d \in \mathbb{Z}_+^R$, a profit $\ell \in \mathbb{Z}_+^R$ and a cost $u \in \mathbb{Z}_+^E$. The trader multiflow problem aims at maximizing $\ell^\top z^1 - u^\top z^2$ over all $(y, z^1, z^2) \in \mathbb{Z}_+^{\mathcal{P}} \times \mathbb{Z}_+^R \times \mathbb{Z}_+^E$ such that $y$ is a multiflow of $(G, H, \tilde{c}, \tilde{d})$ with $\tilde{c} = c + z^2$ and $\tilde{d} = d + z^1$.

This problem contains the maximum multiflow problem as a special case. Let $(G, H, c, d, \ell, u)$ be an instance of the trader multiflow problem with $d = \mathbf{0}$, $\ell = \mathbf{1}$ and $u = +\infty$. In any optimal solution $(\bar{y}, \bar{z}^1, \bar{z}^2)$, since $u = +\infty$, we have $\bar{z}^2 = \mathbf{0}$, that is, capacities remain unchanged. Since $d = \mathbf{0}$ and $\ell = \mathbf{1}$, the problem reduces to find $\bar{z}^1$ such that $\sum_{r \in R} \bar{z}^1_r$ is maximum and there exists a multiflow in $(G, H, c, \bar{z}^1)$. This is nothing but the maximum multiflow problem associated with $(G, H, c)$.

The trader multiflow problem also contains the min-cost multiflow problem as a special case. Let $(G, H, c, d, w)$ be an instance of the min-cost multiflow problem. It is transformed into an instance $(G', H', c', d', \ell', u')$ of the trader multiflow problem as follows. Let $G' = (V', E')$ be the graph obtained from $G$ by subviding every link $e \in E$ into two links $e_1, e_2$ in series. Then, the amount of flow passing by $e_1$ equals the amount of flow passing by $e_2$. Let $c'_{e_1} = c_e$ and $u'_{e_1} = +\infty$. The capacity and cost of $e_1$ is chosen in order to limit the value of the flow passing by $e_1, e_2$ to $c_e$. Let $c'_{e_2} = 0$ and $u'_{e_2} = w_e$. The role of $e_2$ is to charge a fee $w_e$ for each unit of flow passing by $e_1, e_2$. Let $H' = (V', R)$, $d' = d$ and $\ell' = \mathbf{0}$. In an optimal solution $(\bar{y}, \bar{z}^1, \bar{z}^2)$ of the trader multiflow problem, we may suppose without loss of generality that $\bar{z}^1 = \mathbf{0}$ since $\ell' = \mathbf{0}$. Since $u'_{e_1} = +\infty$, the amount of flow passing by $e_1, e_2$ is no more than $c'_{e_1} = c_e$. Since $c'_{e_2} = 0$, for each unit of flow passing by $e_1, e_2$, one has to increase the capacity of $e_2$ by one at cost $u'_{e_2} = w_e$. Hence, $\bar{y}$ defines a multiflow in $(G, H, c, d)$ minimizing the total cost of the flow.

*3.2. Min-max theorem*

Given a graph and integer vectors $\ell$ and $u$ indexed on its edges, the integer vectors $x$ satisfying system (Cut cone) and $\ell \leq x \leq u$ are called *box-multicuts within* $[\ell, u]$. If we are also given a cost vector $c$ defined on the edges, the *minimum box-multicut problem* seeks a box-multicut $x$ within $[\ell, u]$ of minimum cost $c^\top x$.

Box-multicuts are a generalization of multicuts, these latter being box-multicuts within $[\mathbf{0}, \mathbf{1}]$. Box-multicuts also generalize separating multicuts, where, given a supply graph $G$ and a demand graph $H = (V, R)$, a *separating multicut* is a multicut of $G + H$ containing $R$. Indeed, separating multicuts are box-multicuts of $G + H$ within $[\ell, \mathbf{1}]$ where $\ell$ equals 1 for every net of $R$ and 0 otherwise.

The min-max relation between the trader multiflow and minimum box-multicut problems given in the following Corollary 6 is a consequence of Theorem 1. Its statement uses the following notation: given a supply graph $G = (V, E)$ and a demand graph $H = (V, R)$ and two vectors $v^1 \in \mathbb{Z}_+^E$ and $v^2 \in \mathbb{Z}_+^R$, the vector associated with the edges of $G + H$ defined by $v^1$ and $v^2$ is denoted by $(v^1, v^2)$.

**Corollary 6.** *The maximum trader multiflow of* $(G, H, c, d, \ell, u)$ *equals the minimum box-multicut of* $G + H$ *within* $[(\mathbf{0}, \ell), (u, +\infty)]$ *with respect to costs* $(c, -d)$*, if* $G + H$ *is series-parallel.*

*Proof.* First, set $\hat{c} = (c, -d)$, $\hat{\ell} = (\mathbf{0}, \ell)$ and $\hat{u} = (u, +\infty)$. Consider the linear program $(P)$ of the proof of Theorem 1 where $G$, $c$, $\ell$ and $u$ are replaced by $G + H$, $\hat{c}$, $\hat{\ell}$ and $\hat{u}$, respectively. Since $\hat{\ell}_e = 0$, we may suppose, without loss of generality, that $\bar{z}^1_e = 0$ for all links $e \in E$ in an optimal solution $(\bar{y}, \bar{z}^1, \bar{z}^2)$ of the dual $(D)$. Moreover, as $u_r = +\infty$, $\bar{z}^2_r = 0$ for all nets $r \in R$. The dual can then be written as:

$$(D') \begin{cases} \max \sum_{r\in R} \ell_r z^1_r - \sum_{e\in E} u_e z^2_e \\ \displaystyle\sum_{\text{circuit } C \ni r} \Big( y_{C,r} \quad - \sum_{f \in C\setminus\{r\}} y_{C,f} \Big) \geq d_r + z^1_r & \text{for each } r \in R, \\ \displaystyle\sum_{\text{circuit } C \ni e} \Big( \sum_{f \in C\setminus\{e\}} y_{C,f} \quad - \quad y_{C,e} \Big) \leq c_e + z^2_e & \text{for each } e \in E, \\ y \geq \mathbf{0}, \quad z^1, z^2 \geq \mathbf{0}. \end{cases}$$

By strong duality, the optimal values of $(P)$ and $(D')$ are equal, when finite. In this case, we will show that there exists an integer optimal solution for both problems.

We may suppose that $\bar{y}_{C,f} = 0$ if $f \in E$. Otherwise, one may decrease $\bar{y}_{C,f}$ by some $\epsilon > 0$. If the solution becomes infeasible, then there exists a circuit $C' \ni f$ and $f' \in C \setminus f$ with $\bar{y}_{C',f'} \geq \epsilon$ since $c \geq \mathbf{0}$. Decreasing $\bar{y}_{C',f'}$ by $\epsilon$ and increasing $\bar{y}_{C'',f'}$ by $\epsilon$ where $C''$ is the circuit of $C \Delta C'$ containing $f$ restores its feasibility. Similarly, we may suppose that $\bar{y}_{C,f} = 0$ if $C \setminus f$ intersects $R$. Thus, for every $\bar{y}_{C,f} > 0$, $f \in R$ and $C \setminus f \in \mathcal{P}(r)$. Since $G + H$ is series-parallel, system (Cut cone) is box-TDI and $(\bar{y}, \bar{z}^1, \bar{z}^2)$ may be assumed integer. The latter then corresponds to an optimal solution to the trader multiflow problem. Finally, since $\hat{\ell}$ and $\hat{u}$ are integer, the box-TDIness of system (Cut cone) implies that the optimal solution of $(P)$ is integer, that is, a box-multicut of $G + H$ within $[\hat{\ell}, \hat{u}]$. $\square$

Min-max relations involving min-cost multiflow and maximum multiflow stem from Corollary 6 since the transformations described in Section 3.1 preserve series-parallelness. In particular, Corollary 6 implies that the two following min-max relations of [16] that hold if $G + H$ is series-parallel:

- the maximum multiflow equals the minimum separating multicut,
- the minimum multiflow loss equals the maximum multicut,

where the *minimum multiflow loss problem* asks to remove a minimum number of demands of $H$ to ensure the existence of a multiflow in $G + H$.

Applying the arguments used in the proof of $(i) \Rightarrow (ii)$ of Theorem 1, it can be shown that optimizing over $(D')$ amounts to optimize over an integer polyhedron similar to $proj_z(Q)$. For series-parallel graphs, optimizing over such a polyhedron is polynomial-time solvable [20, 21]. It yields an increase of capacities and demands which maximizes the objective function and ensures that the cut condition is satisfied. Then, applying Theorem 8.1 of [14] provides an optimal solution to the trader multiflow problem. To sum up, we have the following complexity result.

**Corollary 7.** *If $G+H$ is series-parallel, then the maximum trader multiflow problem on $(G, H, c, d, \ell, u)$ is polynomial-time solvable for all vectors $\ell$ and $u$ and for all integer vectors $c$ and $d$.*

As seen in Corollary 7, our approach yields a polynomial algorithm, however it relies on the ellipsoid method. We conclude with the question: is there a combinatorial algorithm that solves the trader multiflow problem in series-parallel graphs?

## Appendix A.

The proof of Theorem 1 is based on Claim 2 which is a direct consequence of the following result.

**Lemma 8.** *Let $G = (V, E)$ be a graph. The polyhedron $P(G, c)$ defined by*

$$P(G, c) := \{(x, y) \in \mathbb{R}_+^{E\times E} : x(D) - y(D) \le c(D), \text{ for each cut } D \text{ of } G\}$$

*is integer for all integer weights $c \in \mathbb{Z}^E$ if and only if $G$ is series-parallel.*

*Proof. Necessity.* First, note that $P(\hat{G}, \hat{c})$ has a fractional extreme point if $\hat{G}$ is the complete graph $K_4$ with cost $\hat{c}_e = -1$ on the three edges of a triangle and $\hat{c}_e = +1$ on the remaining star. Indeed, the point $\hat{p} = (\hat{x}, \hat{y})$ defined by $\hat{y}_e = 1/2$ for the edges of the triangle and zero elsewhere is the unique optimal solution of maximizing $\hat{\ell}^\top x - \hat{u}^\top y$ over $P(\hat{G}, \hat{c})$, where $\hat{\ell}$ is zero and $\hat{u}$ is the all-one vector. Now, let $\bar{G}$ be a graph which is not series-parallel, then, by [17], it has a $K_4$-minor, that is we can remove and contract some edges of $\bar{G}$ to obtain $K_4$. Let us extend $(\hat{c}, \hat{\ell}, \hat{u})$ to $(\bar{c}, \bar{\ell}, \bar{u})$ by defining $\bar{\ell}_e = -\infty$ and $\bar{u}_e = +\infty$ for the new edges $e$, with $\bar{c}_e = +\infty$ if $e$ must be contracted, and $\bar{c}_e = 0$ if it must be deleted. Clearly, the point $\bar{p}$ obtained by extending $\hat{p}$ with zero components is the unique optimal solution of maximizing $\bar{\ell}^\top x - \bar{u}^\top y$ over $P(\bar{G}, \bar{c})$.

*Sufficiency.* By contradiction, let $(G, c)$ be a counter-example with a minimum number of edges. Throughout, $\bar{p} = (\bar{x}, \bar{y})$ will denote some fractional extreme point of $P(G, c)$ and

$$\bar{b} := c - \bar{x} + \bar{y}.$$

Note that $\bar{b}(D) \geq 0$, for each cut $D$.

First, note that $G$ has no loops or bridges. Indeed, a loop belongs to no cut, and a bridge $e$ appears exactly in three nonredundant constraints, namely $x_e \geq 0$, $y_e \geq 0$ and $y_e - x_e \geq c_e$, two of which are satisfied with equality by any extreme point.

Moreover, $P(G, c)$ is full-dimensional. To see this, observe that the point $p = (x, y) \in \mathbb{R}^{E \times E}$ defined by $x_e = 1$ and $y_e = +\infty$ for all $e \in E$ belongs to $P(G, c)$. Moreover, for each edge $e \in E$, the point $p_e^x$ (resp. $p_e^y$) obtained from $p$ by resetting $x_e$ to zero (resp. $y_e$ to zero) also belongs to $P(G, c)$ since each cut has size at least two. The $2|E| + 1$ points $p$, $p_e^x$, $p_e^y$, for $e \in E$, are affinely independent, hence the dimension of $P(G, c)$ is $2|E|$.

In consequence, the point $\bar{p}$ is the solution of a system of $2|E|$ equations of the following type, where the left-hand-side forms a full-rank matrix.

$$\bar{x}_e = 0 \quad \text{for some edges } e, \tag{A.1}$$

$$\bar{y}_e = 0 \quad \text{for some edges } e, \tag{A.2}$$

$$\bar{x}(D) - \bar{y}(D) = c(D) \quad \text{for some bonds } D \neq \emptyset. \tag{A.3}$$

Suppose $G$ has two parallel edges $\bar{e}$ and $\bar{f}$. Then, replacing $(\bar{x}_{\bar{e}}, \bar{y}_{\bar{e}})$ by $(\bar{x}_{\bar{e}}, \bar{y}_{\bar{e}}) + (\bar{x}_{\bar{f}}, \bar{y}_{\bar{f}})$ and $(\bar{x}_{\bar{f}}, \bar{y}_{\bar{f}})$ by $(0, 0)$ yields a feasible point $(\tilde{x}, \tilde{y})$ because $\bar{e}$ and $\bar{f}$ belongs to the same cuts. This point $(\tilde{x}, \tilde{y})$ satisfies all the equations (A.1)-(A.3) except possibly the equations (A.1) and (A.2) associated with $\bar{e}$. But these two equations are not satisfied only if $\bar{x}_{\bar{f}} > 0$ or $\bar{y}_{\bar{f}} > 0$ respectively. This implies that $(\tilde{x}, \tilde{y})$ satisfies $2|E|$ equations among (A.1)-(A.3), $\bar{x}_{\bar{f}} = 0$, and $\bar{y}_{\bar{f}} = 0$. Hence, it is also an extreme point of $P(G, c)$. Therefore resetting $c_{\bar{e}} := c_{\bar{e}} + c_{\bar{f}}$ and removing $\bar{f}$ gives a counter-example with a smaller number of edges, a contradiction. We have just proved the following.

$$G \text{ has no parallel edges.} \tag{A.4}$$

Note that, if both $\bar{x}_e > 0$ and $\bar{y}_e > 0$ for some edge $e$, then one could reset $\bar{x}_e := \bar{x}_e - \varepsilon$ and $\bar{y}_e := \bar{y}_e - \varepsilon$ (for some $\varepsilon > 0$) and still satisfy (A.1)-(A.3), contradicting the extremality of $\bar{p}$. Thus,

$$\text{for all } e, \text{ either } \bar{x}_e = 0 \text{ or } \bar{y}_e = 0. \tag{A.5}$$

We can choose $c$ so as to minimize the norm of $\bar{p}$ (*e.g.* Euclidean). Consequently, nonzero coordinates of $\bar{p}$ are fractional. Indeed, we have

$$\mathbf{0} \leq \bar{p} < \mathbf{1}, \tag{A.6}$$

as otherwise, if $\bar{x}_e \geq 1$ (resp. $\bar{y}_e \geq 1$) for some edge $e$, then (A.1)-(A.3) would still be satisfied after resetting $\bar{x}_e := \bar{x}_e - 1$ and $c_e := c_e - 1$ (resp. $\bar{y}_e := \bar{y}_e - 1$ and $c_e := c_e + 1$).

By (A.4) and by construction of series-parallel graphs, there are two edges $\bar{e}$ and $\bar{f}$ in series. We may assume w.l.o.g. that $\bar{b}_{\bar{e}} \leq \bar{b}_{\bar{f}}$. Since $\bar{D} = \{\bar{e}, \bar{f}\}$ is a cut, we have $\bar{b}_{\bar{f}} \geq -\bar{b}_{\bar{e}}$. Denote by $\hat{p} = (\hat{x}, \hat{y}) \in \mathbb{R}^{E\setminus\{\bar{f}\}\times E\setminus\{\bar{f}\}}$ the restriction of $\bar{p}$ to $E\setminus\{\bar{f}\}\times E\setminus\{\bar{f}\}$, and let $\hat{G}$ be the graph obtained from $G$ by contracting $\bar{f}$, and $\hat{c}$ the restriction of $c$ to $E \setminus \{\bar{f}\}$. Clearly, $\hat{p}$ belongs to $P(\hat{G}, \hat{c})$, and the latter is full-dimensional since neither loops nor bridges appeared in $\hat{G}$.

Moreover, since $c$ is integer and $\bar{p}$ fractional, (A.3) and (A.5) imply that at least two edges have a fractional $\bar{x}$ or $\bar{y}$ coordinate. Therefore $\hat{p}$ is fractional, and hence, by minimality of $|E|$, $\hat{p}$ is not an extreme point of $P(\hat{G}, \hat{c})$.

Remark that in fact we have:

$$\bar{b}_{\bar{f}} = |\bar{b}_{\bar{e}}| \tag{A.7}$$

If it is not true, then $\bar{p}$ does not saturate the constraint associated to $\bar{D}$, and moreover $\bar{b}_{\bar{f}} > \bar{b}_{\bar{e}}$. Hence, except maybe for $\bar{x}_{\bar{f}} = 0$ or $\bar{y}_{\bar{f}} = 0$, the edge $\bar{f}$ appears in no equation among (A.1)-(A.3). Then $\hat{p}$ is an extreme point, a contradiction.

By the integrality of $c$, a direct consequence of (A.5)-(A.7) is that:

$$\text{Exactly one of } \bar{x}_{\bar{e}}, \bar{y}_{\bar{e}} \text{ is fractional} \iff \text{exactly one of } \bar{x}_{\bar{f}}, \bar{y}_{\bar{f}} \text{ is fractional} \tag{A.8}$$

Since $\hat{p}$ is not extreme, there is a (nonzero) direction $\hat{d} = (\hat{d}^x, \hat{d}^y) \in \mathbb{R}^{E\setminus\{\bar{f}\}\times E\setminus\{\bar{f}\}}$ and an $\varepsilon > 0$ such that

$$\hat{p} = \frac{1}{2}(\hat{p} + \varepsilon \cdot \hat{d}) + \frac{1}{2}(\hat{p} - \varepsilon \cdot \hat{d})$$

where both $\hat{p} + \varepsilon \cdot \hat{d}$ and $\hat{p} - \varepsilon \cdot \hat{d}$ belong to $P(\hat{G}, \hat{c})$. Extend the direction $\hat{d} = (\hat{d}^x, \hat{d}^y) \in \mathbb{R}^{E\setminus\{\bar{f}\}\times E\setminus\{\bar{f}\}}$ to a direction $\bar{d} = (\bar{d}^x, \bar{d}^y) \in \mathbb{R}^{E\times E}$ by arbitrarily defining the two missing components $\bar{d}^x_{\bar{f}}$ and $\bar{d}^y_{\bar{f}}$. So

$$\bar{p} = \frac{1}{2}(\bar{p} + \varepsilon \cdot \bar{d}) + \frac{1}{2}(\bar{p} - \varepsilon \cdot \bar{d}) \qquad \forall \varepsilon > 0$$

where the points $\bar{p}^+ = \bar{p} + \varepsilon \cdot \bar{d}$ and $\bar{p}^- = \bar{p} - \varepsilon \cdot \bar{d}$ are different. Since $\bar{p}$ is extreme, we can assume that $\bar{p}^+ = (\bar{x}^+, \bar{y}^+) \notin P(G, c)$. Clearly, we have

$$\bar{x}_{\bar{e}} = 0 \text{ (resp. } \bar{y}_{\bar{e}} = 0\text{) implies } \bar{d}^x_{\bar{e}} = 0 \text{ (resp. } \bar{d}^y_{\bar{e}} = 0\text{).} \tag{A.9}$$

Define $\bar{b}^+ := c - \bar{x}^+ + \bar{y}^+$. By (A.7), there are two cases.

*Case 1:* $\bar{b}_{\bar{e}} = \bar{b}_{\bar{f}} \geq 0$.

Define

$$\bar{d}^x_{\bar{f}} = \begin{cases} \bar{d}^x_{\bar{e}} - \bar{d}^y_{\bar{e}} & \text{if } \bar{x}_{\bar{f}} > 0 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \bar{d}^y_{\bar{f}} = \begin{cases} \bar{d}^y_{\bar{e}} - \bar{d}^x_{\bar{e}} & \text{if } \bar{y}_{\bar{f}} > 0 \\ 0 & \text{otherwise.} \end{cases}$$

By definition of $\bar{d}$, and by (A.8)-(A.9), we have

$$\bar{b}^+_{\bar{e}} - \bar{b}_{\bar{e}} = (\bar{y}^+_{\bar{e}} - \bar{y}_{\bar{e}}) - (\bar{x}^+_{\bar{e}} - \bar{x}_{\bar{e}}) = \varepsilon(\bar{d}^y_{\bar{e}} - \bar{d}^x_{\bar{e}}) = (\bar{y}^+_{\bar{f}} - \bar{y}_{\bar{f}}) - (\bar{x}^+_{\bar{f}} - \bar{x}_{\bar{f}}) = \bar{b}^+_{\bar{f}} - \bar{b}_{\bar{f}}.$$

Therefore, $\bar{b}^+_{\bar{e}} = \bar{b}^+_{\bar{f}}$. By (A.9), choosing a small enough $\varepsilon$ ensures the non-negativity of $\bar{p}^+$. Since $\bar{p}^+$ does not belong to $P(G, c)$, we get that $\bar{p}^+$ violates $x(\bar{D}) - y(\bar{D}) \leq c(\bar{D})$, that is,

$$\bar{b}^+_{\bar{e}} + \bar{b}^+_{\bar{f}} = \bar{b}_{\bar{e}} + \bar{b}_{\bar{f}} + 2\varepsilon(\bar{d}^y_{\bar{e}} - \bar{d}^x_{\bar{e}}) < 0, \qquad \forall \varepsilon > 0 \tag{A.10}$$

Notice that exactly one of $\bar{x}_{\bar{e}}$ and $\bar{y}_{\bar{e}}$ is fractional, as otherwise (A.9) would imply $\bar{d}^x_{\bar{e}} = \bar{d}^y_{\bar{e}} = 0$, and then (A.10) would give the contradiction $\bar{b}(\bar{D}) < 0$. Consequently, we have $\bar{b}_{\bar{e}} + \bar{b}_{\bar{f}} > 0$, a contradiction to the fact that (A.10) holds for all $\epsilon > 0$. This settles Case 1.

*Case 2:* $\bar{b}_{\bar{e}} = -\bar{b}_{\bar{f}} < 0$.

Define

$$\bar{d}^x_{\bar{f}} = \begin{cases} \bar{d}^y_{\bar{e}} - \bar{d}^x_{\bar{e}} & \text{if } \bar{x}_{\bar{f}} > 0 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \bar{d}^y_{\bar{f}} = \begin{cases} \bar{d}^x_{\bar{e}} - \bar{d}^y_{\bar{e}} & \text{if } \bar{y}_{\bar{f}} > 0 \\ 0 & \text{otherwise.} \end{cases}$$

By definition of $\bar{d}$, and by (A.8)-(A.9), we have $\bar{b}^+_{\bar{e}} - \bar{b}_{\bar{e}} = \varepsilon(\bar{d}^y_{\bar{e}} - \bar{d}^x_{\bar{e}}) = (\bar{x}^+_{\bar{f}} - \bar{x}_{\bar{f}}) - (\bar{y}^+_{\bar{f}} - \bar{y}_{\bar{f}}) = \bar{b}_{\bar{f}} - \bar{b}^+_{\bar{f}}$. Therefore, $\bar{b}^+_{\bar{f}} = -\bar{b}^+_{\bar{e}}$.

In particular, $\bar{p}^+$ satisfies the constraint of the cut $\bar{D}$, and since nonnegativity is ensured, then $\bar{p}^+$ violates the constraint of a cut $D$ containing $\bar{f}$ but not $\bar{e}$, that is

$$\bar{b}^+(D) = \bar{b}(D) + \varepsilon(\bar{d}^y(D) - \bar{d}^x(D)) < 0 \qquad (\forall \varepsilon > 0) \tag{A.11}$$

Since $D' = D \cup \{\bar{e}\} \setminus \{\bar{f}\}$ is a cut, we have $\bar{b}(D') \geq 0$, thus $\bar{b}(D) = \bar{b}(D') - \bar{b}_{\bar{e}} + \bar{b}_{\bar{f}} > 0$. This contradiction to (A.11) finishes the proof. $\square$

## References


[1] J. Edmonds, R. Giles, A Min-Max Relation for Submodular Functions on Graphs, Annals of Discrete Mathematics 1 (1977) 185–204. doi:10.1016/S0167-5060(08)70734-9.

[2] W. Cook, On box totally dual integral polyhedra, Mathematical Programming 34 (1) (1986) 48–61. doi:10.1007/BF01582162.

[3] A. Schrijver, Theory of linear and integer programming, John Wiley & Sons, Inc., New York, NY, USA, 1986.

[4] A. Schrijver, Combinatorial Optimization Polyhedra and Efficiency, Vol. 24, Springer, 2003.

[5] G. Ding, L. Feng, W. Zang, The complexity of recognizing linear systems with certain integrality properties, Mathematical Programming 114 (2) (2008) 321–334. doi:10.1007/s10107-007-0103-y.

[6] J. Pap, Recognizing conic TDI systems is hard, Mathematical Programming 128 (1) (2011) 43–48. doi:10.1007/s10107-009-0294-5.

[7] X. Chen, Z. Chen, W. Zang, A Unified Approach to Box-Mengerian Hypergraphs, Mathematics of Operations Research 35 (3) (2010) 655–668. doi:10.1287/moor.1100.0458.

[8] X. Chen, G. Ding, W. Zang, A Characterization of Box-Mengerian Matroid Ports, Mathematics of Operations Research 33 (2) (2008) 497–512. doi:10.1287/moor.1070.0306.

[9] X. Chen, G. Ding, W. Zang, The box-TDI system associated with 2-edge connected spanning subgraphs, Discrete Applied Mathematics 157 (1) (2009) 118–125. doi:10.1016/j.dam.2008.05.001.

[10] P. D. Seymour, The matroids with the max-flow min-cut property, Journal of Combinatorial Theory, Series B 23 (2) (1977) 189–222. doi:10.1016/0095-8956(77)90031-4.

[11] G. Ding, L. Tan, W. Zang, When is the matching polytope box-totally dual integral?, Mathematics of Operations Research 43 (1) (2018) 64–99. doi:10.1287/moor.2017.0852.

[12] P. D. Seymour, Sums of circuits, Graph theory and related topics 1 (1979) 341–355.

[13] F. Barahona, A. R. Mahjoub, On the cut polytope, Mathematical Programming 36 (2) (1986) 157–173. doi:10.1007/BF02592023.

[14] P. D. Seymour, Matroids and Multicommodity Flows, European Journal of Combinatorics 2 (3) (1981) 257–290. doi:10.1016/S0195-6698(81)80033-9.

[15] J. Edmonds, E. L. Johnson, Matching, Euler tours and the Chinese postman, Mathematical Programming 5 (1) (1973) 88–124. doi:10.1007/BF01580113.

[16] D. Cornaz, Max-multiflow/min-multicut for G+H series-parallel, Discrete Mathematics 311 (17) (2011) 1957–1967. doi:10.1016/j.disc.2011.05.025.

[17] R. J. Duffin, Topology of series-parallel networks, Journal of Mathematical Analysis and Applications 10 (2) (1965) 303–318. doi:10.1016/0022-247X(65)90125-3.

[18] A. Sebő, The Schrijver system of odd join polyhedra, Combinatorica 8 (1) (1988) 103–116. doi:10.1007/BF02122558.

[19] S. Chopra, The Graph Partitioning Polytope on Series-Parallel and 4-Wheel Free Graphs, SIAM Journal on Discrete Mathematics 7 (1) (1994) 16–31. doi:10.1137/S0895480191199415.

[20] M. Grötschel, L. Lovász, A. Schrijver, The ellipsoid method and its consequences in combinatorial optimization, Combinatorica 1 (2) (1981) 169–197. doi:10.1007/BF02579273.

[21] W.-K. Shih, S. Wu, Y. S. Kuo, Unifying Maximum Cut and Minimum Cut of a Planar Graph, IEEE Trans. Comput. 39 (5) (1990) 694–697. doi:10.1109/12.53581.