Scientific Research

Generalized Powers of Substitution with Pre-Function Operators

Laurent Poinsot

University Paris 13, Paris Sorbonne Cité, LIPN, CNRS, Villetaneuse, France
Email: laurent.poinsot@lipn.univ-paris13.fr

ABSTRACT

An operator on formal power series of the form $S \mapsto \mu S(\sigma)$, where μ is an invertible power series, and σ is a series of the form $t + \mathcal{O}(t^2)$ is called a unipotent substitution with pre-function. Such operators, denoted by a pair (μ, σ), form a group. The objective of this contribution is to show that it is possible to define a generalized powers for such operators, as for instance fractional powers $(\mu, \sigma)^{\frac{a}{b}}$ for every $\frac{a}{b} \in \mathbb{Q}$.

Keywords: Formal Power Series; Formal Substitution; Riordan Group; Generalized Powers; Sheffer Sequences; Umbral Calculus

1. Substitution of Formal Power Series

In this contribution we let $\mathbb{K}$ denote any field of characteristic zero. We recall some basic definitions from [1,2]. The algebra of formal power series in the variable t is denoted by $\mathbb{K}[[t]]$. In what follows we sometimes use the notation $S(t)$ for $S \in \mathbb{K}[[t]]$ to mean that S is a formal power series of the variable t. We recall that any formal power series of the form $\mu = \lambda + tS$ for $\lambda \in \mathbb{K}^*$ and $S \in \mathbb{K}[[t]]$ is invertible with respect to the usual product of series. Its inverse is denoted by μ^{-1} and has the form $\lambda^{-1} + tV$ for some $V \in \mathbb{K}[[t]]$. In particular, the set of all series of the form $\mu = 1 + tT$ forms a group under multiplication, called the group of *unipotent series*. For a series of the form $\sigma = \lambda t + t^2 T$, we may define for any other series $S = \sum_{n \geq 0} \alpha_n t^n$ an operation of *substitution* given by $S(\sigma) = \sum_{n \geq 0} \alpha_n \sigma^n$. A *unipotent substitution* is a series of the form $\sigma = t + t^2 T$. Such series form a group under the operation of substitution, called the *group of unipotent substitutions* (whenever $\lambda \neq 0$, a series $\sigma = \lambda t + t^2 T$ is invertible under substitution, and the totality of such series forms a group under the operation of substitution called the *group of substutions*, and it is clear that the group of unipotent substitutions is a sub-group of this one). The

inverse of σ is then denoted by $\sigma^{[-1]}$ and satisfies $\sigma(\sigma^{[-1]}) = t = \sigma^{[-1]}(\sigma)$. Finally, it is possible to define a semi-direct product of groups by considering pairs (μ, σ) where μ is a unipotent series, and σ is a unipotent substitution, and the operation $(\mu_1, \sigma_1) \ltimes (\mu_2, \sigma_2) = (\mu_1(\sigma_2) \mu_2, \sigma_1(\sigma_2))$. The identity element is $(1, t)$. This group has been previously studied in [3-5], and is called the *group of (unipotent) substitutions with pre-function*. These substitutions with pre-function act on $\mathbb{K}[[t]]$ as follows: $(\mu, \sigma) \cdot S = \mu S(\sigma)$ for every series S. In [3] is associated a doubly-infinite matrix $M_{(\mu, \sigma)}$ to each such operator which defines a matrix representation of the group of substitutions with pre-function, and it is proved that there exists a one-parameter sub-group $\lambda \in \mathbb{K} \mapsto M_{(\mu, \sigma)}^\lambda$. Therefore, it satisfies $M_{(\mu, \sigma)}^{\alpha + \beta} = M_{(\mu, \sigma)}^\alpha M_{(\mu, \sigma)}^\beta$ for every $\alpha, \beta \in \mathbb{K}$, and $M_{(\mu, \sigma)}^\lambda$ is the usual λ-th power of $M_{(\mu, \sigma)}$ whenever λ is an integer. It amounts that for every λ, $M_{(\mu, \sigma)}^\lambda$ is the matrix representation of a substitution with pre-function say $(\mu_\lambda, \sigma_\lambda)$ so that $M_{(\mu, \sigma)}^\lambda = M_{(\mu_\lambda, \sigma_\lambda)}$. The authors of [3] then define $(\mu, \sigma)^\lambda = (\mu_\lambda, \sigma_\lambda)$. Actually in [3] no formal proof is given for the existence of such generalized powers for matrices or unipotent substitu-

tions with pre-function.

In this contribution, we provide a combinatorial proof for the existence of these generalized powers for unipotent substitutions with pre-function, and we show that this even forms a one-parameter sub-group. To achieve this objective we use some ingredients well-known in combinatorics such as delta operators, Sheffer sequences and umbral composition which are briefly presented in what follows (Sections 2, 3, 4 and 5). The Section 6 contains the proof of our result.

2. Differential and Delta Operators, and Their Associated Polynomial Sequences

By *operator* we mean a linear endomorphism of the $\mathbb{K}$-vector space of polynomials $\mathbb{K}[\mathrm{x}]$ (in one indeterminate x). The composition of operators is denoted by a simple juxtaposition. If $p \in \mathbb{K}[\mathrm{x}]$, then we sometimes write $p(\mathrm{x})$ to mean that p is a polynomial in the variable x.

Let $(p_n)_{n \geq 0} \in \mathbb{K}[\mathrm{x}]^{\mathbb{N}}$ be a sequence of polynomials. It is called a *polynomial sequence* if $\deg(p_n) = n$ for every $n \geq 0$ (in particular, $p_0 \in \mathbb{K}^*$). It is clear that a polynomial sequence is thus a basis for $\mathbb{K}[\mathrm{x}]$.

An operator D is called a *differential operator* (see [6]) if

1) $D\lambda = 0$ for every $\lambda \in \mathbb{K}$.

2) $\deg(Dp) = \deg(p) - 1$ for every non-constant polynomial p.

For instance, the usual derivation ∂ of polynomials is a differential operator. Moreover, let $\lambda \in \mathbb{K}$, and let us define the *shift-invariant operator* E^{λ} as the unique linear map such that $E^{\lambda}(\mathrm{x}^n) = (\mathrm{x} + \lambda)^n$ for every $n \geq 0$. Then, $\Delta = E^1 - id$ is also a differential operator. A polynomial sequence $(p_n)_{n \geq 0}$ is said to be a *normal family* if

1) $p_0 = 1$.

2) $p_n(0) = 0$ for every $n > 0$.

Let D be a differential operator. A normal family $(p_n)_{n \geq 0}$ is said to be a *basic family* for D if

$$Dp_{n+1} = (n+1)p_n$$

for every $n \geq 0$. It is proved in [6] that for any differential operator admits is one and only one basic family, and, conversely, any normal family is the basic family of a unique differential operator. As an example, the normal family $(\mathrm{x}^n)_{n \geq 0}$ is the basic family of ∂.

Let L be an operator such that for every non-zero polynomial $p \in \mathbb{K}[\mathrm{x}]$, $\deg(Lp) < \deg(p)$ (in particular, $L(\lambda) = 0$ for every constant $\lambda \in \mathbb{K}$). Such an operator is called a *lowering operator* (see [7]). For instance any differential operator is a lowering operator. Then given a lowering operator L, we may consider the algebra of formal power series $\mathbb{K}[[L]]$ of operators of

the form $\sum_{n \geq 0} \alpha_n L^n$ where $\alpha_n \in \mathbb{K}$ for every $n \geq 0$. The series $\sum_{n \geq 0} \alpha_n L^n$ converges to an operator of $\mathbb{K}[\mathrm{x}]$ in the topology of simple convergence (when $\mathbb{K}$ has the discrete topology) since for every $p \in \mathbb{K}[\mathrm{x}]$, there exists $n_p \in \mathbb{N}$ such that for all $n \geq n_p$, $L^n(p) = 0$, so that we may define

$$\left(\sum_{n \geq 0} \alpha_n L^n \right)(p) = \sum_{n=0}^{n_p} \alpha_n L^n(p).$$

According to [6], if D is a differential operator, then $\phi \in \mathbb{K}[[D]]$ if, and only if, ϕ commutes with D, *i.e.*, $\phi D = D\phi$. Moreover, if $\phi = \sum_{n \geq 0} \phi_n D^n \in \mathbb{K}[[D]]$, then ϕ is also a differential operator if, and only if, $\phi_0 = 0$ and $\phi_1 \neq 0$.

Following [1], let us define a sequence of polynomials $\left(\binom{\mathrm{x}}{n} \right)_{n \geq 0}$ by $\binom{\mathrm{x}}{0} = 0$ and $\binom{\mathrm{x}}{n+1} = \frac{1}{(n+1)!} \prod_{i=0}^{n} (\mathrm{x} - i)$ for every integer n. For $\lambda \in \mathbb{K}$, we denote by $\binom{\lambda}{n}$ the value of the polynomial $\binom{\mathrm{x}}{n}$ for $\mathrm{x} = \lambda$. Let L be a lowering operator, and let $U = id + L \in \mathbb{K}[[L]]$ be its *unipotent part*. Then we may consider *generalized power* $U^{\lambda} = \sum_{n \geq 0} \binom{\lambda}{n} L^n \in \mathbb{K}[[L]]$ (in particular, this explains the notation E^{λ} for the shift operator). We observe that for every integer k, U^k really coincides to the k-th power $\underbrace{U \circ \cdots \circ U}_{k \text{ factors}}$ of U. Moreover, $U^{\alpha + \beta} = U^{\alpha} U^{\beta}$ for every $\alpha, \beta \in \mathbb{K}$. We may also form

$$\log(U) = \log(id + L) = \sum_{n \geq 1} \frac{(-1)^{n+1}}{n} L^n \in \mathbb{K}[[L]]$$

in such a way that for every $\lambda \in \mathbb{K}$,

$$U^{\lambda} = \exp(\lambda \log(U))$$

where for every $\phi = \sum_{n \geq 0} \phi_n L^n \in \mathbb{K}[[L]]$ with $\phi_0 = 0$, $\exp(\phi) = \sum_{n \geq 0} \frac{1}{n!} \phi^n$ (it is a well-defined operator). This kind of generalized powers may be used to compute fractional power of the form $U^{\frac{a}{b}}$ for every $a \in \mathbb{Z}$, $b \in \mathbb{Z}^*$ (for instance, $\sqrt[n]{U} = U^{\frac{1}{n}}$). They satisfy the usual properties of powers: $U^0 = id$, $U^{\alpha + \beta} = U^{\alpha} U^{\beta}$. The objective of this contribution is to provide a proof of the existence of such generalized powers for unipotent substitutions with pre-function.

Following [8], we may consider the following sub-set of differential operators, called delta operators. A polynomial sequence $(p_n)_{n\geq 0}$ is said to be of *binomial-type* if for every $n \geq 0$,

$$p_n(x + y) = \sum_{k=0}^{n} p_k(x) p_{n-k}(y) \in \mathbb{K}[x, y].$$

An operator ϕ is a *shift-invariant* operator if for every $\lambda \in \mathbb{K}$, $\phi E^{\lambda} = E^{\lambda}\phi$. Now, a *delta operator* D is a shift invariant operator such that $Dx \in \mathbb{K}^*$. For instance, the usual derivation ∂ of polynomials is a delta operator. It can be proved that a delta operator is a differential operator. The basic family (uniquely) associated to a delta operator is called its *basic set*. Moreover, the basic set of a delta operator is of binomial-type, and to any polynomial sequence of binomial-type is uniquely associated a delta operator. If D is a delta operator, then there exists a unique $\mathbb{K}$-algebra isomorphism from $\mathbb{K}[[t]]$ to the ring of shift-invariant operators $\mathbb{K}[[D]]$ that maps $S = \sum_{n\geq 0} s_n \dfrac{t^n}{n!}$ to $S(D) = \sum_{n\geq 0} s_n \dfrac{D^n}{n!}$. In [8] is proved that given a delta operator D, and a series $\sigma = \sum_{n\geq 1}\sigma_n t^n$ $(\sigma_0 = 0)$ with $\sigma_1 \neq 0$, then $\sigma(D)$ is also a delta operator. Conversely, if ϕ is a shift-invariant operator (so that $\phi \in \mathbb{K}[[D]]$), then if it is a delta operator, the unique series

$$\sigma = \sum_{n\geq 0}\sigma_n \frac{t^n}{n!} \in \mathbb{K}[[t]]$$

such that $\sigma(D) = \phi$ satisfies $\sigma_0 = 0$ and $\sigma_1 \neq 0$.

3. Sheffer Sequences

In this section, we also briefly recall some definitions and results from [8].

Let $(p_n)_{n\geq 0}$ be a sequence of polynomials in $\mathbb{K}[x]$. We define the *exponential generating function* of $(p_n)_{n\geq 0}$ as

$$EGF((p_n)_n; t) = \sum_{n\geq 0} p_n \frac{t^n}{n!} \in \mathbb{K}[x][[t]].$$

Let D be a delta operator and $(p_n)_{n\geq 0}$ be its basic set. Let $\sigma \in \mathbb{K}[[t]]$ with $\sigma_0 = 0$ and $\sigma_1 \neq 0$ such that $\sigma(\partial) = D$. Then from [8],

$$EGF((p_n)_n; t) = e^{x\sigma^{[-1]}(t)}.$$

A polynomial sequence $(s_n)_{n\geq 0}$ is said to be a *Sheffer sequence* (also called a *polynomial sequence of type zero* in [9] or a *poweroid* in [10]) if there exists a delta operator D such that

1) $s_0 \in \mathbb{K}^*$,

2) $Ds_{n+1} = (n+1)s_n$ for every n.

Following [9], a polynomial sequence $(s_n)_{n\geq 0}$ is a Sheffer sequence if, and only if, there exists a pair

(μ, σ) of formal power series in $\mathbb{K}[[t]]$ with μ invertible, and $\sigma_0 = 0$, $\sigma_1 \neq 0$, such that

$$EGF((s_n)_n; t) = \mu(t)e^{x\sigma(t)}.$$

Remark 1. The basic set of a delta operator D is a Sheffer sequence.

Let D be a delta-operator with basic set $(p_n)_{n\geq 0}$. Following [8], the following result holds.

Proposition 1. A polynomial sequence $(s_n)_{n\geq 0}$ is a Sheffer sequence if, and only if, there exists an invertible shift-invariant operator S such that $s_n = S^{-1}p_n$ for each $n \geq 0$. Moreover, let S be an invertible shift-invariant operator. Let $\mu \in \mathbb{K}[[t]]$ be the unique formal power series such that $\mu(\partial) = S$. Then, μ is invertible, and

$$EGF((s_n)_n; t) = \left(\mu\left(\sigma^{[-1]}\right)\right)^{-1} e^{x\sigma^{[-1]}}$$

where $(s_n)_n$ is the Sheffer sequence defined by $s_n = S^{-1}p_n$ for each $n \geq 0$, and $\sigma \in \mathbb{K}[[t]]$ is the unique formal power series such that $\sigma(\partial) = D$. Finally we also have the following characterization.

Proposition 2. Let $(s_n)_n$ be a polynomial sequence. It is a Sheffer sequence if, and only if, there exists a delta operator D with basic set $(p_n)_n$ such that

$$s_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} s_k(x) p_{n-k}(y) \in \mathbb{K}[x, y].$$

4. Umbral Composition

This section is based on [11].

Let $(p_n)_n$ be a fixed polynomial sequence. Let us define an operator ϕ by $\phi(x^n) = p_n$ for each $n \geq 0$. Since $(p_n)_n$ is a basis of $\mathbb{K}[x]$, this means that ϕ is a linear isomorphism of $\mathbb{K}[x]$. When $(p_n)_n$ is the basic set of a delta operator, then ϕ is referred to as an *umbral operator*, while if $(p_n)_n$ is a Sheffer sequence, then ϕ is said to be a *Sheffer operator*. An umbral operator maps basic sets to basic sets, while a Sheffer operator maps Sheffer sequences to Sheffer sequences.

Let $(p_n)_n$ be a polynomial sequence. For every n, $p_n = \sum_{k=0}^{n} \langle p_n | x^k \rangle x^k$ where $\langle p | x^k \rangle$ is the coefficient of x^k in the polynomial $p \in \mathbb{K}[x]$. Let $(p_n)_n$ and $(q_n)_n$ be two polynomial sequences. Their *umbral composition* is defined as the polynomial sequence $(r_n)_n = (p_n)_n \# (q_n)_n$ defined by

$$r_n = \sum_{k=0}^{n} \langle p_n | x^k \rangle q_k$$

for each $n \geq 0$. By simple computations, it may be proved that $\langle r_n | x^k \rangle = \sum_{\ell=k}^{n} \langle p_n | x^\ell \rangle \langle q_\ell | x^k \rangle$. The set of all polynomial sequences becomes a (non-commutative) monoid under $\#$ with $(x^n)_n$ as identity. We observe

that if T is the operator defined by $T\mathrm{x}^n = q_n$ for each $n \geq 0$, then $(p_n)_n \# (q_n)_n = (Tp_n)_n$. More generally, we have $(T^k \mathrm{x}^n)_n = ((q_n)_n)^{\#(k)}$ where $((q_n)_n)^{\#(k)}$ is the k-th power of $(q_n)_n$ for the umbral composition (it is equal to a sequence say $(r(k)_n)_n$ and we denote $r(k)_n$ by $q_n^{\#(k)}$). Under umbral composition, the set of all Sheffer sequences is a (non-commutative) group, called the *Sheffer group* ([12]), and the set of all basic sequences is a sub-group of the Sheffer group.

From [8] we have the following result that combines delta operators, basis sets, Sheffer sequences and umbral composition.

Theorem 1. Let Q and P be two delta operators with respective basic sets $(q_n)_n$ and $(p_n)_n$. Let S and T be two invertible shift-invariant operators. Let $(s_n)_n$ and $(t_n)_n$ be the Sheffer sequences defined by $s_n = S^{-1} q_n$ and $t_n = T^{-1} p_n$ for each n. Let μ, ν be two invertible series such that $S = \mu(\partial)$, $T = \nu(\partial)$. Let σ, τ be two formal power series with $\sigma_0 = 0 = \tau_0$, $\sigma_1 \neq 0 \neq \tau_1$ such that $Q = \sigma(\partial)$ and $P = \tau(\partial)$. Then, $R = T \circ \mu(P) = \nu(\partial) \mu(\tau(\partial))$ is a shift-invariant operator, $\sigma(\tau(\partial))$ is a delta operator with basic sequence $(v_n)_n = (q_n)_n \# (p_n)_n$. Finally, let $(r_n)_n$ be the Sheffer sequence given by $(r_n)_n = (s_n)_n \# (t_n)_n$. Then, $r_n = R^{-1} v_n$ for each $n \in \mathbb{N}$.

It may be proved that if $(s_n)_n$ is the Sheffer sequence obtained from the delta operator $D = \sigma(\partial)$ with basic set $(p_n)_n$ and the invertible shift-invariant operator $S = \mu(\partial)$, i.e., $s_n = S^{-1} p_n$ for each n, then the inverse $(q_n)_n$ of $(p_n)_n$ with respect to the umbral composition is the basic set of the delta operator $\sigma^{[-1]}(\partial)$, the inverse $(t_n)_n$ of $(s_n)_n$ with respect to the umbral composition is the Sheffer sequence $t_n = \mu\left(\sigma^{[-1]}(\partial)\right) q_n$.

5. Unipotent Sequences

The basic set $(p_n)_n$ of a delta operator D is said to be *unipotent* if the unique series σ such that $D = \sigma(\partial)$ satisfies $\sigma_1 = 1$ (and, obviously, $\sigma_0 = 0$), i.e., σ is a unipotent substitution. A Sheffer sequence $(s_n)_n$ associated to a delta operator $D = \sigma(\partial)$ (with $\sigma_0 = 0$, $\sigma_1 \neq 0$) and an invertible shift-invariant operator $S = \mu(\partial)$ (with μ invertible), i.e., $s_n = S^{-1} p_n$ for every n where $(p_n)_n$ is the basic set of D, is said to be *unipotent* if $(p_n)_n$ is unipotent, and if μ is unipotent, i.e., $\mu_0 = 1$. It is also clear from the previous section (theorem 4) that the (umbral) inverse of a unipotent basic set is unipotent, and the (umbral) inverse of a

Sheffer sequence is also unipotent.

It is clear from theorem 4 that the group of basic sets under umbral composition is isomorphic to the group of substitutions. Moreover, the group of unipotent basic sets also is isomorphic to the group of unipotent substitutions. Likewise, the group of (unipotent) Sheffer sequences is isomorphic to the group of (unipotent) substitutions with pre-function (see also [12]).

Lemma 1. Let (μ, σ) be a substitution with pre-function, and let $((s_n)_n, (p_n)_n)$ be the Sheffer sequence and the basic set associated to the delta operator $\sigma(\partial)$ and the invertible shift-invariant operator $\mu(\partial)$ (this means that $(p_n)_n$ is the basic set of $\sigma(\partial)$, and $s_n = (\mu(\partial))^{-1} p_n$ for each n). Then, (μ, σ) is a unipotent substitution with pre-function if, and only if, $\langle p_n | \mathrm{x}^n \rangle = 1 = \langle s_n | \mathrm{x}^n \rangle$ for every n.

Proof. Let us first assume that (μ, σ) is a unipotent substitution with prefunction. We have $p_0 = 1$ for every basic set, so that $\langle p_0 | 1 \rangle = 1$. Let $n \in \mathbb{N}$. We have $\sigma(\partial) = \partial + \sum_{k \geq 2} \sigma_k \partial^k$. Then, $\sigma(\partial) p_{n+1} = (n+1) p_n$ is equivalent to

$$\sigma(\partial)\left(\sum_{k=0}^{n+1} \langle p_{n+1} | \mathrm{x}^k \rangle\right) = (n+1) \sum_{k=0}^{n} \langle p_n | \mathrm{x}^k \rangle.$$

By identification of the coefficient of x^n on both sides, we obtain $(n+1)\langle p_{n+1} | \mathrm{x}^{n+1} \rangle = (n+1)\langle p_n | \mathrm{x}^n \rangle$ (since σ is assumed to be a unipotent substitution), and, by induction, $\langle p_{n+1} | \mathrm{x}^{n+1} \rangle = 1$. Besides, we have $s_n = (\mu(\partial))^{-1} p_n$ for each n. But $(\mu(\partial))^{-1} = \mu^{-1}(\partial)$ (because there is a ring isomorphism between $\mathbb{K}[[\mathrm{t}]]$ and $\mathbb{K}[[\partial]]$), and $\mu^{-1} = 1 + \mathrm{t}\nu$, where $\nu \in \mathbb{K}[[\mathrm{t}]]$. Then, by identification of the coefficient of x^n, we have $\langle s_n | \mathrm{x}^n \rangle = \langle p_n | \mathrm{x}^n \rangle = 1$ for every n. Conversely, let us assume that $((s_n)_n, (p_n)_n)$ is the Sheffer sequence and the basic set associated to the delta operator $\sigma(\partial)$ and the invertible shift-invariant operator $\mu(\partial)$ with $\langle p_n | \mathrm{x}^n \rangle = 1 = \langle s_n | \mathrm{x}^n \rangle$ for every n. By construction we have $\sigma_1(n+1)\langle p_{n+1} | \mathrm{x}^{n+1} \rangle = (n+1)\langle p_n | \mathrm{x}^n \rangle$ so that $\sigma_1 = 1$. Likewise, $\mu_0 \langle s_n | \mathrm{x}^n \rangle = \langle p_n | \mathrm{x}^n \rangle$, so that $\mu_0 = 1$. □

6. Generalized Powers of Unipotent Substitutions with Pre-Function

The purpose of this section is to define $(\mu, \sigma)^\lambda$ for any $\lambda \in \mathbb{K}$ and any unipotent substitution with pre-function (μ, σ), and to prove that it is also a unipotent substitution with pre-function. Moreover we show that

$\lambda \mapsto (\mu,\sigma)^{\lambda}$ is a one-parameter sub-group, *i.e.*, $(\mu,\sigma)^{\alpha+\beta} = (\mu,\sigma)^{\alpha} \rtimes (\mu,\sigma)^{\beta}$ for every $\alpha,\beta \in \mathbb{K}$, and $(\mu,\sigma)^0 = (1,\mathrm{t})$.

Let (μ,σ) be a unipotent substitution with pre-function of $\mathbb{K}[[\mathrm{t}]]$. Let $(p_n)_n$ be the unipotent basic set of the (unipotent) delta operator $\sigma(\partial)$. Let $(s_n)_n$ be the unipotent Sheffer sequence associated to $\sigma(\partial)$ and the (unipotent) invertible shift-invariant operator $\mu(\partial)$. Let R be the umbral operator given by $R\mathrm{x}^n = p_n$ for all n, and let T be the Sheffer operator defined by $T\mathrm{x}^n = s_n$ for all n. It is easily checked that for every integer k, $p_n^{\#(k)} = R^k(\mathrm{x}^n)$ and $s_n^{\#(k)} = T^k(\mathrm{x}^n)$. In particular, for each n, $T(\mathrm{x}^n) = s_n = \mathrm{x}^n + \sum_{k=0}^{n-1}\langle s_n | \mathrm{x}^n \rangle \mathrm{x}^k$ (by Lemma 5). Therefore, $T = id + L$, where $L\mathrm{x}^n = \sum_{k=0}^{n-1}\langle s_n | \mathrm{x}^n \rangle \mathrm{x}^k$ for each n. The operator L is actually a lowering operator. Then according to section 2, it is possible to define $T^{\lambda} = \sum_{k\geq 0}\binom{\lambda}{k}L^k \in \mathbb{K}[[L]]$ for every $\lambda \in \mathbb{K}$. Moreover, we have $T^{\lambda} = \exp(\lambda \log(T))$. For each $\lambda \in \mathbb{K}$, let us define $s_n(\lambda) = T^{\lambda}(\mathrm{x}^n)$ for every $n \in \mathbb{N}$. When $\lambda = k \in \mathbb{N}$, we have $s_n(k) = T^k(\mathrm{x}^n) = s_n^{\#(k)}$. So that in this case, $(s_n(k))_n$ is the unipotent Sheffer sequence associated to $(\mu,\sigma)^k$. This means that if $(\mu_k,\sigma_k) = (\mu,\sigma)^k$, and $(q_n)_n$ is the unipotent basic set of the (unipotent) delta operator $\sigma_k(\partial)$, then $s_n^{\#(k)} = \mu_k^{-1}(\partial)q_n$ for each n. Similarly, let $R\mathrm{x}^n = p_n = \mathrm{x}^n + \sum_{k=0}^{n-1}\langle p_n | \mathrm{x}^k \rangle \mathrm{x}^k$ for every n. Therefore, $R = id + N$, where $N\mathrm{x}^n = \sum_{k=0}^{n-1}\langle p_n | \mathrm{x}^k \rangle \mathrm{x}^k$ is a lowering operator. Again for every $\lambda \in \mathbb{K}$, we define $R^{\lambda} = \exp(\lambda \log(R)) = \sum_{k\geq 0}\binom{\lambda}{k}N^k \in \mathbb{K}[[N]]$. For each λ, we define $p_n(\lambda) = R^{\lambda}\mathrm{x}^n$ for each n. In particular for $\lambda = k \in \mathbb{K}$, $p_n(k) = p_n^{\#(k)}$, so that it is the basic set of the unipotent delta operator $\sigma^{[k]}(\partial)$. Clearly, $s_n(k) = \mu_k^{-1}(\partial)p_n(k)$ for each n. Thus for every $k \in \mathbb{N}$, we have

$$p_n(k)(\mathrm{x}+\mathrm{y}) = \sum_{i=0}^{n}\binom{n}{i}p_i(k)(\mathrm{x})p_{n-i}(k)(\mathrm{y})$$
$$= \sum_{i=0}^{n}\binom{n}{i}R^k(\mathrm{x}^i)R^k(\mathrm{y}^{n-i}), \quad (1)$$

$$s_n(k)(\mathrm{x}+\mathrm{y}) = \sum_{i=0}^{n}\binom{n}{i}s_i(k)(\mathrm{x})p_{n-i}(k)(\mathrm{y})$$
$$= \sum_{i=0}^{n}\binom{n}{i}T^k(\mathrm{x}^i)R^k(\mathrm{y}^{n-i}), \quad (2)$$

Now, let z be a variable commuting with x and y, and let us define

$$s_n(z) = \left(\sum_{i\geq 0}\binom{z}{i}L^i\right)(\mathrm{x}^n) = \sum_{i=0}^{n}\binom{z}{i}L^i\mathrm{x}^n \in \mathbb{K}[\mathrm{x},z]$$

and similarly,

$$p_n(z) = \left(\sum_{i\geq 0}\binom{z}{i}N^i\right)(\mathrm{x}^n) = \sum_{i=0}^{n}\binom{z}{i}N^i\mathrm{x}^n \in \mathbb{K}[\mathrm{x},z],$$

for each n. As polynomials in the variable z, their degrees are at most n. As polynomials in the variable z, $s_n(z)(\mathrm{x}+\mathrm{y})$, $p_n(z)(\mathrm{x}+\mathrm{y})$,

$$\sum_{i=0}^{n}\binom{n}{i}p_i(z)(\mathrm{x})p_{n-i}(z)(\mathrm{y})$$

and

$$\sum_{i=0}^{n}\binom{n}{i}s_i(z)(\mathrm{x})p_{n-i}(z)(\mathrm{y})$$

have also a degree at most n. Because the equations (1) and (2) hold for every integer k, the polynomials (in the variable z)

$$p_n(z)(\mathrm{x}+\mathrm{y}) - \sum_{i=0}^{n}\binom{n}{i}p_i(z)(\mathrm{x})p_{n-i}(z)(\mathrm{y})$$

and

$$s_n(z)(\mathrm{x}+\mathrm{y}) - \sum_{i=0}^{n}\binom{n}{i}s_i(z)(\mathrm{x})p_{n-i}(z)(\mathrm{y})$$

are identically zero, and the above equations hold for every $\lambda \in \mathbb{K}$. Therefore, $(p_n(\lambda))_n$ is a polynomial sequence of binomial-type, and $(s_n(\lambda))_n$ is a Sheffer sequence for every $\lambda \in \mathbb{K}$. Moreover, for every $\alpha,\beta \in \mathbb{K}$, we have

$$p_n(\alpha+\beta) = R^{\alpha+\beta}(\mathrm{x}^n) = R^{\beta}(R^{\alpha}(\mathrm{x}^n))$$
$$= R^{\beta}(p_n(\alpha)) = \sum_{k=0}^{n}\langle p_n(\alpha)|\mathrm{x}^k \rangle R^{\beta}(\mathrm{x}^k)$$
$$= \sum_{k=0}^{n}\langle p_n(\alpha)|\mathrm{x}^k \rangle p_k(\beta)$$

so that $(p_n(\alpha+\beta))_n = (p_n(\alpha))_n\#(q_n(\beta))_n$. Similarly, $(s_n(\alpha+\beta))_n = (s_n(\alpha))_n\#(s_n(\beta))_n$ for every $\alpha,\beta \in \mathbb{K}$. Moreover,

$$(p_n(0))_n = (R^0(\mathrm{x}^n))_n = (\mathrm{x}^n)_n = (T^0(\mathrm{x}^n))_n = (q_n(0))_n.$$

Therefore, $\lambda \in \mathbb{K} \mapsto (p_n(\lambda))_n$ and $\lambda \in \mathbb{K} \mapsto (s_n(\lambda))_n$

are one-parameter sub-groups. It follows that

$$\left(p_n(-\lambda)\right)_n = \left(\left(p_n(\lambda)\right)_n\right)^{\#(-1)}$$

and

$$\left(s_n(-\lambda)\right)_n = \left(\left(s_n(\lambda)\right)_n\right)^{\#(-1)}$$

(inverses under umbral operation).

We define $(\mu,\sigma)^{\lambda}$ as the pair of formal power series $(\mu_{\lambda},\sigma_{\lambda})$ such that σ_{λ} is the substitution that defines the delta operator $\sigma_{\lambda}(\partial)$ with basic sequence $\left(p_n(\lambda)\right)_n$, and μ_{λ} is the invertible series such that

$$s_n(\lambda) = \left(\mu_{\lambda}(\partial)\right)^{-1} p_n(\lambda)$$

for each $\lambda \in \mathbb{K}$. Since $\left(p_n(\lambda)\right)_n$ and $\left(s_n(\lambda)\right)_n$ are unipotent sequences, it is clear that μ_{λ} is unipotent, and σ_{λ} is a unipotent substitution. It is also clear that whenever $\lambda = k \in \mathbb{N}$, then $(\mu_k,\sigma_k) = (\mu,\sigma)^k$. Let us check that $\lambda \in \mathbb{K} \mapsto (\mu_{\lambda},\sigma_{\lambda})$ is a one-parameter sub-group of the group of unipotent substitutions with pre-function. This means that for every $\alpha,\beta \in \mathbb{K}$,

$$\left(\mu_{\alpha+\beta},\sigma_{\alpha+\beta}\right) = \left(\mu_{\alpha},\sigma_{\alpha}\right) \rtimes \left(\mu_{\beta},\sigma_{\beta}\right)$$
$$= \left(\mu_{\alpha}\left(\sigma_{\beta}\right)\mu_{\beta}, \sigma_{\alpha}\left(\sigma_{\beta}\right)\right)$$

First of all, by definition, $\sigma_{\alpha+\beta}$ is the unipotent substitution associated to the basic set

$$\left(p_n(\alpha+\beta)\right)_n = \left(p_n(\alpha)\right)_n \#\left(p_n(\beta)\right)_n,$$

and therefore $\sigma_{\alpha+\beta} = \sigma_{\alpha}\left(\sigma_{\beta}\right)$. In a similar way, the series $\mu_{\alpha+\beta}$ is uniquely associated to the Sheffer sequence $\left(s_n(\alpha+\beta)\right)_n = \left(s_n(\alpha)\right)_n \#\left(s_n(\beta)\right)_n$ and to the basic set $\left(p_n(\alpha+\beta)\right)_n = \left(p_n(\alpha)\right)_n \#\left(p_n(\beta)\right)_n$. Again this means that $\mu_{\alpha+\beta} = \mu_{\alpha}\left(\sigma_{\beta}\right)\mu_{\beta}$. Therefore, we obtain the expected result. It is also clear that $(\mu_0,\sigma_0) = (1,t)$.

Remark 2. In particular, since $\mathbb{K}$ is a field of characteristic zero, for every $q \in \mathbb{Q}$, we may define fractional powers $(\mu,\sigma)^q$ such as for instance $\sqrt[n]{(\mu,\sigma)}$ for each integer n.

REFERENCES

[1] A. Benhissi, "Rings of Formal Power Series," Queen's Papers in Pure and Applied Mathematics, Queen's University, Kingsone, 2003.

[2] R. P. Stanley, "Enumerative Combinatorics—Volume 1, Volume 49 of Cambridge Studies in Advanced Mathematics," Cambridge University Press, Cambridge, 2000.

[3] G. H. E. Duchamp, K. A. Penson, A. I. Solomon, A. Horzeal and P. Blasiak, "One-Parameter Groups and Combinatorial Physics," *Proceedings of the Symposium CO-PROMAPH3: Contemporary Problems in Mathematical Physics*, Cotonou, 2004, pp. 436-449.

[4] L. Poinsot and G. H. E. Duchamp, "A Formal Calculus on the Riordan near Algebra," *Advances and Applications in Discrete Mathematics*, Vol. 6, No. 1, 2010, pp. 11-44.

[5] L. W. Shapiro, S. Getu, W.-J. Woan and L. C. Woodson, "The Riordan Group," *Discrete Applied Mathematics*, Vol. 34, No. 1-3, 1991, pp. 229-239. doi:10.1016/0166-218X(91)90088-E

[6] G. Markowsky, "Differential Operators and the Theory of Binomial Enumeration," *Journal of Mathematical Analysis and Application*, Vol. 63, No. 1, 1978, pp. 145-155. doi:10.1016/0022-247X(78)90111-7

[7] G. H. E. Duchamp, L. Poinsot, A. I. Solomon, K. A. Penson, P. Blasiak and A. Horzela, "Ladder Operators and Endomorphisms in Combinatorial Physics," *Discrete Mathematics and Theoretical Computer Science*, Vol. 12, No. 2, 2010, pp. 23-46.

[8] G.-C. Rota, D. Kahaner and A. Odlyzko, "Finite Operator Calculus," *Journal of Mathematical Analysis and Its Applications*, Vol. 42, No. 3, 1973, pp. 684-760.

[9] I. M. Sheffer, "Some Properties of Polynomial Sets of Type Zero," *Duke Mathematical Journal*, Vol. 5, No. 3, 1939, pp. 590-622. doi:10.1215/S0012-7094-39-00549-1

[10] J. F. Steffensen, "The Poweroid, an Extension of the Mathematical Notion of Power," *Acta Mathematica*, Vol. 73, No. 1, 1941, pp. 333-366. doi:10.1007/BF02392231

[11] S. Roman, "The Umbral Calculus," Dover Publications, New York, 1984.

[12] T.-X. He, L.C. Hsu and P.J.-S. Shiue, "The Sheffer Group and the Riordan Group," *Discrete Applied Mathematics*, Vol. 155, No. 15, 2007, pp. 1895-1909. doi:10.1016/j.dam.2007.04.006