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On the complexity of the Eulerian closed walk with precedence path constraints problem

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ABSTRACT

The Eulerian closed walk problem in a digraph is a well-known polynomial-time solvable problem. In this paper, we show that if we impose the feasible solutions to fulfill some precedence constraints specified by paths of the digraph, then the problem becomes NP-complete. We also present a polynomial-time algorithm to solve this variant of the Eulerian closed walk problem when the set of paths does not contain some forbidden structure. This allows us to give necessary and sufficient conditions for the existence of feasible solutions in this polynomial-time solvable case.

1. Introduction

The Eulerian closed walk problem is one of the most famous problems in graph theory. It can be presented as follows. Let $D = (V, A)$ be a digraph. A *walk* of D is a sequence $P = (a_1, \dots, a_k)$ of arcs of A with $k \geq 1$, $a_i = (u_i, v_i)$ for all $i = 1, \dots, k$ and $v_i = u_{i+1}$ for $i = 1, \dots, k-1$. A walk having no vertex appearing more than once is called a *path*. If $P = (a_1, \dots, a_k)$ is a walk of D, with $a_1 = (u_1, v_1)$ and $a_k = (u_k, v_k)$, then P is said to be of *length* k. The vertex u_1 (respectively v_k) is called the *starting* (respectively *end*) *vertex* of P; both vertices u_1 and v_k are the *extremities* of P. A *closed walk* is a walk having $v_k = u_1$. A closed walk P is *Eulerian* if each arc of D appears exactly once in P. The *Eulerian Closed Walk Problem (ECWP)* consists of finding an Eulerian closed walk in D. Digraphs having an Eulerian closed walk are called *Eulerian*. Such digraphs are connected and for each vertex, its indegree and outdegree are equal. From the description of Eulerian digraphs, it is easy to see that given an Eulerian digraph D, the ECWP can be solved in linear time.

In this paper, we consider a variant of the ECWP where the Eulerian closed walk has a fixed starting vertex and must fulfill some precedence constraints on the arcs, specified by a partial order on arcs, the latter being defined by a set of paths. Before stating the problem, we introduce some notation. Given a walk P and two distinct arcs a, a' of A, we write $a \prec_P a'$ if a and a' belong to P and a precedes a' in P, that is, if P traverses a before a'. Moreover if we consider a path $Q = (a_1, a_2, \dots, a_k)$ of D, $k \geq 1$, we say that the walk P *respects* the path Q if, for all $a, a' \in A$, $a \neq a'$,

$$a \prec_Q a' \quad \text{and} \quad a, a' \in P \Rightarrow a \prec_P a'.$$

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Note that two adjacent arcs of Q are not necessarily adjacent in P. It is straightforward to see that if a closed walk P is considered, then the ordering relation $\prec_p$ makes sense only if a specific vertex is chosen as the starting vertex of P. This specification of the starting vertex will be omitted wherever this information is obvious.

We now precisely define the problem we consider hereafter. Let $D = (V, A)$ be a loopless Eulerian digraph and $v_0 \in V$ be a specified vertex. (Note that D is not necessarily simple, that is, it may have multiple arcs.) The required partial order on A is given by a set $K = \{Q_1, Q_2, \ldots, Q_q\}$ of paths of D, $q \geq 1$. The *Eulerian Closed Walk with Precedence Path Constraints Problem (ECWPPCP)* consists of finding an Eulerian closed walk P of D whose starting vertex is v_0 and which respects all the paths of K, that is, for $i = 1, 2, \ldots, q$, if $a \prec_{Q_i} a'$, then $a \prec_P a'$, for all $a \neq a' \in A$. An instance of the ECWPPCP then is defined by the (ordered) triple (D, v_0, K). From the definition of the precedence constraints, a path composed of only one arc does not induce any precedence constraint. Therefore, in the rest of the paper, we will only consider in K paths of at least two arcs.

The aim of this paper is to study the complexity of the ECWPPCP. We show that the ECWPPCP is NP-complete in general. We also present a polynomial-time algorithm to solve it when the set of paths does not contain a forbidden structure. This allows us to give necessary and sufficient conditions for the existence of feasible solutions in this polynomial-time solvable case.

Studying the ECWPPCP was originally motivated by the so-called *Single-vehicle Preemptive Pickup and Delivery Problem (SPPDP)* [1,2]. In this vehicle routing problem with a single vehicle having limited capacity, each demand may be temporarily unloaded elsewhere than its destination and picked up later, generating what is called a *reload*. This *preemptive* variant of the Single-vehicle Pickup and Delivery Problem (SPDP) [3] aims to achieve some significant transportation cost savings, since it tends to decrease the distance traveled by the vehicle.

A common way of representing a feasible solution to a vehicle routing problem consists of specifying, on one hand, the sequence of arcs of the vehicle route (that is, of a walk), and on the other hand, the sets of arcs of the demand paths. Moreover, for the (classical) pickup and delivery problem, it is well-known that every vertex of the given digraph is visited exactly once by the vehicle. Consequently, the set of arcs traversed by the vehicle is sufficient to represent a feasible solution to the SPDP. Unfortunately, this property does not hold for our preemptive version of the SPDP; a vertex may now be visited more than once and then, the demand paths are no more implicitly given by the vehicle route.

For the single-vehicle preemptive pickup and delivery problem, some synchronization constraints have to be taken into account in order to deal with the reloads, that is, with vertices visited more than once. In terms of representing a feasible solution to the SPPDP, one might consider the sequence (rather than the set) of arcs of the vehicle route. Kerivin et al. [1] proved that this additional information is not sufficient to guarantee the feasibility of the vehicle route with respect to the reloads; one then needs to also consider the set of arcs of the demand paths in a representation of a feasible solution to the SPPDP. To avoid carrying too much information (and then variables when formulating the SPPDP as a mixed-integer linear program), a natural question that may be posed is whether or not one can get rid of the sequence of arcs of the vehicle route, and therefore only represent a feasible solution to the SPPDP by the sets of arcs of the vehicle route and the demand paths. In other words, can one find, in polynomial time, the sequence of arcs of the vehicle route (satisfying the reloads) from the knowledge of its set of arcs and the paths the demands are carried along? Since the vehicle starts and finishes its route at a specified depot, this is exactly the Eulerian closed walk with precedence path constraints problem when D is the digraph induced by the set of arcs of the vehicle route, v_0 is the vertex representing the depot of the vehicle, and each path in K corresponds to a demand path.

To the best of our knowledge, the ECWPPCP has not been considered yet. Nevertheless, some variants of the Eulerian closed walk problem have been already considered, three of them being related to the so-called DNA Fragment Assembly Problem (DFAP) and de Bruijn graphs [4–6].

Pevzner et al. [4] actually considered the *Eulerian Superpath Problem (ESP)* which has almost the same input as the ECWPPCP (that is, digraph D, starting vertex v_0 and set K of paths), yet seeks an Eulerian closed walk starting at v_0 and having all the paths specified in K as subpaths. A main difference lies in the definition of the elements of K; in the ESP, each path in K is specified by a sequence of adjacent vertices, whereas a sequence of incident arcs is used to specify each path in K in the ECWPPCP. Pevzner et al. [4] proved that the ESP is NP-complete by reducing the DFAP, known for being NP-hard [7], to it. They also pointed out that the ESP can be solved in polynomial time whenever the digraph D is simple. Note that despite looking alike, the ESP and ECWPPCP are quite different; for instance, as we will see in Section 2, the ECWPPCP remains NP-complete even when D is a simple digraph.

A second variant of the ECWP, called the *Eulerian Closed Walk of Lexicographically Minimal Label Problem (ECWLMLP)*, was considered by Moreno and Matamala [5]. An instance of the ECWLMLP is specified by an Eulerian digraph $D = (V, A)$, a vertex v_0 in V, an arc-labeling function over an alphabet Σ and a word X of $\Sigma^{|A|}$. The ECWLMLP looks for an Eulerian closed walk P starting at v_0 and so that the word induced by P is lexicographically before X. Moreno and Matamala proved the NP-completeness of the ECWLMLP by reducing the directed Hamiltonian circuit problem, which is a well-known NP-hard problem [8], to its decision problem. They also gave a greedy algorithm, running in linear time, to solve the ECWLMLP when for each vertex $v \in V$, all the arcs leaving v have different labels.

Another problem which is worth being mentioned is the *Positional Eulerian Path Problem (PEPP)* which was considered by Hannenhalli et al. [6]. Given a digraph $D = (V, A)$ and an interval $I_a \subseteq \{1, 2, \ldots, |A|\}$ associated with every arc a of D, the PEPP consists of finding an Eulerian walk Q in D so that the position of arc a in Q belongs to I_a, for any $a \in A$. Using a quite simple reduction from the Hamiltonian path problem in a digraph of indegrees and outdegrees exactly two [9], Hannenhalli et al. showed that the PEPP is NP-complete.

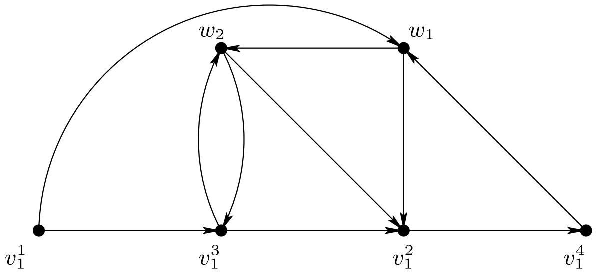

Fig. 1. Transformation of vertex v_1.

The paper is organized as follows. In Section 2 we prove that the ECWPPCP is NP-complete, even when the digraph D is simple and K satisfies some additional conditions. Section 3 is then dedicated to a polynomial-time solvable case of the ECWPPCP, based on some forbidden structures for the path set K. Some concluding remarks are given in Section 4.

The rest of this section is devoted to some notation and terminology used throughout the paper. The reader is referred to [10] for any used terminology not defined in the paper.

Given a digraph $D = (V, A)$, for any vertex $v \in V$, we denote by $\delta^{\text{in}}(v)$ ($\delta^{\text{out}}(v)$) the set of arcs of A entering (leaving) v. The cardinality of this arc set is called the *indegree* (*outdegree*) of v. Δ^{in} and Δ^{out} denote the maximum indegree and outdegree of the vertices of D, respectively. Note that we may need to specify the graph as a subscript in the notation (e.g., $\delta^{\text{out}}_D(v)$), whenever the considered graph may not be clearly deduced from the context.

Given a vertex subset U of V, $D[U]$ represents the subgraph of D induced by U, that is, the subgraph $(U, A[U])$ where $A[U]$ consists of all the arcs of D spanned by U. Given an arc subset F of A, $D[F]$ represents the subgraph of D induced by F, that is, the subgraph $(V[F], F)$ where $V[F]$ is the set of nodes covered by F. If $P = (a_1, a_2, \ldots, a_k)$ and $Q = (b_1, b_2, \ldots, b_l)$ are two walks of D so that the head of the arc a_k equals the tail of the arc b_1, the *concatenation* of P and Q is the walk $(a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_l)$ and it is denoted by (P, Q). If $P = (a_1, a_2, \ldots, a_k)$ is a walk of D, then a *subpath* of P is a set $(a_i, a_{i+1}, \ldots, a_{i+s})$ of contiguous arcs in P, where $i \in \{1, 2, \ldots, k\}$ and $s \in \{0, 1, \ldots, k - i\}$. Let C be a collection of walks of D. We denote by $V[C]$ the set of vertices of D spanned by the walks in C.

Given a path set K of D and an arc $a \in A$, the path set $K \setminus a$ is the path set obtained from K by removing the arc a in any path of K. If a belongs to a path, say Q, but is not the first or last arc of Q, that is, if $Q = (a_1, a_2, \ldots, a_l, a, a_{l+1}, a_{l+2}, \ldots, a_q)$, then, removing a from Q leads to the two paths $(a_1, a_2, \ldots, a_l)$ and $(a_{l+1}, a_{l+2}, \ldots, a_q)$.

Consider now the set K of paths of D involved in the definition of an instance of the ECWPPCP. Let a and a' be two distinct arcs of D. For a sake of conciseness, if there exists a path P in K so that $a \prec_P a'$, then we may write $a \prec_K a'$ instead of $a \prec_P a'$ whenever specifying path P is not relevant. If a and a' are incident in D, and $a \prec_K a'$, then a is called a *predecessor* of a' with respect to K and a' is called a *successor* of a with respect to K. Since an arc a of D may have several predecessors and successors with respect to K, we denote by $P_K(a)$ and $S_K(a)$ the sets of predecessors and successors, respectively.

2. NP-completeness of the ECWPPCP

In this section, we prove the NP-completeness of the Eulerian closed walk with precedence path constraints problem. To do so, we use a polynomial reduction from the *Directed Hamiltonian Circuit of indegrees and outdegrees exactly two Problem (2DHCP)*, which can be stated as follows. Let $D_H = (V_H, A_H)$ be a digraph having all its vertices of indegree and outdegree two. The 2DHCP consists of asserting whether or not D_H contains a Hamiltonian circuit, that is, a closed walk traversing all the vertices of V_H exactly once. The 2DHCP is known to be NP-complete [9]. We remark that the proof given by Plesnik [9] was devised for digraphs with indegrees and outdegrees at most two. By considering some additional arcs, we can easily extend this result to digraphs (with possible multiple arcs) with indegrees and outdegrees two.

The first step of our reduction is the construction of an Eulerian digraph $D = (V, A)$ from $D_H = (V_H, A_H)$, along with the definition of a set K of paths of D and the specification of a starting vertex v_0 of D. Suppose that $V_H = \{v_1, v_2, \ldots, v_n\}$. With vertex v_1 of V_H, we associate six vertices $v_1^1, v_1^2, v_1^3, v_1^4, w_1, w_2$, together with the following ten arcs: (v_1^1, v_1^3), (v_1^3, v_1^2), (v_1^2, v_1^4), (v_1^1, w_1), (w_1, w_2), (w_2, v_1^2), (v_1^4, w_1), (w_1, v_1^2),(v_1^3, w_2), (w_2, v_1^3). Let A_1 be the set composed of these ten arcs. (See Fig. 1.)

For any $i \in \{2, 3, \ldots, n\}$, we associate with v_i four vertices $v_i^1, v_i^2, v_i^3, v_i^4$, together with the eight following arcs (v_i^1, v_i^3), (v_i^3, v_i^2), (v_i^2, v_i^4), (v_i^1, v_i^2), (v_i^4, w_1), (w_1, v_i^2), (v_i^3, w_2), (w_2, v_i^3). Let A_i be the set composed of these eight arcs, for any $i = 2, 3, \ldots, n$. (See Fig. 2.) Note that the only difference between the transformations of vertex v_1 and vertices v_i, for $i = 2, 3, \ldots, n$, is that the path $((v_1^1, w_1), (w_1, w_2), (w_2, v_1^2))$ is replaced by the arc (v_i^1, v_i^2) in A_i, $i = 2, 3, \ldots, n$.

Let

$$V = \{v_i^j : i = 1, 2, \ldots, n,\ j = 1, 2, 3, 4\} \cup \{w_1, w_2\}.$$

At this point, all the vertices in V have their indegree which equals their outdegree, except the vertices v_i^1 and v_i^2, for $i = 1, 2, , \ldots, n$, which have a difference of two between their indegree and outdegree.

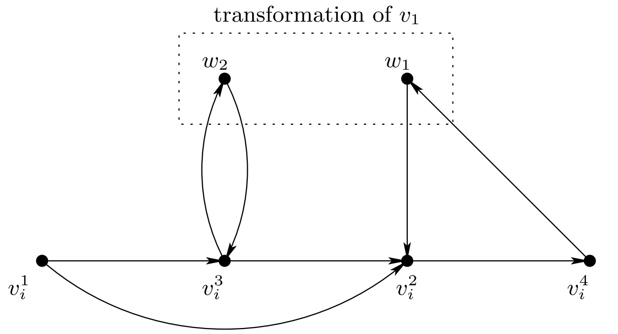

Fig. 2. Transformation of vertex v_i, $i = 2, 3, \ldots, n$.

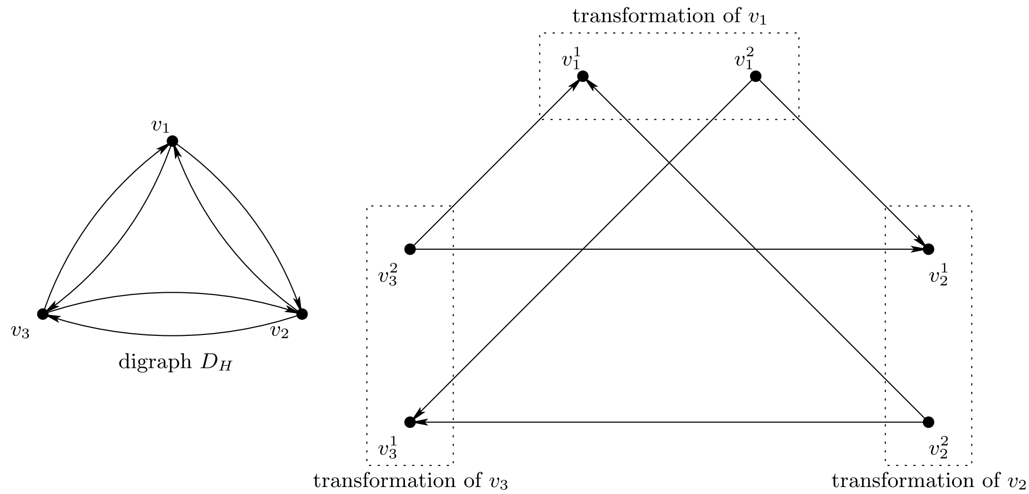

Fig. 3. Transformation of arcs in A_H.

We now consider the arcs of D_H in the following manner. With every arc (v_i, v_j) in A_H, we associate the arc (v_i^2, v_j^1). (See Fig. 3.)
Let

$$A = \left(\bigcup_{i=1}^{n} A_i\right) \cup \{(v_i^2, v_j^1) : (v_i, v_j) \in A_H\}.$$

The digraph $D = (V, A)$ clearly is connected. Moreover, since D_H has all its vertices of indegree and outdegree two, the vertices v_i^1 and v_i^2, for $i = 1, 2, , \ldots, n$, now have the same indegree and outdegree. Therefore, the digraph D is Eulerian.
To obtain an instance of the ECWPPCP, we need to be given a specific vertex v_0 of D (which will be the starting vertex of the Eulerian closed walk) as well as a set K of paths of D. Let $v_0 = v_1^2$. For any $i = 1, 2, \ldots, n$, let P_i and Q_i be the paths of D defined as

$$P_i = \left((v_i^4, w_1), (w_1, w_2), (w_2, v_i^3), (v_i^3, v_i^2)\right),$$
$$Q_i = \left((v_i^4, w_1), (w_1, v_i^2)\right).$$

The set K of paths which needs to be respected by the Eulerian closed walk of D is the following

$$K = \{P_i : i = 1, 2, \ldots, n\} \cup \{Q_i : i = 1, 2, \ldots, n\} \cup \{\left((v_1^1, w_1), (w_1, w_2), (w_2, v_1^2)\right)\}.$$

The digraph D has $(4n + 2)$ vertices and $(10n + 2)$ arcs, and can be obviously constructed in polynomial time. (Note that $|A_H| = 2n$.) The set K contains $(2n + 1)$ paths.

Lemma 1. *If D_H has a Hamiltonian circuit, then D has an Eulerian closed walk which, starting at v_1^2, respects the precedence path constraints specified by K.*

Proof. Let C_H be a Hamiltonian circuit of D_H. Without loss of generality, we suppose that $C_H = ((v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_1))$. Our proof consists of constructing an Eulerian closed walk C of D in several steps. At each of them, we check that no path in K is not respected by the current sequence of selected arcs.

The first arc sequence we consider is obtained by substituting arc (v_i, v_{i+1}) of C_H by the walk

$$\left((v_i^2, v_i^4), (v_i^4, w_1), (w_1, v_i^2), (v_i^2, v_{i+1}^1), (v_{i+1}^1, v_{i+1}^2)\right)$$

for any $i = 1, 2, \ldots, n-1$, and arc (v_n, v_1) by the walk

$$\left((v_n^2, v_n^4), (v_n^4, w_1), (w_1, v_n^2), (v_n^2, v_1^1), (v_1^1, w_1), (w_1, w_2)\right).$$

Let C_1 be the resulting sequence of arcs. Clearly, C_1 corresponds to a walk starting at v_1^2. Since (w_1, w_2) is the last arc of C_1, (v_i^4, w_1) belongs to C_1 for $i = 1, 2, \ldots, n$, and C_1 does not go through any vertex v_i^3, $i = 1, 2, \ldots, n$, C_1 respects the paths P_i, $i = 1, 2, \ldots, n$. Moreover, as the paths Q_i and $\left((v_1^1, w_1), (w_1, w_2)\right)$ are subpaths of C_1 and (w_2, v_1^2) does not belong to C_1, the latter straightforwardly respects K.
Consider now the digraph $D^* = (V^*, A^*)$ induced by the remaining arcs of D, that is, by $A \setminus C_1$. We have that $V^* = V \setminus \left(\{v_i^4 : i = 1, 2, \ldots, n\} \cup \{w_1\}\right)$ and

$$|\delta_{D^*}^{\text{in}}(v)| - |\delta_{D^*}^{\text{out}}(v)| = \begin{cases} 1 & \text{if } v = v_1^2, \\ -1 & \text{if } v = w_2, \\ 0 & \text{if } v \in V^* \setminus \{v_1^2, w_2\}. \end{cases}$$

Moreover, D^* is connected since it contains the arcs (w_2, v_i^3), (v_i^1, v_i^3) and (v_i^3, v_i^2) for $i = 1, 2, \ldots, n$, and they span all the vertices in V^*. Removing arc (w_2, v_1^2) does not disconnect any pair of vertices of V^*; therefore, $D^* - (w_2, v_1^2) = (V^*, A^* \setminus \{(w_2, v_1^2)\})$ is an Eulerian digraph. Furthermore, the digraph $D^*[V^* \setminus \{w_2\}]$ may not be connected, yet each vertex in $V^* \setminus \{w_2\}$ clearly has equal indegree and outdegree. Let $D_1^*, D_2^*, \ldots, D_p^*$, $p \geq 1$, be the strongly connected components of $D^*[V^* \setminus \{w_2\}]$ which obviously are Eulerian. Since w_2 and v_i^3 are adjacent in D^* for $i = 1, 2, \ldots, n$, there must exist $i_k \in \{1, 2, \ldots, n\}$ so that $v_{i_k}^3$ is a vertex of D_k^*, for any $k = 1, 2, \ldots, p$. Consider any strongly connected component D_k^*, $k = 1, 2, \ldots, p$. Any Eulerian closed walk C_k^* of D_k^* starting at $v_{i_k}^3$ can be transformed into a closed walk $\tilde{C}_k$ of D^* starting at w_2 in the following manner. The first arc of $\tilde{C}_k$ is $(w_2, v_{i_k}^3)$. All the arcs of C_k^* are then added to $\tilde{C}_k$ sequentially, according to if the head of the added arc is a vertex v_i^3, for some $i \in \{1, 2, \ldots, n\}$, then the circuit $\left((v_i^3, w_2)(w_2, v_i^3)\right)$ is added to $\tilde{C}_k$ before moving to the next arc of C_k^*. Once we have dealt with all the arcs of C_k^*, we complete $\tilde{C}_k$ by adding $(v_{i_k}^3, w_2)$. Since any $\tilde{C}_k$, for $k = 1, 2, \ldots, p$, starts at vertex w_2, the concatenation $(\tilde{C}_1, \tilde{C}_2, \ldots, \tilde{C}_p)$ of these Eulerian closed walks clearly forms an Eulerian closed walk C_2 of $D^* - (w_2, v_1^2)$.
Let C be the concatenation of C_1, C_2 and arc (w_2, v_1^2). Note that C is composed of all the arcs in A. Moreover, since C_1 starts at v_1^2 and ends at w_2 which is the starting vertex of closed walk C_2, C is an Eulerian closed walk of D starting at v_1^2. For any $i = 1, 2, \ldots, n$, the paths $\left((v_i^4, w_1), (w_1, w_2)\right)$ and $\left((w_2, v_i^3), (v_i^3, v_i^2)\right)$ are respected by C_1 and C_2, respectively. Recalling that all the paths Q_i, for $i = 1, 2, \ldots, n$, are subpaths of C_1, we can conclude that C respects all the paths of K, and our proof is complete. $\square$

In order to prove the converse of Lemma 1, we need to give the following technical result.

Proposition 2. *If the digraph D has an Eulerian closed walk C which starts at v_1^2 and respects the precedence path constraints specified by K, then the closed walk $\left((v_i^2, v_i^4), (v_i^4, w_1), (w_1, v_i^2)\right)$ is a subpath of C for $i = 1, 2, \ldots, n$.*

Proof. For any $i = 1, 2, \ldots, n$, vertex v_i^4 has exactly one entering arc and one leaving arc in D. Therefore $\left((v_i^2, v_i^4), (v_i^4, w_1)\right)$ is a subpath of the Eulerian closed walk C. Moreover, since C respects all the paths Q_i of K, we have

$$(v_i^2, v_i^4) \prec_C (v_i^4, w_1) \prec_C (w_1, v_i^2) \quad \text{for } i = 1, 2, \ldots, n.$$

Suppose there exists $j \in \{1, 2, \ldots, n\}$ so that $\left((v_j^2, v_j^4), (v_j^4, w_1), (w_1, v_j^2)\right)$ is not a subpath of C, that is, an arc $a_1 \in \delta^{\text{out}}(w_1) \setminus \{(w_1, v_j^2)\}$ directly follows (v_j^4, w_1) in C, and then $a_1 \prec_C (w_1, v_j^2)$. Let $I \subset \{1, 2, \ldots, n\}$ represent the subscripts i of the vertices v_i^4 which appear in C before v_j^4. Without loss of generality, we suppose that $\left((v_i^2, v_i^4), (v_i^4, w_1), (w_1, v_i^2)\right)$ is a subpath of C for all $i \in I$. We clearly have

$$(w_1, v_i^2) \prec_C (v_j^4, w_1) \prec_C (v_{i'}^4, w_1) \quad \text{for } i \in I \text{ and } i' \notin I \cup \{j\}. \tag{1}$$

Therefore, a_1 cannot obviously be any of the arcs (w_1, v_i^2) with $i \in I$. If $a_1 = (w_1, v_{i'}^2)$ with $i' \notin I \cup \{j\}$, then because of path $Q_{i'}$ being respected by C, one would have $(v_{i'}^4, w_1) \prec_C (w_1, v_{i'}^2)$, that is, $(v_{i'}^4, w_1) \prec_C (v_j^4, w_1)$, a contradiction with (1). Consequently, a_1 has to be arc (w_1, w_2). Since C respects all the paths P_i of K, we must have $(v_i^4, w_1) \prec_C (w_1, w_2)$ for $i = 1, 2, \ldots, n$. From C being an Eulerian closed walk and $\delta^{\text{out}}(w_1) = \{(w_1, v_i^2) : i = 1, 2, \ldots, n\} \cup \{(w_1, w_2)\}$, we straightforwardly have $(w_1, v_i^2) \prec_C (w_1, w_2)$ for $i = 1, 2, \ldots, n$. Therefore, $(w_1, v_j^2) \prec_C a_1 = (w_1, w_2)$ which contradicts the definition of a_1. The arc following (v_j^4, w_1) in C must then be (w_1, v_j^2), and our proof is complete. $\square$

We can now state the converse of Lemma 1.

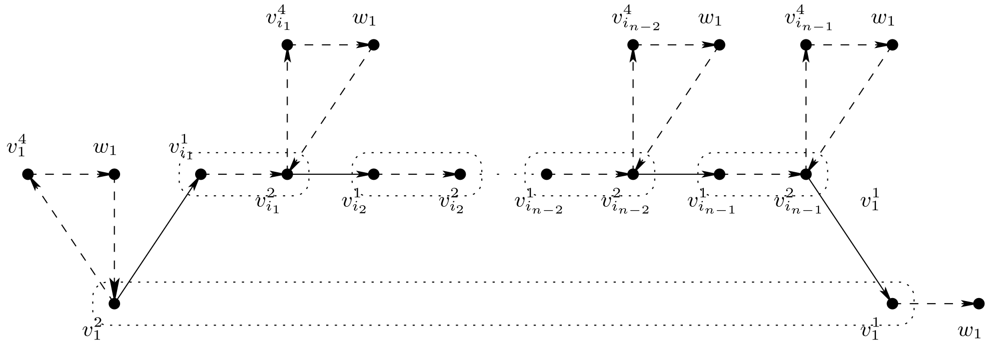

Fig. 4. Walk $\overline{C}$; the solid arcs are the ones in the Hamiltonian circuit; the dashed arcs are the deleted ones; each pair of vertices within a dotted box is contracted.

Lemma 3. *If D has an Eulerian closed walk which starts at v_1^2 and respects the precedence path constraints specified by K, then D_H has a Hamiltonian circuit.*

Proof. Let C be such an Eulerian closed walk of D. Since C respects all the paths of K, arc (w_1, w_2) appears in C after all the arcs of D entering w_1, that is,

$$a \prec_C (w_1, w_2) \quad \text{for all } a \in \delta^{\text{in}}(w_1), \tag{2}$$

and before all the arcs of D leaving w_2, that is,

$$(w_1, w_2) \prec_C a \quad \text{for all } a \in \delta^{\text{out}}(w_2). \tag{3}$$

Moreover, due to $|\delta^{\text{out}}(v)| = |\delta^{\text{in}}(v)|$ for all $v \in V$, we have that

$$a \prec_C (w_1, w_2) \quad \text{for all } a \in \delta^{\text{out}}(w_1) \setminus \{(w_1, w_2)\}, \tag{4}$$

and

$$(w_1, w_2) \prec_C a \quad \text{for all } a \in \delta^{\text{in}}(w_2) \setminus \{(w_1, w_2)\}. \tag{5}$$

Furthermore, by considering all the paths P_i of K, we also have

$$(w_1, w_2) \prec_C (v_i^3, v_i^2) \quad \text{for } i = 1, 2, \ldots, n,$$

which, combined with (5), implies that all the vertices v_i^3, $i = 1, 2, \ldots, n$, appear after arc (w_1, w_2) in C.
Let $\overline{C}$ be the walk obtained from C by only considering the arcs of D preceding (w_1, w_2) in C. We will show that $\overline{C}$ contains a Hamiltonian circuit of D_H after a series of arc deletions and vertex contractions. (See Fig. 4.) Note that $\overline{C}$ starts at v_1^2, ends at w_1 and does not contain any vertex in $\{v_i^3 : i = 1, 2, \ldots, n\}$. Moreover, from (2) and Proposition 2, we deduce that closed walk $((v_i^2, v_i^4), (v_i^4, w_1), (w_1, v_i^2))$, $i \in \{1, 2, \ldots, n\}$, is a subpath of $\overline{C}$. Removing such closed walks in $\overline{C}$ leads to a closed walk, say $\tilde{C}$, starting at v_1^2, ending at w_1, spanning node set $\{v_i^2 : i = 1, 2, \ldots, n\}$, and which does not contain any vertex of $\{v_i^3, v_i^4 : i = 1, 2, \ldots, n\}$. Moreover, we have $\delta^{\text{in}}(v_i^2) \cap \tilde{C} = \{(v_i^1, v_i^2)\}$ for all $i = 2, 3, \ldots, n$, and $\delta^{\text{in}}(v_1^2) \cap \tilde{C} = \emptyset$ due to (3). Furthermore, the last arc of $\tilde{C}$ is (v_1^1, w_1). $\tilde{C} \setminus \{(v_1^1, w_1\}$ is then a walk starting at v_1^2, ending at v_1^1 and being composed of an alternate sequence of arcs of $\{(v_i^2, v_j^1) : (v_i, v_j) \in A_H\}$ and of $\{(v_i^1, v_i^2) : i = 2, 3, \ldots, n\}$. As any arc (v_i^2, v_j^1) corresponds to an arc (v_i, v_j) of D_H, contracting v_i^1 and v_i^2 to a vertex v_i for all $i = 1, 2, \ldots, n$ leads to a Hamiltonian circuit of D_H. □

Proposition 4. *The Eulerian closed walk with precedence path constraints problem is NP-complete.*

Proof. Clearly the problem is in NP. Moreover, the construction from an instance of the NP-complete 2DHCP into an instance of the ECWPPCP can be performed in polynomial time. Therefore, the NP-completeness of the Eulerian closed walk with precedence path constraints problem directly follows from Lemmas 1 and 3. □

One can remark that the vertices v_i^4, for $i = 1, 2, \ldots, n$, are not necessary in the reduction proof. However, if they are removed, each path Q_i, $i = 1, 2, \ldots, n$, has to be replaced by $((v_i^2, w_1), (w_1, v_i^2))$, which corresponds to a circuit and not to a path. Therefore, in order to obtain an instance with a path set K, the vertices v_i^4, $i = 1, 2, \ldots, n$, must be kept.

In Section 1, we mentioned that the ECWPPCP and the Eulerian Superpath Problem considered by Pevzner et al. [4] are quite different, despite some similarities in their statement. In fact, the ESP has been proved to be polynomial-time solvable when the digraph D is simple [4]. The next result shows that for the ECWPPCP, this restriction on D does not make the problem be easier to solve.

Corollary 5. *The Eulerian closed walk with precedence path constraints problem remains NP-complete when D is a simple digraph.*

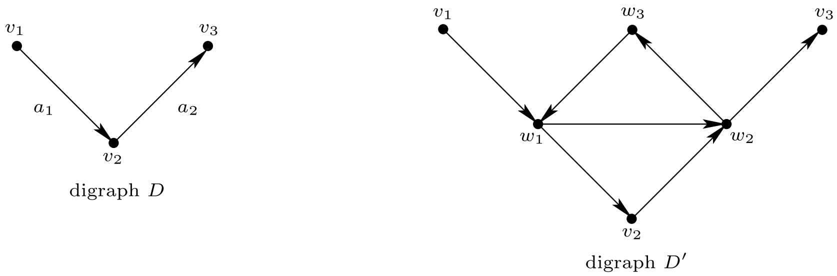

Fig. 5. Set A_w of arcs of D'.

Proof. Consider a NP-complete instance of the ECWPPCP defined by an Eulerian digraph $D = (V, A)$ having multiple arcs, a starting vertex v_0 and a set K of paths of D. Let $D' = (V', A')$ be the simple digraph obtained from D by sequentially replacing each multiple arc (u, v) by the two arcs (u, w) and (w, v) where w is a new vertex with indegree and outdegree one. The same substitutions are applied to the paths of K, and let K' be the set of resulting paths of D'. The construction of D' and K' can obviously be done in polynomial time. Moreover, it is straightforward to see that both instances of the ECWPPCP (i.e., the one defined by D, v_0 and K, and the one defined by D', v_0 and K') are equivalent. □

The remaining of this section is devoted to proving that the ECWPPCP remains NP-complete even if, for each arc of D, its total number of predecessors and successors with respect to K is at most two. This is motivated by the fact that, as will be shown in Section 3, and pointed out in the concluding remarks, the problem is polynomial-time solvable if each arc of D has at most one successor (predecessor) with respect to K.

Recall that for any arc a in A, $S_K(a)$ and $P_K(a)$ represent the sets of successors and predecessors of a with respect to K, respectively. We now introduce the following notation which will make our statements clearer. Let $\mathcal{S}_3(K)$ be the sets of arcs of A having at least three successors with respect to K, that is,

$$\mathcal{S}_3(K) = \{a \in A : |S_K(a)| \geq 3\}.$$

We denote by $\sigma_3(K)$ the cardinality of the family $\{S_K(a) : a \in \mathcal{S}_3(K)\}$, that is,

$$\sigma_3(K) = \sum_{a \in \mathcal{S}_3(K)} |S_K(a)|.$$

Similarly for the sets of at least three predecessors, we define

$$\mathcal{P}_3(K) = \{a \in A : |P_K(a)| \geq 3\},$$

and

$$\pi_3(K) = \sum_{a \in \mathcal{P}_3(K)} |P_K(a)|.$$

Lemma 6. *Consider a digraph $D = (V, A)$, a vertex v_0 of D, and a set K of paths of D. If $\sigma_3(K) \geq 3$, then the instance (D, v_0, K) of the ECWPPCP can be polynomially reduced to an instance (D', v_0, K') of the ECWPPCP wherein $\sigma_3(K') \leq \sigma_3(K) - 1$ and $\pi_3(K') \leq \pi_3(K)$.*

Proof. Since $\sigma_3(K) \geq 3$, we obviously have $\mathcal{S}_3(K) \neq \emptyset$. Let $a_1 = (v_1, v_2)$ be an arc in $\mathcal{S}_3(K)$, and consider an arc $a_2 = (v_2, v_3)$ in $S_K(a_1)$.
First, we construct the digraph $D' = (V', A')$ from D by

(i) replacing arc a_1 by the two arcs (v_1, w_1) and (w_1, v_2), where w_1 is a new vertex,
(ii) replacing arc a_2 by the two arcs (v_2, w_2) and (w_2, v_3), where w_2 is a new vertex,
(iii) adding the three arcs (w_1, w_2), (w_2, w_3) and (w_3, w_1), where w_3 is a new vertex.

Let A_w be the set of new arcs, that is,

$$A_w = \{(v_1, w_1), (w_1, v_2), (v_2, w_2), (w_2, v_3), (w_1, w_2), (w_2, w_3), (w_3, w_1)\}.$$

We then have $V' = V \cup \{w_1, w_2, w_3\}$ and $A' = (A \setminus \{a_1, a_2\}) \cup A_w$. (See Fig. 5.) Note that any vertex in $\{v_i, w_i : i = 1, 2, 3\}$ has equal indegree and outdegree in D'. Since D is an Eulerian digraph, so is D'.
Second, we define the paths of D' which compose the set K'. We remark that since K is only composed of paths of D, arcs a_1 and a_2 cannot both belong to a path of K without a_1 being the predecessor of a_2. With any path P in K, we then associate a path in the following manner,

(iv) keep P unchanged, if P does not contain either a_1 or a_2,
(v) replace a_1 by subpath $((v_1, w_1), (w_1, v_2))$, if P contains a_1 but not a_2,

(vi) replace a_2 by subpath $((v_2, w_2), (w_2, v_3))$, if P contains a_2 but not a_1,
(vii) replace subpath (a_1, a_2) by subpath $((v_1, w_1), (w_1, w_2), (w_2, v_3))$, if P contains both a_1 and a_2.

We also add to K' the two paths $((w_1, w_2), (w_2, w_3))$ and $((v_2, w_2), (w_2, v_3))$. This results in a set K' composed of $|K| + 2$ paths of D'. Note that transforming (D, v_0, K) into (D', v_0, K') can be straightforwardly performed in polynomial time.
Since the vertices w_1, w_2 and w_3 have their indegree and outdegree at most two, neither the arcs (v_1, w_1), (v_2, w_2), (w_1, w_2), (w_2, w_3), and (w_3, w_1) belong to $\mathcal{S}_3(K')$ nor the arcs (w_1, v_2), (w_2, v_3) , (w_1, w_2), (w_2, w_3) and (w_3, w_1) belong to $\mathcal{P}_3(K')$. Furthermore from the construction of K', we clearly have

$$|S_{K'}(a)| = |S_K(a)| \quad \text{for all } a \in A' \setminus A_w, \tag{6}$$

$$|P_{K'}(a)| = |P_K(a)| \quad \text{for all } a \in A' \setminus A_w. \tag{7}$$

Moreover because of (v) and (vii), we easily obtain

$$|S_{K'}((w_1, v_2))| = |S_K(a_1)| - 1, \tag{8}$$

$$|P_{K'}((v_1, w_1))| = |P_K(a_1)|. \tag{9}$$

Using (vi) and (vii), we also deduce

$$|S_{K'}((w_2, v_3))| = |S_K(a_2)|, \tag{10}$$

$$|P_{K'}((v_2, w_2))| = |P_K(a_2)| - 1. \tag{11}$$

From (6), we clearly have $\mathcal{S}_3(K') \cap (A' \setminus A_w) = \mathcal{S}_3(K) \cap (A \setminus \{a_1, a_2\})$, and from (7), $\mathcal{P}_3(K') \cap (A' \setminus A_w) = \mathcal{P}_3(K) \cap (A \setminus \{a_1, a_2\})$. Therefore, to compare $\sigma_3(K')$ with $\sigma_3(K)$, and $\pi_3(K')$ with $\pi_3(K)$, we only have to consider the arcs a_1 and a_2 for K and the arcs of A_w for K'. Suppose $a_2 \in \mathcal{S}_3(K)$; the argument is similar if $a_2 \notin \mathcal{S}_3(K)$. By (10), we also have $(w_2, v_3) \in \mathcal{S}_3(K')$. From (6), (8) and (10), we then obtain

$$\begin{aligned}
\sigma_3(K') &= \sum_{\substack{a \in \mathcal{S}_3(K') \\ a \notin A_w}} |S_{K'}(a)| + \sum_{\substack{a \in \mathcal{S}_3(K') \\ a \in \{(w_1, v_2), (w_2, v_3)\}}} |S_{K'}(a)| \\
&\le \sum_{\substack{a \in \mathcal{S}_3(K) \\ a \notin \{a_1, a_2\}}} |S_K(a)| + |S_K(a_1)| - 1 + |S_K(a_2)| \\
&= \sigma_3(K) - 1,
\end{aligned}$$

the inequality coming from $(w_1, v_2) \notin \mathcal{S}_3(K')$ if $|S_K(a_1)| = 3$. Using (7), (9) and (11), we can similarly prove that $\pi_3(K') \le \pi_3(K)$.
We now need to prove that (D, v_0, K) has a feasible solution if and only if (D', v_0, K') has one. Consider an Eulerian closed walk C of D which respects K. Since $a_2 \in S_K(a_1)$, we must have $a_1 \prec_C a_2$. Let C' be the Eulerian closed walk of D' obtained from C by substituting $((v_1, w_1), (w_1, w_2), (w_2, w_3), (w_3, w_1), (w_1, v_2))$ and $((v_2, w_2), (w_2, v_3))$ for a_1 and a_2, respectively. Since C respects K, it is straightforward to see that C' respects all the paths of K' generated by (iv), (v) and (vi). Moreover from the substitutions for a_1 and a_2, C' clearly respects the paths $((w_1, w_2), (w_2, w_3))$ and $((v_2, w_2), (w_2, v_3))$. Because of $a_1 \prec_C a_2$, we also have $(w_1, w_2) \prec_{C'} (w_2, v_3)$, which implies that C' respects all the paths of K' generated by (vii). Therefore, C' is a feasible solution to instance (D', v_0, K') of the ECWPPCP.
Conversely, let $\overline{C}'$ be an Eulerian closed walk of D' which respects K'. If path $((w_1, w_2), (w_2, v_3))$ was a subpath of $\overline{C}'$, then by considering the paths of K' $((v2, w2), (w2, v3))$ and $((w_1, w_2), (w_2, w_3))$, one would have

$$(v_2, w_2) \prec_{\overline{C}'} (w_1, w_2) \prec_{\overline{C}'} (w_2, v_3) \prec_{\overline{C}'} (w_2, w_3).$$

Yet, this contradicts the Eulerian-walk property of $\overline{C}'$. Therefore, both paths $((w_1, w_2), (w_2, w_3))$ and $((v_2, w_2), (w_2, v_3))$ represent subpaths of $\overline{C}'$. Since $\overline{C}'$ respects all the paths of K' generated by (vii), we know that $(v_1, w_1) \prec_{\overline{C}'} (w_1, w_2) \prec_{\overline{C}'} (w_2, w_3)$. Consequently, we can easily deduce that walk $((v_1, w_1), (w_1, w_2), (w_2, w_3), (w_3, w_1), (w_1, v_2))$ is a subpath of $\overline{C}'$. Moreover due to $\overline{C}'$ respecting subpath $((v_1, w_1), (w_1, w_2), (w_2, v_3)$ considered in (vii), walk $((v_1, w_1), (w_1, w_2), (w_2, w_3), (w_3, w_1), (w_1, v_2))$ appears before path $((v_2, w_2), (w_2, v_3))$ in $\overline{C}'$. Let $\overline{C}$ be the Eulerian closed walk of D obtained from $\overline{C}'$ by replacing walk $((v_1, w_1), (w_1, w_2), (w_2, w_3), (w_3, w_1), (w_1, v_2))$ and path $((v_2, w_2), (w_2, v_3))$ by arcs a_1 and a_2, respectively. Since all the paths of K' except $((w_1, w_2), (w_2, w_3))$ and $((v_2, w_2), (w_2, v_3))$ are derived from all the paths of K, $\overline{C}$ obviously respects K, and our proof is complete. □

A similar result can be obtained for the whole number of predecessors $\pi_3(K)$ associated with the arcs of $\mathcal{P}_3(K)$. The next lemma then is given without proof.

Lemma 7. *Consider a digraph $D = (V, A)$, a vertex v_0 of D, and a set K of paths of D. If $\pi_3(K) \ge 3$, the instance (D, v_0, K) of the ECWPPCP can be polynomially reduced to an instance (D', v_0, K') of the ECWPPCP wherein $\sigma_3(K') \le \sigma_3(K)$ and $\pi_3(K') \le \pi_3(K) - 1$.* □

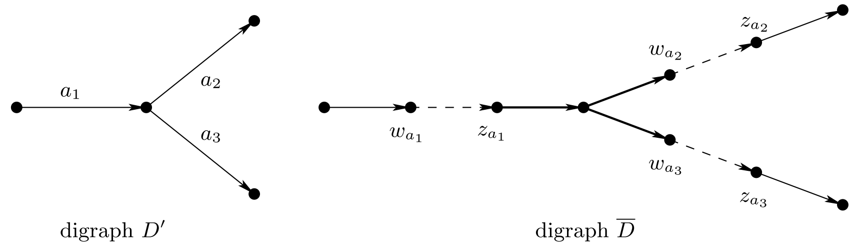

Fig. 6. Transformation of arcs of $A'_{K'}$; $S_{K'}(a_1) = \{a_2, a_3\}$, the solid arcs may belong to $\overline{K}$, the bold arcs belong to $\overline{K}$, and the dashed arcs cannot belong to $\overline{K}$.

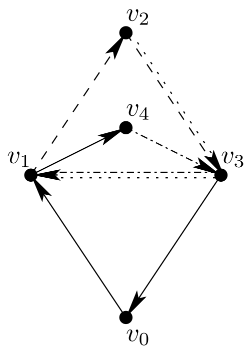

Fig. 7. A MSF path set K on a simple digraph.

We can now state our main complexity result.

Theorem 8. *The Eulerian closed walk with precedence path constraints problem is NP-complete even if every arc has at most two successors and predecessors with respect to K, that is, $|S_K(a)| + |P_K(a)| \leq 2$ for all $a \in A$.*

Proof. Consider any instance (D, v_0, K) of the ECWPPCP. By Lemmas 6 and 7, there exists a polynomial-time reduction which transforms (D, v_0, K) into another instance (D', v_0, K') of the ECWPPCP so that $\mathcal{S}_3(K') = \emptyset$, $\mathcal{P}_3(K') = \emptyset$, and (D', v_0, K') is as hard to solve as (D, v_0, K). (Note that neither $\sigma_3(K)$ nor $\pi_3(K)$ exceeds $|A|^2$.) Moreover, each arc a of D' satisfies $|S_{K'}(a)| \leq 2$ and $|P_{K'}(a)| \leq 2$.

To obtain an instance satisfying the condition of the theorem, we need to transform (D', v_0, K') into a new instance $(\overline{D}, v_0, \overline{K})$ of the ECWPPCP as follows. (See Fig. 6.) Let $A'_{K'}$ be the set of arcs of D' which appear in paths of K'. Digraph $\overline{D} = (\overline{V}, \overline{A})$ is obtained from D' by sequentially replacing any arc $a = (u_a, v_a)$ of $A'_{K'}$ by the path $((u_a, w_a), (w_a, z_a), (z_a, v_a))$, where w_a and z_a are two new vertices. For any path $P = (a_1, a_2, \dots, a_k)$ of K', substitute the $(k-1)$ paths $((z_{a_i}, v_{a_i}), (u_{a_{i+1}}, w_{a_{i+1}}))$, $i = 1, 2, \dots, k-1$, to generate the set $\overline{K}$ of paths of $\overline{D}$. (Recall that $v_{a_i} = u_{a_{i+1}}$ for $i = 1, 2, \dots, k-1$.) Once again, this transformation can be performed in polynomial time. Since $\mathcal{S}_3(K') = \emptyset$ and $\mathcal{P}_3(K') = \emptyset$, we also have $\mathcal{S}_3(\overline{K}) = \emptyset$ and $\mathcal{P}_3(\overline{K}) = \emptyset$. Moreover from the generation of the paths of $\overline{K}$, we know that none of the arcs (w_a, z_a), $a \in A'_{K'}$, appears in a path of $\overline{K}$. Therefore, each arc a of $\overline{D}$ satisfies $|S_{\overline{K}}(a)| + |P_{\overline{K}}(a)| \leq 2$.

Since the vertices in $\{w_a, z_a : a \in A'_{K'}\}$ have indegree and outdegree one in $\overline{D}$, any Eulerian closed walk of $\overline{D}$ contains $((u_a, w_a), (w_a, z_a), (z_a, v_a))$ as a subpath, for $a \in A'_{K'}$. It therefore follows that an Eulerian closed walk of D' respecting K' exists if and only if an Eulerian closed walk of $\overline{D}$ respecting $\overline{K}$ exists. The instance $(\overline{D}, v_0, \overline{K})$ is then as hard to solve as the original instance (D, v_0, K). $\square$

3. A polynomial-time solvable case

In this section, we show that every instance (D, v_0, K) of the ECWPPCP can be solved in polynomial time if there does not exist an arc of D having more than one successor with respect to K.

Throughout this section, the triple (D, v_0, K) always refers to a digraph $D = (V, A)$ with $|V| = n$ and $|A| = m$, a vertex v_0 of V, and a path set K of D with $|K| = q$. We remark that (D, v_0, K) corresponds to an instance of the ECWPPCP if D is Eulerian. The path set K is called *Multiple Successor Free (MSF)* if every arc of D has at most one successor with respect to K. Fig. 7 shows an example of a MSF set K composed of three paths represented by dashed arcs, dotted arcs, and dot-dashed arcs, respectively.

We now introduce the definition of impregnable subgraphs which is the keystone of our analysis towards devising a polynomial-time algorithm for the ECWPPCP. Consider a triple (D, v_0, K). Let $D' = (V', A')$ be a subgraph of D without isolated vertex. A vertex v of V' is called D'-*impregnable* with respect to D, v_0, and K if at least one of the following conditions holds:

(i) for any arc a in $\delta^{\text{in}}_{D'}(v)$ there exists an arc a' in $\delta^{\text{out}}_{D'}(v)$ with $a \prec_K a'$,
(ii) $\delta^{\text{out}}(v) \subseteq A'$ and $v \neq v_0$.

The subgraph D' is then called *impregnable* with respect to D, v_0, and K if each vertex of V' is D'-impregnable with respect to D, v_0, and K. Note that the notion of impregnability depends on the original digraph D, and v_0 and K as well. Fig. 7 gives an example of a subgraph $D' = D \setminus \{(v_3, v_0)\}$ of D which is impregnable with respect to D, v_0, and K. Indeed, the vertices v_0 and v_3 are D'-impregnable because of (i), and the vertices v_1, v_2 and v_4 are D'-impregnable because of (ii). Note that the D'-impregnability of v_0 is clear since it has no entering arc and v_2 is also D'-impregnable by (i).

Lemma 9. *An instance (D, v_0, K) of the ECWPPCP has no solution if D contains an impregnable subgraph, say $D' = (V', A')$, with respect to D, v_0, and K.*

Proof. Indeed, the last arc of D' traversed by any Eulerian closed walk C of D starting from v_0, say a, is an arc entering either v_0 or a vertex $v \in V'$ with $\delta^{\text{out}}(v) \setminus A' \neq \emptyset$. Therefore, this vertex is D'-impregnable with respect to D, v_0, and K because of (i). Hence there exists an arc a' of $\delta^{\text{out}}_{D'}(v)$ with $a \prec_K a'$. Since a is the last arc of D' traversed by C, we have $a' \prec_C a$, implying that C does not respect K. □

In what follows, we will show that checking whether D contains an impregnable subgraph with respect to D, v_0, and K can be done in polynomial time if K is MSF. Indeed, this is shown using Algorithm 1 given below, hereafter referred as the *Impregnable Subgraph Detection Algorithm*. This algorithm successively removes all the arcs of the digraph D which have no successor with respect to K or for which all the successors with respect to K have already been removed. If, at the end of the algorithm, there still exist some arcs which have not been removed, these latter form an impregnable subgraph. Otherwise, we can claim that D does not contain an impregnable subgraph with respect to D, v_0, and K.

The algorithm uses a vertex stack. Two operations, namely Push and Pop, are used for the stack. The first one consists of pushing a vertex into the top of the stack whereas the second one consists of removing from the stack the vertex which is at the top of the stack.

Algorithm 1: Impregnable Subgraph Detection Algorithm

```
Input: Triple (D, v0, K) with K MSF.
Output: An impregnable subgraph of D with respect to D, v0, and K if it exists.
begin
    A' = A;
    Stack S = {v0};
    while S ≠ ∅ do
        v = Pop(S);
        foreach (u, v) ∈ δ^in_{D[A']}(v) do
            if ∄(v, w) ∈ A' with (u, v) ≺_K (v, w) then
                A' = A' \ {(u, v)};
                Push(S, u);
    if A' ≠ ∅ then
        return D[A'];
end
```

Proposition 10. *The Impregnable Subgraph Detection Algorithm works correctly in time $O(mq\Delta^{\text{in}})$.*

Proof. We first prove the correctness of the Algorithm. First, assume that at the end of Algorithm 1, $A' \neq \emptyset$. Let W_1 and W_2 be the sets of vertices of $V[A']$ which appeared and did not appear in the stack S, respectively. Since $v_0 \notin W_2$ and a vertex in $V \setminus \{v_0\}$ is pushed into S whenever one of its leaving arcs is removed from A', we clearly have $\delta^{\text{out}}_D(v) \subseteq A'$ for $v \in W_2$. The vertices in W_2 are then $D[A']$-impregnable with respect to D, v_0, and K due to Condition (ii). Let $v_1 \in W_1$. Consider the iteration of Algorithm 1 where v_1 is popped out of S for the last time. Any arc in $\delta^{\text{in}}_D(v_1) \cap A'$ having no successor in $\delta^{\text{out}}_D(v_1) \cap A'$ is then removed from A'. Therefore, at the end of Algorithm 1, all the arcs in $\delta^{\text{in}}_{D[A']}(v_1)$ have a successor in $\delta^{\text{out}}_{D[A']}(v_1)$, meaning that v_1 is $D[A']$-impregnable with respect to D, v_0, and K due to Condition (i). It then follows that $D[A']$ is impregnable with respect to D, v_0, and K.

Assume now that D contains a subgraph, say $\bar{D} = (\bar{V}, \bar{A})$, which is impregnable with respect to D, v_0 and K, and suppose that $A' = \emptyset$ at the end of the algorithm. Let (u, v) be the first arc of $\bar{A}$ to be removed from A'. The algorithm implies that v has been pushed into S. Since any vertex different from v_0 is pushed into S whenever one of its leaving arc is removed from A', and (u, v) is the first arc of $\bar{A}$ being removed from A', it follows that v corresponds either to v_0 or to a vertex having at least one leaving arc which does not belong to $\bar{A}$. The $\bar{D}$-impregnability of v with respect to D, v_0 and K then follows by

Condition (i). This means that there exists an arc (v, w) of $\bar{A}$ with $(u, v) \prec_K (v, w)$. Note that (v, w) is removed from A' before (u, v). For otherwise (u, v) could not be removed at this step. But this contradicts the fact that (u, v) is the first arc of $\bar{D}$ being removed. Therefore, we have $A' \neq \emptyset$ and the algorithm then works correctly.

Anytime a vertex is pushed into S, an arc is removed from A'. The number of iterations of our algorithm is then at most m. Each iteration involves checking for every entering arc of the current vertex v whether it has a successor in $\delta^{\text{out}}_{D[A']}(v)$. Since K contains q paths and the indegree of v is at most Δ^{in}, the Impregnable Subgraph Detection Algorithm has a complexity of $O(mq\Delta^{\text{in}})$. □

We now devise a second algorithm, Algorithm 2, later referred to as the *ECWPPCP algorithm*, which permits to solve the ECWPPCP when the path set is MSF. Given an instance (D, v_0, K) of the ECWPPCP, the ECWPPCP algorithm first checks whether or not D contains an impregnable subgraph with respect to D, v_0, and K, using Algorithm 1. If no such subgraph exists, then a solution of the ECWPPCP is constructed as follows. We begin with an empty closed walk C. At each iteration of the algorithm, we add to C an arc leaving the end vertex of C until C becomes an Eulerian closed walk of D. Moreover, we ensure that C starts from v_0 and respects K at each iteration, which implies that C is a solution.

During the algorithm, the arc set A' corresponds to the arcs of A which have not been yet added to C. The digraph $D' = (V, A')$ is the spanning digraph of D given by A', and K' corresponds to the restriction of K on D'.

At each iteration, an arc (u, v), where u corresponds to the end vertex of C, is removed from A' and added at the end of C. The arc (u, v) is chosen in order to ensure that $(C, (u, v))$ respects K, and the digraph $D' \setminus (u, v)$ does not contain an impregnable subgraph with respect to $D' \setminus (u, v)$, v_0, and $K' \setminus (u, v)$. The algorithm ends when A' is empty, which implies that C is a solution of the ECWPPCP for (D, v_0, K).

Algorithm 2: ECWPPCP algorithm

Input: Instance (D, v_0, K) of the ECWPPCP with K MSF
Output: Either an impregnable subgraph of D with respect to D, v_0, and K, or a solution of ECWPPCP for (D, v_0, K)
begin
  Apply Algorithm 1 to check if D contains an impregnable subgraph with respect to D, v_0, and K;
  if *D contains an impregnable subgraph D^* with respect to D, v_0 and K* **then**
    return D^*;
  $A' = A$;
  $K' = K$;
  $C = \emptyset$;
  $u = v_0$;
  while $A' \neq \emptyset$ **do**
    $D' = (V, A')$;
    Choose $(u, v) \in A'$ so that :
    (i) $(C, (u, v))$ respects K,
    (ii) $D' \setminus (u, v)$ does not contain an impregnable subgraph with respect to $D' \setminus (u, v)$, v_0, and $K' \setminus (u, v)$.
    $C = (C, (u, v))$;
    $A' = A' \setminus (u, v)$;
    $K' = K' \setminus (u, v)$;
    $u = v$;
  return C;
end

Proposition 11. *The ECWPPCP algorithm works correctly in $O(m^2 q\Delta^{\text{in}}\Delta^{\text{out}})$.*

Proof. By Lemma 9, an instance (D, v_0, K) has no solution if D contains an impregnable subgraph with respect to D, v_0, and K. We now suppose that D does not contain such impregnable subgraph, and then show that the ECWPPCP Algorithm ends and returns a solution of (D, v_0, K), say C. For this, we have to show that, at each iteration of the algorithm, if C is a walk that respects K, u is the end vertex of C and $A' = A \setminus C$ is such that $D' = (V, A')$ does not contain any impregnable subgraph with respect to D', v_0 and K', then there exists an arc (u, v) of A' leaving u such that:

1. $(C, (u, v))$ respects K,
2. $D' \setminus (u, v) = (V, A' \setminus (u, v))$ does not contain any impregnable subgraph with respect to $D' \setminus (u, v)$, v_0, and $K' \setminus (u, v)$.

We distinguish two cases, depending on whether or not vertex u corresponds to v_0. First, suppose that $u = v_0$. Hence, C is a closed walk. Since $A' = A \setminus C$ and D is Eulerian, we have $|\delta^{\text{in}}_{D'}(v_0)| = |\delta^{\text{out}}_{D'}(v_0)|$. As $A' \neq \emptyset$, we have that $\delta^{\text{out}}_{D'}(v_0) \neq \emptyset$. Otherwise, every vertex of $D[A']$ satisfies $\delta^{\text{out}}_{D'}(v) \subseteq A'$ and, as v_0 is not a vertex of $D[A']$, it follows that $D[A']$ is impregnable with respect to D', v_0, and K', a contradiction.

Moreover, there must exist an arc (v_0, v_1) of A' leaving v_0 and having no predecessor in A' with respect to K'. Otherwise, as K' is MSF, we obtain that for any arc a of A' entering v_0, there exists an arc a' of A' leaving v_0 with $a \prec_{K'} a'$. This implies that v_0 is $D[A']$-impregnable with respect to D', v_0, and K'. It follows that $D[A']$ is impregnable with respect to D', v_0, and K' which contradicts the hypothesis.

Since every arc of $A \setminus A'$ satisfies $a \prec_C (v_0, v_1)$, C respects K, and (v_0, v_1) has no predecessor in A' with respect to K', we deduce that $(C, (u, v))$ respects K. Suppose now that $D' \setminus (v_0, v_1)$ contains an impregnable subgraph, say $\bar{D} = (\bar{V}, \bar{A})$, with respect to $D' \setminus (v_0, v_1)$, v_0, and $K' \setminus (v_0, v_1)$. Since (v_0, v_1) is not an arc of $\bar{A}$, every vertex of $\bar{V}$ which is $\bar{D}$-impregnable with respect to $D' \setminus (v_0, v_1)$, v_0, and $K' \setminus (v_0, v_1)$ due to Condition (i) is still $\bar{D}$-impregnable with respect to D', v_0 and K'. In the same way, since (v_0, v_1) is not an arc leaving a vertex different from v_0, every vertex $\bar{D}$-impregnable with respect to $D' \setminus (v_0, v_1)$, v_0, and $K' \setminus (v_0, v_1)$ due to Condition (ii) is also $\bar{D}$-impregnable with respect to D', v_0 and K'. Thus, $\bar{D}$ is an impregnable subgraph of D' with respect to D', v_0, and K', a contradiction. Therefore, if $u = v_0$, then there exists an arc of A' satisfying Conditions 1. and 2. and leaving v_0.

Suppose now that $u \neq v_0$. We denote by B the set of arcs leaving u and having no predecessor in A' with respect to K'. Since C is a walk starting at v_0 and ending at u, and D is Eulerian, we have $|\delta^{\text{in}}_{D'}(u)| < |\delta^{\text{out}}_{D'}(u)|$. As K is MSF, it follows that B is not empty. Moreover, since C respects K, the path $(C, (u, u'))$ also respects K for every arc $(u, u') \in B$. We now have to show that there exists at least one arc (u, u') of B so that the digraph $D' \setminus (u, u')$ does not contain a subgraph which is impregnable with respect to $D' \setminus (u, u')$, v_0, and $K' \setminus (u, u')$.

Suppose the contrary, that is, for every arc $(u, u') \in B$, there exists an impregnable subgraph, say $D^{(u,u')} = (V^{(u,u')}, A^{(u,u')})$, of $D' \setminus (u, u')$ with respect to $D' \setminus (u, u')$, v_0, and $K' \setminus (u, u')$. We then obtain the following result.

Claim 12. *B contains at least two arcs. Moreover, each arc (u, u') of B is the only arc leaving u which does not belong to $A^{(u,u')}$.*

Proof. Let (u, u') be an arc of B. By hypothesis, $D^{(u,u')}$ is impregnable with respect to $D' \setminus (u, u')$, v_0 and $K' \setminus (u, u')$, but not with respect to D', v_0, and K'. Since $D^{(u,u')}$ is a subgraph of $D' \setminus (u, u')$, the arc (u, u') does not belong to $A^{(u,u')}$. From the definition of impregnable subgraphs, every vertex of $D^{(u,u')}$ which is $D^{(u,u')}$-impregnable with respect to $D' \setminus (u, u')$, v_0, and $K' \setminus (u, u')$ due to Condition (i) is also $D^{(u,u')}$-impregnable with respect to D', v_0, and K'. Moreover, every vertex of $D^{(u,u')}$ different from u which is $D^{(u,u')}$-impregnable with respect to $D' \setminus (u, u')$, v_0, and $K' \setminus (u, u')$ due to Condition (ii) is also $D^{(u,u')}$-impregnable with respect to D', v_0, and K'. We then deduce that u is $D^{(u,u')}$-impregnable with respect to $D' \setminus (u, u')$, v_0, and $K' \setminus (u, u')$, but not with respect to D', v_0, and K'. This implies that

(a) $\delta^{\text{out}}_{D'}(u) \not\subseteq A^{(u,u')}$,
(b) there exists an arc, say $(\bar{u}, u)$, of $A^{(u,u')}$ with no successor in $A^{(u,u')}$ with respect to K',
(c) $\delta^{\text{out}}_{D' \setminus (u,u')}(u) \subseteq A^{(u,u')}$.

Implications (a) and (b) come from the non $D^{(u,u')}$-impregnability of u with respect to D', v_0, and K' whereas Implication (c) is given by the $D^{(u,u')}$-impregnability of u with respect to $D' \setminus (u, u')$, v_0, and $K' \setminus (u, u')$.

From Implications (a) and (c), we deduce that $\delta^{\text{out}}(u) \setminus A^{(u,u')} = \{(u, u')\}$. Moreover, as $|\delta^{\text{in}}_{D'}(u)| < |\delta^{\text{out}}_{D'}(u)|$ and $(\bar{u}, u)$ is an arc of $\delta^{\text{in}}_{D'}(u)$, we have $|\delta^{\text{out}}_{D'}(u)| \geq 2$. Since the arcs of $\delta^{\text{out}}_{D'}(u) \setminus (u, u')$ belong to $D^{(u,u')}$, from Implication (b), they do not have $(\bar{u}, u)$ as predecessor with respect to K'. The number of predecessors of $\delta^{\text{out}}_{D'}(u) \setminus \{(u, u')\}$ with respect to K' is then at most $|\delta^{\text{in}}_{D'}(u)| - 1 < |\delta^{\text{out}}_{D'}(u) \setminus (u, u')|$. As K is MSF, there exists an arc of $\delta^{\text{out}}_{D'}(u) \setminus (u, u')$ having no predecessor in A' with respect to K'. This arc then belongs to B and is different from (u, u').

Let $\bar{A}$ be the union of the arc sets $A^{(u,u')}$ for all arcs (u, u') of B. Consider the subgraph $\bar{D} = (V(\bar{A}), \bar{A})$ induced by $\bar{A}$. Clearly, u is $\bar{D}$-impregnable with respect to D', v_0, and K'. Indeed, Claim 12 implies that $\delta^{\text{out}}_{D'}(u) \subseteq \bar{A}$. As $u \neq v_0$, the impregnability of u is given by Condition (ii).

Consider any vertex v of $\bar{V} \setminus u$. If, for some arc (u, u') of B, vertex v is $D^{(u,u')}$-impregnable with respect to $D' \setminus (u, u')$, v_0, and $K' \setminus (u, u')$ due to Condition (ii), it means that $\delta^{\text{out}}_{D' \setminus (u,u')}(v) \subseteq A^{(u,u')}$. Since $v \neq u$, we have $\delta^{\text{out}}_{D' \setminus (u,u')}(v) = \delta^{\text{out}}_{D'}(v)$. Moreover, as $A^{(u,u')} \subseteq \bar{A}$, we have $\delta^{\text{out}}_{D'}(v) \subseteq \bar{A}$ which implies that v is $\bar{D}$-impregnable with respect to D', v_0, and K'.

Suppose now that the impregnability of v is always given by Condition (i). Then, consider any arc (v', v) of $\delta^{\text{in}}_{\bar{D}}(v)$. By construction, there exists an arc (u, u') of B such that (v', v) belongs to $\delta^{\text{in}}_{D^{(u,u')}}(v)$. Since v is $D^{(u,u')}$-impregnable with respect to $D' \setminus (u, u')$, v_0, and $K' \setminus (u, u')$ due to Condition (i), there exists (v, v'') belonging to $\delta^{\text{out}}_{D^{(u,u')}}(v)$ with $(v', v) \prec_{K' \setminus (u,u')} (v, v'')$. By construction, (v, v'') also belongs to $\bar{A}$ and $(v', v) \prec_{K'} (v, v'')$. Thus, every arc of $\bar{A}$ entering v has a successor in $\bar{A}$, which implies that v is $\bar{D}$-impregnable with respect to D', v_0, and K. Consequently, we conclude that $\bar{D}$ is a subgraph of D' which is impregnable with respect to D', v_0, and K', a contradiction. Therefore, there exists one arc of B satisfying Conditions 1. and 2., which ends the proof of the correctness of the algorithm.

We now establish the complexity of the algorithm. The loop is executed at most m times. At each iteration, we have to test, for every arc a leaving u, if $D' \setminus a$ does not contain any impregnable subgraph with respect to $D' \setminus a$, v_0, and $K' \setminus a$, and if (C, a) respects K. The complexities of these two operations are $O(mq\Delta^{\text{in}})$ and $O(q)$, respectively. Since every vertex has at most Δ^{out} leaving arcs, the complexity of each iteration is $O(mq\Delta^{\text{in}}\Delta^{\text{out}})$. □

Algorithm 2 gives necessary and sufficient conditions for an instance of the ECWPPCP to admit a solution when the path set is MSF.

Theorem 13. *An instance (D, v_0, K) of the ECWPPCP where K is MSF has a solution if and only if D does not contain a subgraph which is impregnable with respect to D, v_0, and K.* $\square$

4. Concluding remarks

In this paper, we studied the Eulerian closed walk problem where some precedence constraints are specified by a set of paths K. We first proved its NP-completeness by a polynomial reduction from the directed Hamiltonian circuit of indegrees and outdegrees exactly two problem. We then refined this result by showing that the NP-completeness is preserved if every arc has at most two successors and predecessors with respect to K. We finally presented a polynomial-time algorithm to solve the problem when each arc of D has at most one successor with respect to K. For this polynomial case, we also gave necessary and sufficient conditions for the problem to admit a solution.

We can point out a further polynomial-time solvable case for the ECWPPCP when each arc has at most one predecessor with respect to K. For this, it suffices to see that every instance (D, v_0, K) where each arc has at most one predecessor can be transformed into the instance (D_r, v_0, K_r) of the ECWPPCP by reversing all the arcs of A and all the paths of K, that is, $(u, v) \in A_r$ if and only if $(v, u) \in A$, and $((v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k)) \in K_r$ if and only if $((v_k, v_{k-1}), (v_{k-1}, v_{k-2}), \ldots, (v_2, v_1)) \in K$. Therefore, each arc of A_r has at most one successor with respect to K_r and the ECWPPCP Algorithm applies to this new instance. Any solution returned by the algorithm can be transformed into a solution of (D, v_0, K) by reversing its arcs (*i.e.*, $((v_0, v_1), (v_1, v_2), \ldots, (v_r, v_0))$ of D_r is transformed into $((v_0, v_r), (v_r, v_{r-1}), \ldots, (v_1, v_0))$ of D). If an impregnable subgraph is found, this implies that the instance (D, v_0, K) has no solution.

One can also consider the node-precedence variant of the ECWPPCP defined as follows. In this variant, we suppose that the underlying graph is simple and walks and paths are defined by sequences of vertices instead of sequences of arcs. Remark that the vertex sequence of a walk may contain several copies of a same vertex. A closed walk then *respects* a path P if its contains a copy of every vertex of P in the same order as in P. In a more formal way, given a closed walk $C = (u_1, u_2, \ldots, u_k)$ and a path $P = (v_1, v_2, \ldots, v_l)$, C respects P if there exists a mapping $\pi : \{1, 2, \ldots, l\} \to \{1, 2, \ldots, k\}$ such that $v_i = u_{\pi(i)}$ for all $i = 1, 2, \ldots, l$ and $\pi(i) < \pi(j)$ for all $i < j \in \{1, 2, \ldots, l\}$.

Given an instance (D, v_0, K), the node-precedence variant of the ECWPPCP then consists of finding whether or not there exists an Eulerian closed walk of D starting from v_0 and respecting all the paths of K. This problem is clearly NP-complete due to the NP-completeness of the ECWPPCP. To see this, one can transform any instance (D, v_0, K) of the ECWPPCP into an instance (D', v_0, K') of the node-precedence variant as follows. Replace each arc (u_i, v_i) by the two arcs (u_i, w_i) and (w_i, v_i), where w_i is a new vertex, for all $i = 1, 2, \ldots, m$. Also replace in K' every path $((u_1, v_1), (u_2, v_2), \ldots, (u_k, v_k))$ of K by the path $(w_1, v_1, w_2, v_2, \ldots, w_k, v_k)$. Clearly, any Eulerian closed walk C' of D' can be transformed into an Eulerian closed walk C of D by replacing each subpath $((u_i, w_i), (w_i, v_i))$ by the arc (u_i, v_i), for all $i = 1, 2, \ldots, m$. Moreover, since C' goes exactly once through each vertex w_i, $i = 1, 2, \ldots, m$, C' traverses w_i before w_j if and only if $(u_i, v_i) \prec_C (u_j, v_j)$. This implies that if C' respects K', then C respects K. The converse can be shown along the same line. Moreover, the construction of (D', v_0, K') from (D, v_0, K) can be done in polynomial time.

This work has been motivated by the question of representing a solution of the single-vehicle preemptive pickup and delivery problem as briefly mentioned in the introduction. An outcome of the NP-completeness of the ECWPPCP is that in the general case, a solution of this routing problem needs to be defined by the sequence (and not only by the set) of arcs of the vehicle route, together with the sets of arcs of the demand paths. However, if the vehicle cannot carry more than one demand at a time, then paths of K are pairwise arc-disjoint, which means that each arc of D has at most one predecessor and one successor with respect to K. Therefore in this case, the main algorithm of Section 3 allows to check in polynomial time whether or not a solution, defined by the sets of arcs of the vehicle route and the demand paths, is feasible for this unitary case of the SPPDP. Consequently, it is possible to avoid the order on the arc set associated with the vehicle route in the representation of a solution of the unitary SPPDP. All these results are the basis of the different integer linear programs given in [1] to model various cases of the SPPDP, and of the ongoing polyhedral study on the unitary case as well.

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