# Renormalizability in (noncommutative) field theories

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in collaboration with:
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Paris, 26th of November 2010

# Plan

- Renormalizability in QFT
- Connes-Kreimer approach for renormalizability in QFT
- Noncommutative QFT (NCQFT) and renormalizability
- Connes-Kreimer approach for NCQFT
- Perspectives

# Renormalizability

first computations in QFT end in infinite results

a cure for these infinities (such that the theoretical results can be compared with experiments) - renormalization.

huge experimental success!

renormalizable theories - building block of mathematical physics

# Main ingredients of renormalizability

1. power counting theorem: indicates which Feynman graphs are primitively divergent
   superficial degree of divergence $\omega$ -
   should not depend on the internal structure
   *exemple:* the $\phi^4$ model

   $$\omega = N - 4.$$

   $N$ - number of external legs of the graph
   primitively divergent graphs: $2-$ and $4-$point graphs

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2. locality

$\hookrightarrow$ Bogoliubov subtraction operator $R$
(defined as a *sum over forests*)

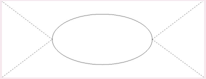
subtraction of divergences

# The physical principle of locality (Feynman graph level)

connected graphs can be reduced to points

graph made of internal propagators of
high energy (or short distance) (ultraviolet (UV) regime) - local

*example:*



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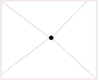

subtraction of local counterterms -
*i. e.* counterterms have the same form as the terms of the action

*via* Taylor expansion

$\Longrightarrow$ renormalized amplitude $\mathcal{A}_R$: finite!

# Connes-Kreimer Hopf algebra

A. Connes and D. Kreimer, *Commun. Math. Phys.*, '00

$\hookrightarrow$ definition of a coproduct $\Delta$

$\mathcal{H}$ - the algebra generated by Feynman graphs
multiplication: disjoint union of graphs

$$\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}, \quad \Delta(G) = G \otimes 1 + 1 \otimes G + \sum_{\gamma \in \underline{G}} \gamma \otimes G/\gamma,$$

$\underline{G}$ – primitively divergent subgraphs of $G$

*renormalization as a factorization issue*

$$\varepsilon : \mathcal{H} \to \mathbb{K}, \quad \varepsilon(1) = 1, \; \varepsilon(G) = 0, \; \forall G \neq 1,$$

$$S : \mathcal{H} \to \mathcal{H},$$

$$S(1_{\mathcal{H}}) = 1_{\mathcal{H}}, \; G \mapsto -G - \sum_{\gamma \in \underline{G}} S(\gamma) G/\gamma.$$

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$$S(1_{\mathcal{H}}) = 1_{\mathcal{H}}, \ G \mapsto -G - \sum_{\gamma \in \underline{G}} S(\gamma) G/\gamma.$$

Theorem: $(\mathcal{H}, \Delta, \varepsilon, S)$ *is a Hopf algebra.*

# Algebraic framework for renormalization

$R$ - the map which given a formal integral returns it evaluated at the subtraction point
$R\mathcal{A}(G)$ - the singular part of the Feynman amplitude $\mathcal{A}(G)$

twisted antipode (recursive definition)

$$S_R^{\mathcal{A}}(1_{\mathcal{H}}) = 1,$$
$$S_R^{\mathcal{A}}(G) = -R(\mathcal{A}(G)) - \sum_{\gamma \in \underline{G}} S_R^{\mathcal{A}}(\gamma) R(\mathcal{A}(G/\gamma)).$$

the renormalized amplitude of the graph

$$\mathcal{A}_R = S_R^{\mathcal{A}} \star \mathcal{A}.$$

*Connes-Kreimer Hopf algebra structure -*
*the combinatorial backbone of renormalization*

# NCQFT - glimpse of the mathematical setup

*the Moyal space*

The *Moyal algebra* is the linear space of smooth and rapidly decreasing functions $\mathcal{S}(\mathbb{R}^{\mathcal{D}})$ equipped with the *Moyal product*:

$$(f \star g)(x) = \int \frac{d^D k}{(2\pi)^D} d^D y\, f(x + \frac{1}{2}\Theta \cdot k) g(x + y) e^{ik \cdot y}.$$

$\star$ - Moyal product
(non-local, noncommutative, associative product)

$$\Theta = \begin{pmatrix} \Theta_2 & 0 \\ 0 & \Theta_2 \end{pmatrix}, \quad \Theta_2 = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}.$$

# Field theory on Moyal space

$\Phi^4$ model:

$$\mathcal{S} = \int d^4x \, [\frac{1}{2}\partial_\mu \Phi \star \partial^\mu \Phi + \frac{1}{2} m^2 \Phi \star \Phi + \frac{\lambda}{4!} \Phi \star \Phi \star \Phi \star \Phi],$$

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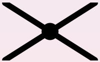
$\int d^4x \, (\Phi\star\Phi)(x) = \int d^4x \, \Phi(x)\,\Phi(x)$
(same propagation as in the commutative theory)

# Implications of the use of the Moyal product in QFT

$$\int d^4x \Phi^{\star 4}(x) \propto \int \prod_{i=1}^{4} d^4 x_i \Phi(x_i) \delta(x_1 - x_2 + x_3 - x_4) \, e^{2i(x_1 - x_2)\Theta^{-1}(x_3 - x_4)}$$

oscillation $\propto$ area of parallelogram



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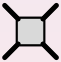

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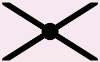

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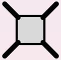
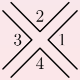


$\hookrightarrow$ non-locality

$\hookrightarrow$ restricted invariance: only under cyclic permutation




$\rightarrow$ ribbon graphs

$\rightarrow$ clear distinction between planar and non-planar graphs

# Feynman graphs in NCQFT

$n$ - number of vertices,
$L$ - number of internal lignes,
$F$ - number of faces,

$$2 - 2g = n - L + F$$

$g \in \mathbb{N}$ - genus

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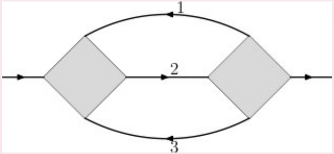
$$2 - 2g = n - L + F$$

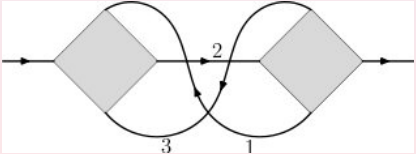
$g \in \mathbb{N}$ - genus

$g = 0$ - planar graph
*example:*




$n = 2, \; L = 3, \; F = 3, \; g = 0$

$g \geq 1$ - non-planar graph
*example:*

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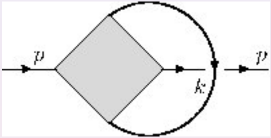
# Renormalization on the Moyal space

UV/IR mixing (S. Minwalla *et. al.*, *JHEP*, '00)

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B - number of faces broken by external lignes
$B > 1$, *planar irregular graph*

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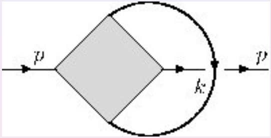


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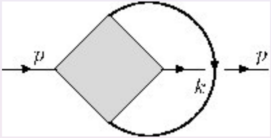


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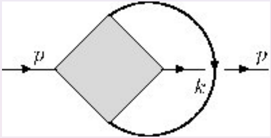


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$$\lambda \int d^4 k \frac{e^{ik_\mu \Theta^{\mu\nu} p_\nu}}{k^2 + m^2} \rightarrow_{|p| \rightarrow 0} \frac{1}{\theta^2 p^2}$$

same type of behavior at any order in perturbation theory

J. Magnen, V. Rivasseau and A. T., *Europhys. Lett.* '09

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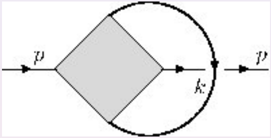


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$\to$ non-renormalizability!

# A first solution to this problem - the Grosse-Wulkenhaar model

*additional harmonic term*

(H. Grosse and R. Wulkenhaar, *Comm. Math. Phys.*, '05)

$$s[\phi(x)] = \int d^4x \left( \frac{1}{2}\partial_\mu\phi \star \partial^\mu\phi + \frac{\Omega^2}{2}(\tilde{x}_\mu\phi) \star (\tilde{x}^\mu\phi) + \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi \right),$$

$$\tilde{x}_\mu = 2(\Theta^{-1})_{\mu\nu}x^\nu.$$

*modification of the propagator* - the model becomes renormalizable

# Translation-invariant renormalizable scalar model

(R. Gurău, J. Magen, V. Rivasseau and A. T., *Commun. Math. Phys.* 2009)

the Grosse-Wulkenhaar model breaks translation-invariance !

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the complete propagator:

$$C(p, m, \theta) = \frac{1}{p^2 + a\frac{1}{\theta^2 p^2} + m^2}$$

*arbitrary planar irregular 2-point function*: same type of $\frac{1}{p^2}$ behavior !

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$\hookrightarrow$ other modification of the action:

$$S = \int d^4p\, [\frac{1}{2}p_\mu\phi \star p^\mu\phi + \frac{1}{2}a\frac{1}{\theta^2 p^2}\phi\star\phi + \frac{1}{2}m^2\phi\star\phi + \frac{\lambda}{4!}V^\star[\phi]]. \quad (1)$$

renormalizability at any order in perturbation theory !

# Glimpse of renormalizability proof - BPHZ scheme

*power counting theorem:*

$\hookrightarrow$ $2-$ and $4-$point planar functions *(primitively divergent)*

- planar regular $2-$point function: wave function and mass renormalization
- planar regular $4-$point function: coupling constant renormalization
- planar irregular $2-$point function: *finite* renormalization of the constant $a$
- planar irregular $4-$point function: convergent

# Comparison between noncommutative models

| | *the "naive" model* | | *GW model* | | *model* (1) | |
|---|---|---|---|---|---|---|
| | 2P | 4P | 2P | 4P | 2P | 4P |
| planar regular | ren. | ren. | ren. | ren. | ren. | ren. |
| planar irregular | UV/IR | log UV/IR | conv. | conv. | finite ren. | conv. |
| non-planar | IR div. | IR div. | conv. | conv. | conv. | conv. |

# Scales - noncommutative renormalization group

definition of the RG scales:

- locus where $C^{-1}(p)$ is big
- locus where $C^{-1}(p)$ is low

$$C^{-1}_{\mathrm{comm}}(p) = p^2$$

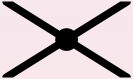
$$C^{-1}_{GW} = p^2 + \Omega^2 x^2$$

$$C^{-1}(p) = p^2 + \frac{a}{\theta^2 p^2}$$

mixing of the UV and IR scales - key of the renormalization

# Renormalizability of NCQFT: locality $\rightarrow$ “Moyality”

| | | |
|---|---|---|
| QFT | $\rightarrow$ | NCQFT |
| locality | $\rightarrow$ | “Moyality” |



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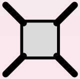
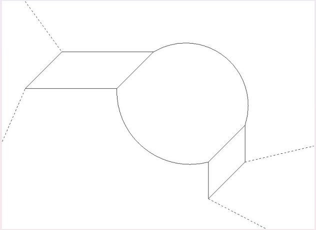
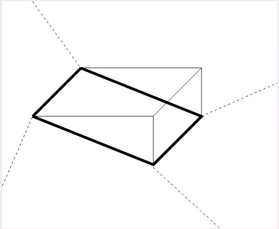


# The principle of “Moyality” - ribbon Feynman graph level





valid iff the graph is planar

renormalization necessary only for the planar sector !

# Hopf algebra for renormalizable NCQFTs

A. T. and F. Vignes-Tourneret, *J. Noncomm. Geom.*, 2008

A. T. and D. Kreimer, arXiv: 0907.2182, submitted

$\hookrightarrow$ definition of a coproduct $\Delta$

$\mathcal{H}$ - the algebra generated by Feynman ribbon graphs

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$$\varepsilon : \mathcal{H} \to \mathbb{K}, \quad \varepsilon(1) = 1, \; \varepsilon(G) = 0, \; \forall G \neq 1,$$

$$S : \mathcal{H} \to \mathcal{H}, \quad G \mapsto -G - \sum_{\gamma \in \underline{G}} S(\gamma) G/\gamma.$$

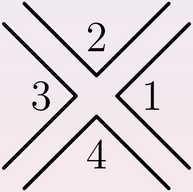
Theorem: $(\mathcal{H}, \Delta, \varepsilon, S)$ *is a Hopf algebra.*

$\hookrightarrow$ 2− and 4−point graphs (in commutative $\phi^4$)
$\rightarrow$ 2− and 4−point planar regular graph

*this Hopf algebra structure - the combinatorial backbone of noncommutative renormalization*

(pre-)Lie algebra structures

Hochschild cohomology - combinatorial Dyson-Schwinger equation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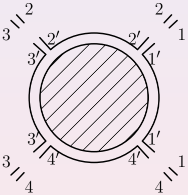
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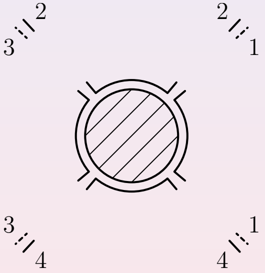
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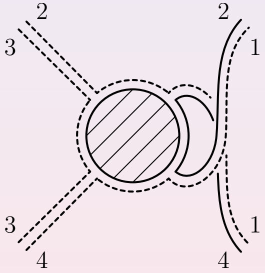
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# Perspectives - can things be (even) more complicated?

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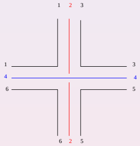
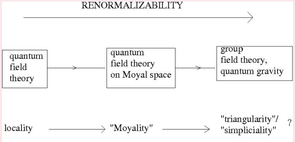
- renormalizable gauge theories? (see next talk)
- generalization to tensor models (recent approaches - group field theory - for a theory of quantum gravity)




various non-trivial combinatorial structures:
  - topological graph polynomial (A. T., work in progress)
  - combinatorial Hopf algebras
- applications of these techniques for the renormalizability study of quantum gravity models



Thank you for your attention!

# Introduction - QFT

QFT - quantum description of particles and interactions, compatible with Einstein's special relativity

$\hookrightarrow$ elementary particle physics (high energy physics)
(Standard Model of Elementary Particle Physics)

greatest experimental success

*QFT formalism* applies also to:
statistical mechanics, condensed matter *etc.*

*"QFT has remained throughout the years one of the most important tools in understanding the microscopic world."*

C. Itzykson and J.-B. Zuber, "QFT"

# Scalar field theory and Feynman graphs

$\Phi : \mathbb{R}^4 \to \mathbb{K}$ - a scalar field

$\mathbb{R}^4$ - the $4-$dimensional space(time), Euclidean metric

the action (functional in the field)

$$S[\Phi(x)] = \int d^4x \left[ \frac{1}{2} \sum_{\mu=1}^{4} \left( \frac{\partial}{\partial x_\mu} \Phi(x) \right)^2 + \frac{1}{2} m^2 \Phi^2(x) + \frac{\lambda}{4!} \Phi^4(x) \right]$$

$m$ - the mass of the particle,
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$m$ - the mass of the particle,
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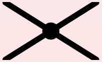
- quadratic part - propagation - edges
- non-quadratic part - interaction potential $V[\Phi(x)] = \frac{\lambda}{4!}\Phi^4(x)$ - vertices:



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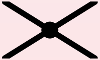
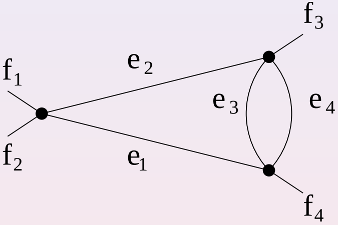
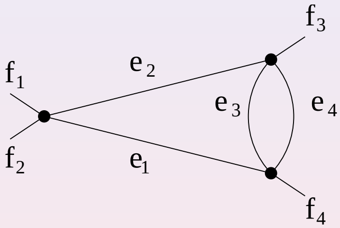


$\Longrightarrow$ (Feynman) graphs of valence 4 - perturbation theory (in $\lambda$)

*example of a Feynman graph:*



*example of a Feynman graph:*




*Fourier transform*: position space (x) $\rightarrow$ momentum space (p)

$$S[\Phi] = \int d^4 p \left[\frac{1}{2}\sum_{\mu=1}^{4}(p_\mu \Phi)^2 + \frac{1}{2}m^2\Phi^2 + \tilde{V}_{\rm int}\right]$$

- propagation - $\frac{1}{p^2+m^2}$
- interaction - $\lambda\,\delta$(sum of incoming/outgoing momentae)

$\Longrightarrow$ Feynman amplitude (in momentum space)

*"The amount of theoretical work one has to cover before being able to solve*
*problems of real practical value is rather large, but this circumstance is [...] likely to become more pronounced in the theoretical physics of the future."*
P.A.M. Dirac, *"The principles of Quantum Mechanics"*, 1930

*renormalization conditions*

$$\Gamma^4(0,0,0,0) = -\lambda_r, \;\; G^2(0,0) = \frac{1}{m^2}, \;\; \frac{\partial}{\partial p^2} G^2(p,-p)|_{p=0} = -\frac{1}{m^4}. \quad (2)$$

where $\Gamma^4$ and $G^2$ are the connected functions and

$0 \to p_m$ (the minimum of $p^2 + \frac{a}{\theta^2 p^2}$)

- most of the techniques of QFT extend to Grosse-Wulkenhaar-like models:
  - the parametric representation

    (R. Gurău and V. Rivasseau, *Commun. Math. Phys.*, '07, A. T. and V. Rivasseau, *Commun. Math. Phys.*, '08, A. T., *J. Phys. Conf. Series*, '08, A. T., solicited by de *Modern Encyclopedia Math. Phys.*)

    (algebraic geometric properties P. Aluffi and M. Marcolli, 0807.1690[math-ph])
  - the Mellin representation

    (R. Gurău, A. Malbouisson, V. Rivasseau and A. T., *Lett. Math. Phys.*, '07)
  - dimensional regularization

    (R. Gurău and A. T., *Annales H. Poincaré*, '08)
- study of vacuum configurations (A. de Goursac, A. T. and J-C. Wallet, *EPJ* **C**, 2008)
- gauge model propositions

  $\hookrightarrow$ non-trivial vacuum state

  (A. de Goursac, J-C. Wallet and R. Wulkenhaar *EPJ* **C**, 2007,2008, H. Grosse and M. Wohlgenannt *EPJ* **C**, 2007 )

# Other translation-invariant field theoretical techniques

- parametric representation (A. T., *J. Phys.* **A** 2009)
    - power counting dependence on the graph genus
- relation with Bollobás-Riordan topologic ribbon graph polynomial

  (T. Krajewski, V. Rivasseau, A. T. and Z. Wang, *J. Noncomm. Geom.* (2010)

  trees $\to$ $\star-$trees (quasi-trees)
- renormalization group flow

  (J. Ben Geloun and A. T., *Lett. Math. Phys.* 2008)

$$\beta_\lambda \propto \beta_\lambda^{comm}, \; \beta_a = 0$$

- commutative limit

  (J. Magnen, V. Rivasseau and A. T., *Lett. Math. Phys.* 2008)
- field theories with other products
  (*ex.:* the Wick-Voros product)

  $\hookrightarrow$ A. T. and P. Vitale, *Phys. Rev.* **D** (2010)