On k-edge-connected Polyhedra: Box-TDIness in Series-parallel Graphs

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Abstract

Given a connected graph $G = (V, E)$ and an integer $k \geq 1$, the connected graph $H = (V, F)$, where F is a family of elements of E, is a k-edge-connected spanning subgraph of G if H remains connected after the removal of any $k-1$ edges. The convex hull of the k-edge-connected subgraphs of a graph G forms the k-edge-connected subgraph polyhedron of G. We prove that this polyhedron is box-totally dual integral if and only if G is series-parallel.

Introduction

Totally dual integral systems—introduced in the late 70's—are strongly connected to min-max relations in combinatorial optimization (see [30]). A rational system of linear inequalities $Ax \geq b$ is *totally dual integral (TDI)* if the maximization problem in the linear programming duality:

$$\min\{c^\top x : Ax \geq b\} = \max\{b^\top y : A^\top y = c, y \geq \mathbf{0}\}$$

admits an integer optimal solution for each integer vector c such that the optimum is finite. Every rational polyhedron can be described by a TDI system (see [24]). For instance, $\frac{1}{q}Ax \geq \frac{1}{q}b$ is TDI for some positive q. However, only integer polyhedra can be described by TDI systems with integer right-hand side (see [19]). TDI systems with only integer coefficients yield min-max results that have combinatorial interpretation.

A stronger property is the box-total dual integrality, where a system $Ax \geq b$ is *box-totally dual integral (box-TDI)* if $Ax \geq b, \ell \leq x \leq u$ is TDI for all rational vectors ℓ and u (possibly with infinite components). General properties of such systems can be found in [10] and Chapter 22.4 of [30]. Note that, although every rational polyhedron $\{x : Ax \geq b\}$ can be described by a TDI system, not every polyhedron can be described by a box-TDI system. A polyhedron which can be described by a box-TDI system is called a *box-TDI polyhedron.* As proved by [10], every TDI system describing such a polyhedron is actually box-TDI.

Recently, several new box-TDI systems were exhibited. [5] characterized box-Mengerian matroid ports. [16] characterized the graphs for which the TDI system of [14] describing the matching polytope is actually box-TDI. [17] introduced new subclasses of box-perfect graphs.

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[11] provided several box-TDI systems in series-parallel graphs. For these graphs, [3] gave the box-TDI system for the flow cone having integer coefficients and the minimum number of constraints. [6] provided a box-TDI system describing the 2-edge-connected spanning subgraph polyhedron for the same class of graphs.

In this paper, we are interested in integrality properties of systems related to k-edge-connected spanning subgraphs. Given a positive integer k, a *k-edge-connected spanning subgraph* of a connected graph $G = (V, E)$ is a connected graph $H = (V, F)$, with F a family of elements of E, that remains connected after the removal of any $k - 1$ edges.

These objects model a kind of failure resistance of telecommunication networks. More precisely, they represent networks which remain connected when $k - 1$ links fail. The underlying network design problem is the *k-edge-connected spanning subgraph problem* (*k-ECSSP*): given a graph G, and positive edge costs, find a k-edge-connected spanning subgraph of G of minimum cost. Special cases of this problem are related to classic combinatorial optimization problems. The 2-ECSSP is a well-studied relaxation of the traveling salesman problem (see [20]) and the 1-ECSSP is nothing but the well-known minimum spanning tree problem. While this latter is polynomial-time solvable, the k-ECSSP is **NP**-hard for every fixed $k \geq 2$ (see [23]).

Different algorithms have been devised in order to deal with the k-ECSSP. Notable examples are branch-and-cut procedures [12], approximation algorithms [22]. Cutting plane algorithms [26], and heuristics [9]. [32], introduced a linear-time algorithm solving the 2-ECSSP on series-parallel graphs. Most of these algorithms rely on polyhedral considerations.

The *k-edge-connected spanning subgraph polyhedron* of G, hereafter denoted by $P_k(G)$, is the convex hull of all the k-edge-connected spanning subgraphs of G. [13] gave a system describing $P_2(G)$ for series-parallel graphs. [31] characterized in terms of forbidden minors the graphs for which this system describes $P_2(G)$. [8] described $P_k(G)$ for outerplanar graphs when k is odd. [15] extended these results to series-parallel graphs for all $k \geq 2$. By a result of [1], the inequalities in these descriptions can be separated in polynomial time, which implies that the k-ECSSP is solvable in polynomial time for series-parallel graphs.

When studying the k-edge-connected spanning subgraphs of a graph G, we can add the constraint that each edge of G can be taken at most once. We denote the corresponding polyhedron by $Q_k(G)$. [2] described $Q_2(G)$ for Halin graphs. Further polyhedral results for the case $k = 2$ have been obtained by [4], [28], and [29]. [25] described several basic facets of $Q_k(G)$. Moreover, [21] extensively studied the extremal points of $Q_k(G)$ and characterized the class of graphs for which this polytope is described by cut inequalities and $\mathbf{0} \leq x \leq \mathbf{1}$.

The polyhedron $P_1(G)$ is known to be box-TDI for all graphs (see [27]). For series-parallel graphs, the system given in [13] describing $P_2(G)$ is not TDI. [6] showed that dividing each inequality by 2 yields a TDI system for such graphs. Actually, they proved that this system is box-TDI if and only if the graph is series-parallel.

Contribution. Our starting point is the result of [6]. First, their result implies that $P_2(G)$ is a box-TDI polyhedron for series-parallel graphs. However, this leaves open the question of the box-TDIness of $P_2(G)$ for non series-parallel graphs. More generally, for which integers k and graphs G is $P_k(G)$ a box-TDI polyhedron? In this paper, we answer this question and prove that, for $k \geq 2$, $P_k(G)$ is a box-TDI polyhedron if and only if G is series-parallel.

1 Definitions and Preliminary Results

This section is devoted to the definitions, notation, and preliminary results used throughout the paper.

1.1 Graphs

Let $G = (V, E)$ be a loopless undirected graph. The graph G is 2-connected if it remains connected whenever a vertex is removed. A 2-connected graph is called *trivial* if it is composed of a single edge. The graph obtained from two disjoint graphs by identifying two vertices, one of each graph, is called a *1-sum*. A subset of edges of G is called a *circuit* if it induces a connected graph in which every vertex has degree 2. Given a subset U of V, the *cut* $\delta(U)$ is the set of edges having exactly one endpoint in U. A *bond* is a minimal nonempty cut. Given a partition $\{V_1, \ldots, V_n\}$ of V, the set of edges having endpoints in two distinct V_i's is called *multicut* and is denoted by $\delta(V_1, \ldots, V_n)$. We denote respectively by $\mathcal{M}_G$ and $\mathcal{B}_G$ the set of multicuts and the set of bonds of G. For every multicut M, there exists a unique partition $\{V_1, \ldots, V_{d_M}\}$ of vertices of V such that $M = \delta(V_1, \ldots, V_{d_M})$, and $G[V_i]$ – the graph induced by the vertices of V_i – is connected for all $i = 1, \ldots, d_M$; we say that d_M is the *order* of M.

We denote the symmetric difference of two sets S and T by $S\Delta T$. It is well-known that the symmetric difference of two cuts is a cut.

We denote by K_n the complete graph on n vertices, that is the simple graph with n vertices and one edge between each pair of distinct vertices.

A graph is *series-parallel* if its 2-connected components can be constructed from an edge by repeatedly adding edges parallel to an existing one, and subdividing edges, that is, replacing an edge by a path of length two. [18] showed that series-parallel graphs are those having no K_4-minor. By construction, simple nontrivial 2-connected series-parallel graphs have least one vertex of degree 2.

Proposition 1.1. *For a simple nontrivial 2-connected series-parallel graph, at least one of the following holds:*

(a) two vertices of degree 2 are adjacent,

(b) a vertex of degree 2 belongs to a circuit of length 3,

(c) two vertices of degree 2 belong to a same circuit of length 4.

Proof. We proceed by induction on the number of edges. The base case is K_3 for which (a) holds.

Let G be a simple 2-connected series-parallel graph such that for every simple, 2-connected series-parallel graph with fewer edges at least one among (a), (b), and (c) holds. Since G is simple, it can be built from a graph H by subdividing an edge e into a path f, g. Let v be the vertex of degree 2 added with this operation. By the induction hypothesis, either H is not simple, or one among (a), (b), and (c) holds for H.

Let fist suppose that H is not simple, then, by G being simple, e is parallel to exactly one edge e_0. Hence, e_0, f, g is a circuit of G length 3 containing v, hence (b) holds for G.

From now on, suppose that H is simple. If (a) holds for H, then it holds for G.

Suppose that (b) holds for H, that is, in H there exists a circuit C of length 3 containing a vertex w of degree 2. Without loss of generality, we suppose that $e \in C$, as otherwise (b) holds for G. By subdividing e, we obtain a circuit of length 4 containing v and w, and hence (c) holds for G.

At last, suppose that (c) holds for H, that is, H has a circuit C of length 4 containing two vertices of degree 2. Without loss of generality, we suppose that $e \in C$, as otherwise (c) holds for G. By subdividing e, we obtain a circuit of length 5 containing three vertices of degree 2. Then, at least two of them are adjacent, and so (a) holds for G. ■

1.2 Box-Total Dual Integrality

Let $A \in \mathbb{R}^{m \times n}$ be a full row rank matrix. This matrix is *equimodular* if all its $m \times m$ non-zero determinants have the same absolute value. The matrix A is *face-defining for* a face F of a polyhedron $P \subseteq \mathbb{R}^n$ if $\text{aff}(F) = \{x \in \mathbb{R}^n : Ax = b\}$ for some $b \in \mathbb{R}^m$. Such matrices are the *face-defining matrices of* P.

Theorem 1.2 ([7]). *Let P be a polyhedron, then the following statements are equivalent:*

(i) P is box-TDI.

(ii) Every face-defining matrix of P is equimodular.

(iii) Every face of P has an equimodular face-defining matrix.

The equivalence of conditions *(ii)* and *(iii)* stems from the following observation.

Observation 1.3 ([7]). *Let F be a face of a polyhedron. If a face-defining matrix of F is equimodular, then so are all face-defining matrices of F.*

Observation 1.4. *Let $A \in \mathbb{R}^{I \times J}$ be a full row rank matrix, $j \in J$, $\mathbf{c}$ be a column of A, and $\mathbf{v} \in \mathbb{R}^I$. If A is equimodular, then so are:*
(i) $\begin{bmatrix} A & \mathbf{c} \end{bmatrix}$, (ii) $\begin{bmatrix} A \\ \pm\chi^j \end{bmatrix}$ *if it is full row rank,* (iii) $\begin{bmatrix} A & \mathbf{v} \\ \mathbf{0}^\top & \pm 1 \end{bmatrix}$, *and* (iv) $\begin{bmatrix} A & \mathbf{0} \\ \pm\chi^j & \pm 1 \end{bmatrix}$.

Observation 1.5 ([7]). *Let $P \subseteq \mathbb{R}^n$ be a polyhedron and let $F = \{x \in P : Bx = b\}$ be a face of P. If B has full row rank and $n - \dim(F)$ rows, then B is face-defining for F.*

1.3 k-edge-connected Spanning Subgraph Polyhedron

The *dominant* of a polyhedron P is $\text{dom}(P) = \{x : x = y + z, \text{ for } y \in P \text{ and } z \geq \mathbf{0}\}$. Note that $P_k(G)$ is the dominant of the convex hull of all k-edge-connected spanning subgraphs of G that have each edge taken at most k times. Since the dominant of a polyhedron is a polyhedron, $P_k(G)$ is a polyhedron even though it is the convex hull of an infinite number of points.

From now on, $k \geq 2$. [15] gave a complete description of $P_k(G)$ for all k, when G is series-parallel.

Theorem 1.6. *Let G be a series-parallel graph and k be a positive integer. Then, when k is even, $P_k(G)$ is described by:*

$$(1) \qquad \begin{cases} x(D) \geq k & \text{for all cuts } D \text{ of } G, & \text{(1a)} \\ x \geq \mathbf{0}, & & \text{(1b)} \end{cases}$$

and, when k is odd, $P_k(G)$ is described by:

$$(2)\quad \begin{cases} x(M) \geq \dfrac{k+1}{2} d_M - 1 & \text{for all multicuts } M \text{ of } G, \quad (2a) \\ x \geq \mathbf{0}. & (2b) \end{cases}$$

The incidence vector of a family F of E is the vector χ^F of $\mathbb{Z}^E$ such that e's coordinate is the multiplicity of e in F for all e in E. Since there is a bijection between families and their incidence vectors, we will often use the same terminology for both. Since the incidence vector of a multicut $\delta(V_1, \ldots, V_{d_M})$ is the half-sum of the incidence vectors of the bonds $\delta(V_1), \ldots, \delta(V_{d_M})$, we can deduce an alternative description of $P_{2h}(G)$.

Corollary 1.7. *Let G be a series-parallel graph and k be a positive even integer. Then $P_k(G)$ is described by:*

$$(3)\quad \begin{cases} x(M) \geq \dfrac{k}{2} d_M & \text{for all multicuts } M \text{ of } G, \quad (3a) \\ x \geq \mathbf{0}. & (3b) \end{cases}$$

We call constraints (2a) and (3a) *partition constraints.* A multicut M is *tight for a point* of $P_k(G)$ if this point satisfies with equality the partition constraint (2a) (resp. (3a)) associated with M when k is odd (resp. even). Moreover, M is *tight for a face* F of $P_k(G)$ if it is tight for all the points of F.

The following results give some insight on the structure of tight multicuts.

Theorem 1.8 ([15])**.** *Let $k > 1$ be odd, let x be a point of $P_k(G)$, and let $M = \delta(V_1, \ldots, V_{d_M})$ be a tight multicut for x. Then, the following hold:*

(i) if $d_M \geq 3$, then $x(\delta(V_i) \cap \delta(V_j)) \leq \frac{k+1}{2}$ for all $i \neq j \in \{1, \ldots, d_M\}$.

(ii) $G \setminus V_i$ is connected for all $i = 1, \ldots, d_M$.

Observation 1.9. *Let M be a multicut of G strictly containing $\delta(v) = \{f, g\}$. If M is tight for a point of $P_k(G)$, then both $M \setminus f$ and $M \setminus g$ are multicuts of G of order $d_M - 1$.*

[8] gave sufficient conditions for an inequality to be facet defining. The following proposition is a direct consequence of [8, Theorem 2.4].

Proposition 1.10. *Let G be a graph having K_4 as a minor and let $k > 1$ be an odd integer. Then, there exist two disjoint nonempty subsets of edges of G, E' and E'', and a rational b such that*

$$\chi^{E'} + 2\chi^{E''} \geq b, \quad (4)$$

is a facet-defining inequality of $P_k(G)$.

[6] provided a box-TDI system for $P_2(G)$ for series-parallel graphs.

Theorem 1.11 ([6])**.** *The system:*

$$\begin{cases} \frac{1}{2}x(D) \geq 1 & \text{for all cuts } D \text{ of } G, \\ x \geq \mathbf{0} \end{cases} \quad (5)$$

is box-TDI if and only if G is a series-parallel graph.

This result proves that $P_2(G)$ is box-TDI for all series-parallel graphs, and gives a TDI system describing this polyhedron in this case. At the same time, Theorem 1.11 is not sufficient to state that $P_2(G)$ is a box-TDI polyhedron if and only if G is series-parallel.

2 Box-TDIness of $P_k(G)$

In this section we show that, for $k \geq 2$, $P_k(G)$ is a box-TDI polyhedron if and only if G is series-parallel.

When $k \geq 2$, $P_k(G)$ is not box-TDI for all graphs as stated by the following lemma.

Lemma 2.1. *For $k \geq 2$, if $G = (V, E)$ contains a K_4-minor, then $P_k(G)$ is not box-TDI.*

Proof. When k is odd, Proposition 1.10 shows that there exists a facet-defining inequality that is described by a non equimodular matrix. Thus, $P_k(G)$ is not box-TDI by Statement (ii) of Theorem 1.2.

We now prove the case when k is even. Since G is connected and has a K_4-minor, there exists a partition $\{V_1, \ldots, V_4\}$ of V such that $G[V_i]$ is connected and $\delta(V_i, V_j) \neq \emptyset$ for all $i < j \in \{1, \ldots, 4\}$. We prove that the matrix T whose three rows are $\chi^{\delta(V_i)}$ for $i = 1, 2, 3$ is a face-defining matrix for $P_k(G)$ which is not equimodular. This will end the proof by Statement (ii) of Theorem 1.2.

Let e_{ij} be an edge in $\delta(V_i, V_j)$ for all $i < j \in \{1, \ldots, 4\}$. The submatrix of T formed by the columns associated with edges e_{ij} is the following:

$$\begin{array}{c} \\ \chi^{\delta(V_1)} \\ \chi^{\delta(V_2)} \\ \chi^{\delta(V_3)} \end{array} \begin{array}{c} \begin{array}{cccccc} e_{12} & e_{13} & e_{23} & e_{14} & e_{24} & e_{34} \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

The matrix T is not equimodular as the first three columns form a matrix of determinant -2 whereas the last three ones have determinant 1.

To show that T is face-defining, we exhibit $|E| - 2$ affinely independent points of $P_k(G)$ satisfying the partition constraint (3a) associated with the multicut $\delta(V_i)$, that is $x(\delta(V_i)) = k$, for $i = 1, 2, 3$.

Let $D_1 = \{e_{12}, e_{14}, e_{23}, e_{34}\}$, $D_2 = \{e_{12}, e_{13}, e_{24}, e_{34}\}$, $D_3 = \{e_{13}, e_{14}, e_{23}, e_{24}\}$ and $D_4 = \{e_{14}, e_{24}, e_{34}\}$. First, we define the points $S_j = \sum_{i=1}^{4} k\chi^{E[V_i]} + \frac{k}{2}\chi^{D_j}$, for $j = 1, 2, 3$, and $S_4 = \sum_{i=1}^{4} k\chi^{E[V_i]} + k\chi^{D_4}$. Note that they are affinely independent.

Now, for each edge $e \notin \{e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}\}$, we construct the point S_e as follows. When $e \in E[V_i]$ for some $i = 1, \ldots, 4$, we define $S_e = S_4 + \chi^e$. Adding the point S_e maintains affine independence as S_e is the only point not satisfying $x_e = k$. When $e \in \delta(V_i, V_j)$ for some i, j, we define $S_e = S_\ell - \chi^{e_{ij}} + \chi^e$, where S_ℓ is S_1 if $e \in \delta(V_1, V_4) \cup \delta(V_2, V_3)$ and S_2 otherwise. Affine independence comes because S_e is the only point involving e. ■

Theorem 2.2. *For $k \geq 2$, $P_k(G)$ is a box-TDI polyhedron if and only if G is series-parallel.*

Proof. Necessity stems from Lemma 2.1. Let us now prove sufficiency. When $k = 2$, the box-TDIness of System (5) has been shown by [6]. This implies box-TDIness for all even k: multiplying the right-hand side of a box-TDI system by a positive rational preserves its box-TDIness (see [30, Section 22.5]). The system obtained by multiplying the right-hand side of System (5) by $\frac{k}{2}$ describes $P_k(G)$ when k is even. Hence, the latter is a box-TDI polyhedron.

The rest of the proof is dedicated to the case where $k = 2h + 1$ for some $h \geq 1$. For this purpose, we prove that every face of $P_{2h+1}(G)$ admits an equimodular face-defining matrix. The characterization of box-TDIness given in Theorem 1.2 concludes. We proceed by induction on the number of edges of G.

As a base-case of the induction we consider the series-parallel graph G consisting of two vertices connected by a single edge. Then, $P_{2h+1}(G) = \{x \in \mathbb{R}_+ : x \geq 2h+1\}$ is box-TDI.

(1-sum) Let G be the 1-sum of two series-parallel graphs $G^1 = (W^1, E^1)$ and $G^2 = (W^2, E^2)$. By induction, there exist two box-TDI systems $A^1 y \geq b^1$ and $A^2 z \geq b^2$ describing respectively $P_{2h+1}(G^1)$ and $P_{2h+1}(G^2)$. If v is the vertex of G obtained by the identification, $G \setminus v$ is not connected, hence, by Statement (ii) of Theorem 1.8, a multicut M of G is tight for a face of $P_{2h+1}(G)$ only if $M \subseteq E^i$ for some $i = 1, 2$. It follows that for every face F of $P_{2h+1}(G)$ there exist two faces F^1 and F^2 of $P_{2h+1}(G^1)$ and $P_{2h+1}(G^2)$ respectively, such that $F = F^1 \times F^2$. Then $P_{2h+1}(G) = \{(y,z) \in \mathbb{R}_+^{E^1} \times \mathbb{R}_+^{E^2} : A^1 y \geq b^1, A^2 z \geq b^2\}$ and so it is box-TDI.

(Parallelization) Let now G be obtained from a series-parallel graph H by adding an edge g parallel to an edge f of H and suppose that $P_{2h+1}(H)$ is box-TDI. Note that $P_{2h+1}(G)$ is obtained from $P_{2h+1}(H)$ by duplicating f's column and adding $x_g \geq 0$. Hence, by [6, Lemma 3.1], $P_{2h+1}(G)$ is a box-TDI polyhedron.

(Subdivision) Let $G = (V, E)$ be obtained by subdividing an edge uw of a series-parallel graph $G' = (V', E')$ into a path of length two uv, vw. By contradiction, suppose there exists a non-empty face $F = \{x \in P_{2h+1}(G) : A_F x = b_F\}$ such that A_F is a face-defining matrix of F which is not equimodular. Take such a face with maximum dimension. Then, every face-defining submatrix of A_F is equimodular. We may assume that A_F is given by the left-hand side of a subset of constraints of System (2). We denote by $\mathcal{M}_F$ the set of multicuts associated with the left-hand sides of constraints (2a) appearing in A_F, and by $\mathcal{E}_F$ the set of edges associated with the nonnegativity constraints (2b) appearing in A_F.

Claim 2.2.1. *$\mathcal{E}_F = \emptyset$.*

Proof. Suppose there exists an edge $e \in \mathcal{E}_F$. Let $H = G \setminus e$ and let $A_{F_H} x = b_{F_H}$ be the system obtained from $A_F x = b_F$ by removing the column and the nonnegativity constraint associated with e. The matrix A_F being of full row rank, so is A_{F_H}. Since $M \setminus e$ is a multicut of H for all M in $\mathcal{M}_F$, the set $F_H = \{x \in P_{2h+1}(H) : A_{F_H} x = b_{F_H}\}$ is a face of $P_{2h+1}(H)$. Moreover, deleting e's coordinate of aff(F) gives aff(F_H) so A_{F_H} is face-defining for F_H. By the induction hypothesis, A_{F_H} is equimodular, and hence so is A_F by Observation 1.4-(iii). ■

Claim 2.2.2. *For all $e \in \{uv, vw\}$, at least one multicut of $\mathcal{M}_F$ different from $\delta(v)$ contains e.*

Proof. Suppose that uv belongs to no multicut of $\mathcal{M}_F$ different from $\delta(v)$.

First, suppose that $\delta(v)$ does not belong to $\mathcal{M}_F$. Then, the column of A_F associated with uv is zero. Let A'_F be the matrix obtained from A_F by removing this column. Every multicut of G not containing uv is a multicut of G' (relabelling vw by uw), so the rows of A'_F are associated with multicuts of G'. Thus, $F' = \{x \in P_k(G') : A'_F x = b_F\}$ is a face of $P_{2h+1}(G')$. Removing uv's coordinate from the points of F gives a set of points of F' of affine dimension at least $\dim(F) - 1$. Since A'_F has the same rank of A_F and one column less than A_F, then A'_F is face-defining for F' by Observation 1.5. By induction hypothesis, A'_F is equimodular, hence so is A_F.

Suppose now that $\delta(v)$ belongs to $\mathcal{M}_F$. Then, the column of A_F associated with uv has zeros in each row but $\chi^{\delta(v)}$. Let $A^\star_F x = b^\star_F$ be the system obtained from $A_F x = b_F$ by removing the row associated with $\delta(v)$. Then $F^\star = \{x \in P_k(G) : A^\star_F x = b^\star_F\}$ is a face of $P_k(G)$ of dimension $\dim(F) + 1$. Indeed, it contains F and $z + \alpha\chi^{uv}$ for every point z of F and $\alpha > 0$. Hence, $A^\star_F$ is face-defining for $F^\star$. This matrix is equimodular by the maximality assumption on F, and so is A_F by Observation 1.4-(iv). ■

Claim 2.2.3. $|M \cap \delta(v)| \neq 1$ *for every multicut* $M \in \mathcal{M}_F$.

Proof. Suppose there exists a multicut M tight for F such that $|M \cap \delta(v)| = 1$. Without loss of generality, suppose that M contains uv and not vw. Then, $F \subseteq \{x \in P_{2h+1}(G) : x_{vw} \geq x_{uv}\}$ because of the partition inequality (2a) associated with the multicut $M \Delta \delta(v)$. Moreover, the partition inequality associated with $\delta(v)$ and the integrality of $P_{2h+1}(G)$ imply $F \subseteq \{x \in P_{2h+1}(G) : x_{vw} \geq h + 1\}$. The proof is divided into two cases.

Case 1. $F \subseteq \{x \in P_{2h+1}(G) : x_{vw} = h+1\}$. We prove this case by exhibiting an equimodular face-defining matrix for F. By Observation 1.3, this implies that A_F equimodular, which contradicts the assumption on F.

Equality $x_{vw} = h + 1$ can be expressed as a linear combination of rows of $A_F x = b_F$. Let $A'_F x = b'_F$ denote the system obtained by replacing a row of $A_F x = b_F$ by $x_{vw} = h+1$ in such a way that the underlying affine space remains unchanged. Denote by $\mathcal{N}$ the set of multicuts of $\mathcal{M}_F$ containing vw but not uv. If $\mathcal{N} \neq \emptyset$, then let N be in $\mathcal{N}$. We now modify the system $A'_F x = b'_F$ by performing the following operations.

1. Every row associated with a multicut M strictly containing $\delta(v)$ is replaced by the partition constraint (2a) associated with $M \setminus vw$ set to equality.
2. Whenever $\delta(v) \in \mathcal{M}_F$, replace the row associated with $\delta(v)$ by the box constraint $x_{uv} = h$.
3. Replace every row associated with $M \in \mathcal{N} \setminus N$ by the partition constraint (2a) associated with $M \Delta \delta(v)$ set to equality.
4. Whenever $\mathcal{N} \neq \emptyset$, replace the row associated with N by the box constraint $x_{uv} = h + 1$.

These operations do not modify the underlying affine space. Indeed, in Operation 1, $M \setminus vw$ is tight for F because of Observation 1.9 and $F \subseteq \{x \in P_{2h+1}(G) : x_{vw} = h + 1\}$. Operation 2 is applied only if $F \subseteq \{x \in P_{2h+1}(G) : x_{uv} = h\}$. Operations 3 and 4 are applied only if $\mathcal{N} \neq \emptyset$, which implies that $F \subseteq \{x \in P_{2h+1}(G) : x_{uv} = h + 1\}$ because of the constraint (2a) associated with $N \Delta \delta(v)$ and $F \subseteq \{x \in P_{2h+1}(G) : x_{vw} \geq x_{uv}\}$. Note that Operations 2 and 4 cannot be applied both, hence the rank of the matrix remains unchanged.

Let $A''_F x = b''_F$ be the system obtained by removing the row $x_{vw} = h + 1$ from $A'_F x = b'_F$. By construction, $A''_F x = b''_F$ is composed of constraints (2a) set to equality and possibly $x_{uv} = h$ or $x_{uv} = h + 1$. Moreover, the column of A''_F associated with vw is zero. Let $F'' = \{x \in P_{2h+1}(G) : A''_F x = b''_F\}$. For every point z of F and $\alpha \geq 0$, $z + \alpha\chi^{vw}$ belongs to F'' because the column of A''_F associated with vw is zero, and $z + \alpha\chi^{vw} \in P_{2h+1}(G)$. This implies that $\dim(F'') \geq \dim(F) + 1$.

If F'' is a face of $P_{2h+1}(G)$, then A''_F is face-defining for F'' by Observation 1.5 and by A'_F being face-defining for F. By the maximality assumption on F, A''_F is equimodular, and hence so is A'_F by Observation 1.4-(ii).

Otherwise, by construction, $F'' = F^\star \cap \{x \in \mathbb{R}^E : x_{uv} = t\}$ where $F^\star$ is a face of $P_{2h+1}(G)$ strictly containing F and $t \in \{h, h+1\}$. Therefore, there exists a face-defining matrix of F'' given by a face-defining matrix of $F^\star$ and the row χ^{uv}. Such a matrix is equimodular by the maximality assumption of F and Observation 1.4-(ii). Hence, A''_F is equimodular by Observation 1.3, and so is A'_F by Observation 1.4-(ii).

Case 2. $F \not\subseteq \{x \in P_{2h+1}(G) : x_{vw} = h+1\}$. Thus, there exists $z \in F$ such that $z_{vw} > h+1$. By Claim 2.2.2, there exists a multicut $N \neq \delta(v)$ containing vw which is tight for F. By Statement (i) of Theorem 1.8, the existence of z implies that N is a bond. Thus, $uv \notin N$ and $F \subseteq \{x \in P_{2h+1}(G) : x_{vw} = x_{uv}\}$. Consequently, $L = N\Delta\delta(v)$ is also a bond tight for F. Moreover, N is the unique multicut tight for F containing vw. Suppose indeed that there exists a multicut B containing vw tight for F. Then, B is a bond by Statement (i) of Theorem 1.8 and the existence of z. Moreover, $B\Delta N$ is a multicut not containing vw. This implies that no point x of F satisfies the partition constraint associated with $B\Delta N$ because $x(B\Delta N) = x(B) + x(N) - 2x(B \cap N) = 2(2h+1) - 2x(B \cap N) \leq 4h + 2 - 2x_e \leq 2h$, a contradiction.

Consider the matrix $A^\star_F$ obtained from A_F by removing the row associated with N. Matrix $A^\star_F$ is a face-defining matrix for a face $F^\star \supseteq F$ of $P_{2h+1}(G)$ because $F^\star$ contains F and $z + \alpha\chi^{uv}$ for every point z of F and $\alpha > 0$. By the maximality assumption, the matrix $A^\star_F$ is equimodular. Let B_F be the matrix obtained from A_F by replacing the row χ^N by the row $\chi^N - \chi^L$. Then, B_F is face-defining for F. Moreover, B_F is equimodular by Observation 1.4-(iv) — a contradiction. ■

Let $A'_F x = b'_F$ be the system obtained from $A_F x = b_F$ by removing uv's column from A_F and subtracting $h+1$ times this column to b_F. We now show that $\{x \in P_{2h+1}(G') : A'_F x = b'_F\}$ is a face of $P_{2h+1}(G')$ if $\delta(v) \notin \mathcal{M}_F$, and $P_{2h+1}(G') \cap \{x : x_{uw} = h\}$ otherwise. Indeed, consider a multicut M in $\mathcal{M}_F$. If $M = \delta(v)$, then the row of $A'_F x = b'_F$ induced by M is nothing but $x_{uw} = h$. Otherwise, by Observation 1.9 and Claim 2.2.3, the set $M \setminus uv$ is a multicut of G' (relabelling vw by uw) of order d_M if $uv \notin M$ and $d_M - 1$ otherwise. Thus, the row of $A'_F x = b'_F$ induced by M is the partition constraint (2a) associated with $M \setminus uv$ set to equality.

By construction, A'_F has full row rank and one column less than A_F. We prove that A'_F is face-defining by exhibiting $\dim(F)$ affinely independent points of $P_{2h+1}(G')$ satisfying $A'_F x = b'_F$. Because of the integrality of $P_{2h+1}(G)$, there exist $n = \dim(F) + 1$ affinely independent integer points $z^1, \ldots, z^n$ of F. By Claim 2.2.3, every multicut in $\mathcal{M}_F$ contains either both uv and vw or none of them. Then, Claim 2.2.2 and Statement (i) of Theorem 1.8 imply that $F \subseteq \{x \in \mathbb{R}^E : x_{uv} \leq h+1, x_{vw} \leq h+1\}$. Combined with the partition inequality $x_{uv} + x_{vw} \geq 2h+1$ associated with $\delta(v)$, this implies that at least one of z^i_{uv} and z^i_{vw} is equal to $h+1$ for $i = 1, \ldots, n$. Since exchanging the uv and vw coordinates of any point of F gives a point of F by Claim 2.2.3, the hypotheses on $z^1, \ldots, z^n$ are preserved under the assumption that $z^i_{uv} = h+1$ for $i = 1, \ldots, n-1$. Let $y^1, \ldots, y^{n-1}$ be the points obtained from $z^1, \ldots, z^{n-1}$ by removing uv's coordinate. Since every multicut of G' is a multicut of G with the same order, $y^1, \ldots, y^{n-1}$ belong to $P_{2h+1}(G')$. By construction, they satisfy $A'_F x = b'_F$ so they belong to a face of $P_{2h+1}(G')$ or $P_{2h+1}(G') \cap \{x : x_{uw} = h\}$. This implies that A'_F is a face-defining matrix of $P_{2h+1}(G')$ if $\delta(v) \notin \mathcal{M}_F$, and $P_{2h+1}(G') \cap \{x : x_{uw} = h\}$ otherwise.

By induction, $P_{2h+1}(G')$ is a box-TDI polyhedron and hence so is $P_{2h+1}(G') \cap \{x : x_{uw} = h\}$. Hence, A'_F is equimodular by Theorem 1.2. Since the columns of A_F associated with uv and vw are equal, Observation 1.4-(i) implies that A_F is equimodular — a contradiction to its assumption of non-equimodularity. ■

3 Conclusions

In this paper, we studied strong integrality properties of the k-edge-connected spanning subgraph polyhedron, $P_k(G)$. We first showed that, for every $k \geq 2$, $P_k(G)$ is a box-TDI polyhedron if and only if G is a series-parallel graph. This result extends and strengthens the work of [6], who provided a box-TDI system when $k = 2$. When G is series-parallel and k is even, the box-total dual integrality of $P_k(G)$ stems from their result. For k odd, we used a different approach, which relies on the recent characterization of box-TDI polyhedra given in [7].

Further, we mention that, for series-parallel graphs, Theorem 2.2 implies that $Q_k(G)$ is a box-TDI polytope.

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