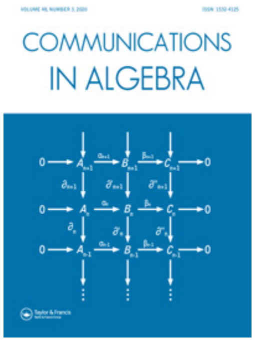

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Hilbertian (function) algebras

Laurent Poinsot[a,b]

[a]CREA, French Air Force Academy, BA 701, Salon-de-Provence, France; [b]LIPN – UMR CNRS 7030, University Paris 13, Sorbonne Paris Cité, Villetaneuse, France

ABSTRACT

A Hilbertian (co)algebra is defined as a (co)semigroup object in the monoidal category of Hilbert spaces. The carrier Hilbert space of such an algebra $(\mathcal{H}, \mu)$ splits as an orthogonal direct sum of its Jacobson radical and the closure of the linear span of a special class of elements, the group-like elements of its adjoint coalgebra $(\mathcal{H}, \mu^\dagger)$, which by the Riesz representation, correspond to closed maximal modular ideals. When the coproduct is isometric, that is, when $\mu \circ \mu^\dagger = id$, semisimplicity is shown to be equivalent to the existence of adjoints in the sense of Ambrose's H^*-algebras. We also prove that the category of semisimple *special* Hilbertian algebras, that is, semisimple Hilbertian algebras with an isometric coproduct, i.e., essentially the algebras of the form $\ell^2(X)$ with the pointwise product, are dually equivalent to a subcategory of pointed sets and base-point preserving maps.

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1. Introduction

This text discusses what one proposes to call a *Hilbertian algebra*: a semigroup object in the symmetric monoidal category of Hilbert spaces and bounded linear maps with the usual tensor product, $\hat{\otimes}_2$, that is, a "Hilbertian algebra" is a Hilbert space $\mathcal{H}$ with a bounded linear map $\mu : \mathcal{H} \hat{\otimes}_2 \mathcal{H} \to \mathcal{H}$ such that $\mu(a \otimes \mu(b \otimes c)) = \mu(\mu(a \otimes b) \otimes c)$ for all $a, b, c \in \mathcal{H}$. One of the main results is a description of the carrier Hilbert space of a commutative Hilbertian algebra (Theorem 22) in terms of "group-like" elements, which is obtained by combining the Gelfand transform from the theory of Banach algebras with basic results on Hilbert spaces.

Any Frobenius algebra (see e.g. [2]) in the monoidal category of Hilbert spaces, and in particular any finite-dimensional C^*-algebra (its underlying vector space becomes a Hilbert space under the inner product $\langle a, b \rangle = \mathrm{tr}(b^* a)$) is a commutative Hilbertian algebra. This suggests that studying Hilbertian algebras might be useful for the study of categorical quantum mechanics (see e.g. [1]) in the infinite-dimensional case.

A result in [2, Proposition 23, p. 16] similar to Theorem 22, is even obtained for (nonunital) Frobenius algebras. It explicitly states that such an algebra is the direct sum of subalgebras consisting of its radical and a H^*-algebra, and that this decomposition is also a direct sum of subcoalgebras for the adjoint coalgebra. This in particular implies that both summands are subalgebras and subcoalgebras. Sadly the proof of this claim is false and the remaining valid part of the proof only states that the summand corresponding to the H^*-algebra is a subcoalgebra,

CONTACT Laurent Poinsot laurent.poinsot@lipn.univ-paris13.fr CREA, French Air Force Academy, BA 701, Salon-de-Provence 13361, France.
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and not a subalgebra (but a quotient H^*-algebra).[1] The flaw in the proof is due to the use of [14, Lemma 6, p. 866] whose validity is itself based on [14, Lemma 1, p. 860] which is manifestly false[2] since it states in the context of Hilbert spaces, that $\ker(f \hat{\otimes}_2 f) = \ker(f) \hat{\otimes}_2 \ker(f)$ for every bounded linear map f. It is one of the purpose of, and the original motivation for, this note to get a corrected and improved result.

Some important points that could be extracted from the text are:

1. On every Hilbertian algebra there is a norm $||\cdot||'$ (possibly different from the norm induced by the inner product but equivalent to) with respect to which the underlying Banach space becomes a Banach algebra with multiplication $a \cdot b = \mu(a \otimes b)$.
2. Any Hilbertian algebra $(\mathcal{H}, \mu)$ gives rise to an *adjoint* Hilbertian coalgebra $(\mathcal{H}, \mu^\dagger)$, that is, a cosemigroup object in the monoidal category of Hilbert spaces. Moreover the orthogonal complement $I^\perp$ of a closed two-sided ideal of $(\mathcal{H}, \mu)$ is a closed subcoalgebra of $(\mathcal{H}, \mu^\dagger)$, and $I \mapsto I^\perp$ is even a bijection between closed two-sided ideals and closed subcoalgebras.
3. Every *character* of a Hilbertian algebra $(\mathcal{H}, \mu)$, that is, nonzero multiplicative linear map f: $\mathcal{H} \to \mathbb{C}$, is of the form $f \equiv \langle \cdot, x \rangle$ for a unique group-like element x of $(\mathcal{H}, \mu)$, that is, a nonzero element of $\mathcal{H}$ with $\mu^\dagger(x) = x \otimes x$.

To get Theorem 22, these three points are combined as follows. Let $(\mathcal{H}, \mu)$ be a commutative Hilbertian algebra. Then by (1) it is a Banach algebra, and so the Gelfand transform G is available. Since its kernel $\ker G$, is a closed ideal, the Hilbert space $\mathcal{H}$ splits as $\mathcal{H} = (\ker G) \oplus_2 (\ker G)^\perp$. By (2) $(\ker G)^\perp$ is a closed subcoalgebra. Moreover, since $\ker G$ is the intersection of the kernels of all characters, and the kernel of a character is equal to $(\mathbb{C}x)^\perp$ for some group-like element x by point (3), it follows that $(\ker G)^\perp$ is the closure of the linear span of the group-like elements. Finally, the kernel of the Gelfand transform is also the radical, and so we have Theorem 22. In particular a commutative Hilbertian algebra is semisimple if, and only if, the linear span of its group-like elements is dense.

In the special case where the coproduct is isometric, that is, when $\mu \circ \mu^\dagger = id$, semisimplicity is shown to be equivalent to the existence of adjoints in the sense of Ambrose's H^*-algebras [5] (Theorem 32 and Corollary 33).

To a pointed set (X, x_0) is associated the Hilbert space $\ell^2_\bullet(X, x_0)$ of all square-summable maps $X \to \mathbb{C}$ annihilated at x_0. With pointwise multiplication it is even a Hilbertian algebra with an isometric coproduct. This correspondence is extended into a functor from the dual of a subcategory of pointed sets to the category of Hilbertian algebras with an isometric coproduct, hereafter referred to as *special*, which is shown to be part of an adjunction (Proposition 36) in which the set of group-like elements – seen as a pointed set with 0 added – provides a left adjoint. Moreover, this adjunction restricts to a dual equivalence between the subcategory of pointed sets and the category of semisimple special Hilbertian algebras (Theorem 40).

The paper is organized as follows:

Section 2 deals with the theory of tensor products of Hilbert and Banach spaces for the sake of the readers not already familiar with it, which leads to the introduction of the main objects of the paper namely the Hilbertian (co)algebras (Definition 1). It also contains a presentation of some functorial relations between the different tensor products (Propositions 3 and 5) and an

[1]At the time of completing this paper its author does not know if [2, Proposition 23, p. 16] is valid in its whole generality, that is, if its proof can be corrected.

[2]For example, consider $H = \mathbb{C} \times \mathbb{C}$, where $\mathbb{C}$ is the complex field, and let $\pi_1 : H \to \mathbb{C}$ be the canonical projection onto the first factor, then $\ker(\pi_1 \hat{\otimes}_2 \pi_1)$ is 3-dimensional as it is equal to the direct sum of the one-dimensional and mutually orthogonal subspaces of $H \hat{\otimes}_2 H$, $((0) \times \mathbb{C}) \hat{\otimes}_2 (\mathbb{C} \times (0))$, $(\mathbb{C} \times (0)) \hat{\otimes}_2 ((0) \times \mathbb{C})$, $((0) \times \mathbb{C}) \hat{\otimes}_2 ((0) \times \mathbb{C})$, while $\ker(\pi_1) \hat{\otimes}_2 \ker(\pi_1) = ((0) \times \mathbb{C}) \hat{\otimes}_2 ((0) \times \mathbb{C}) \simeq \mathbb{C}$ is just a vector line.

analysis of Ambrose's H^*-algebras from the viewpoint of Hilbertian algebras (see in particular Theorem 9).

The main result (Theorem 18) of Section 3 is the one-to-one correspondence from the set of closed two-sided ideals of a Hilbertian algebra onto the set of closed subcoalgebras of its adjoint Hilbertian coalgebra, provided by orthogonal complementation.

From Section 4 one only considers commutative algebras. Section 4 discusses the general concept of semisimplicity for (commutative) Hilbertian algebras. Here a structure theorem (Theorem 22) is provided, which makes use of the group-like elements (Definition 10) corresponding, under the Riesz isomorphism, to the usual structure space.

Section 5 is entirely devoted to commutative special Hilbertian algebras (i.e., with a coisometric multiplication; Definition 12). Subsections 5.1–5.2 reach for these algebras, to the equivalence between semisimplicity and the property of having adjoints in the sense of H^*-algebras (Theorem 32 and Corollary 33), while Subsection 5.3 provides a dual equivalence between the above algebras and a category of pointed sets, seemingly similar to the contravariant equivalence between commutative C^*-algebras and pointed compact spaces.

2. Hilbertian and Banach (co)algebras

The main notion of the paper, namely Hilbertian (co)algebras, is introduced hereafter. It is most naturally presented in the realm of monoidal categories and one also recalls for the reader's convenience some useful facts about the tensor products of Hilbert spaces and Banach spaces (both the injective and the projective) and some of their functorial relations. Most of the notations and notions from the theory of monoidal categories needed hereafter are taken from [25, Sect. 2, pp. 4874–4876] with the following exceptions: $\mathbf{Sem}(\mathbb{C})$ and $\mathbf{Cosem}(\mathbb{C})$ stand for the categories of semigroup and of cosemigroup objects of a monoidal category $\mathbb{C}$, while ${}_c\mathbf{Sem}(\mathbb{C})$ and ${}_{coc}\mathbf{Cosem}(\mathbb{C})$ are the categories of commutative semigroup and of cocommutative cosemigroup objects in a symmetric ([33, p. 69]) monoidal category $\mathbb{C}$. By convention, the underlying category of a (symmetric) monoidal category $\mathbb{C}$ is denoted $\mathbf{C}$, and in general one denotes the same way a(n op)monoidal functor from $\mathbb{C}$ to $\mathbb{D}$ and its underlying functor $F : \mathbf{C} \to \mathbf{D}$; in this situation one says that F *lifts to a(n op)monoidal functor*. A(n op)monoidal functor $F : \mathbb{C} \to \mathbb{D}$ *induces a functor* again denoted $F : \mathbf{Sem}(\mathbb{C}) \to \mathbf{Sem}(\mathbb{D})$ (resp. $F : \mathbf{Cosem}(\mathbb{C}) \to \mathbf{Cosem}(\mathbb{D})$) [25, Prop. 3, p. 4876], and symmetric ones induce functors between the respective categories of (co)commutative (co)semigroup objects ([3, Prop. 3.37, p. 79]). Note that "(op)monoidal functor" always means a lax one. Concerning other standard category-theoretic notions the main reference is [21]. In particular given $\mathbf{C}$-objects A, B, $\mathbf{C}(A, B)$ stands for the hom-set of all $\mathbf{C}$-morphisms with domain A and codomain B.

2.1. The category of Hilbert spaces

In this contribution, vector spaces (algebras and coalgebras) are over a fixed $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ without further ado; in particular $\otimes$ stands for the algebraic tensor product over $\mathbb{K}$. $\alpha \mapsto \bar{\alpha}$ stands for the involution of $\mathbb{K}$ (the usual complex one or the identity in the real case).

By an *inner product* $\langle \cdot, \cdot \rangle$ is meant a Hermitian-symmetric sesquilinear form (conjugate-linearity in the second argument) when $\mathbb{K} = \mathbb{C}$, or just a symmetric bilinear form when $\mathbb{K} = \mathbb{R}$, which is positive-definite. Thus, a vector space with an inner product, i.e., a pre-Hilbert space, has a norm induced by its inner product. The inner product of a Hilbert space $\mathcal{H}$ will sometimes be denoted by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. The norm induced by the inner product of the Hilbert space $\ell^2(X)$ of square-summable functions from X to $\mathbb{K}$ is denoted as usually by $|| \cdot ||_2$.

Categories: Ban and **Hilb** are respectively the categories of Banach and Hilbert spaces, each with bounded linear maps as morphisms (in particular by an isomorphism in these categories is meant a *topological isomorphism*, i.e., a bounded bijective operator; its inverse is also bounded according to the open mapping theorem). The usual operator norm of a (bounded linear) operator between any normed spaces is denoted by $||\cdot||_{op}$. If $\mathbf{C}$ is one of the above categories, then $\mathbf{C}_1$ denotes the subcategory of $\mathbf{C}$ with the same objects and with linear contractions (i.e., of operator norm ≤ 1) as morphisms. Let $U:\mathbf{Hilb}\to\mathbf{Ban}$ (and also $U:\mathbf{Hilb}_1\to\mathbf{Ban}_1$) be the obvious forgetful functor.

Banach and Hilbert adjoint functors: The Banach adjoint functor is denoted $^*:\mathbf{Ban}^{\mathrm{op}}\to\mathbf{Ban}$. As usually it acts on a morphism $f:E\to F$ as $f^*:F^*\to E^*, f^*(\ell)=\ell\circ f$. Of course, one also has a "contractive" version $^*:\mathbf{Ban}_1^{\mathrm{op}}\to\mathbf{Ban}_1$ (since $||f||_{op}=||f^*||_{op}$ [28, Theorem 4.10, p. 98]).

The conjugate $\bar{\mathcal{H}}$ of a Hilbert space $\mathcal{H}$ shares the same abelian group as $\mathcal{H}$ but the scalar action is given by $\alpha\cdot x=\bar{\alpha}x$, while $\langle x,y\rangle_{\bar{\mathcal{H}}}=\langle y,x\rangle_{\mathcal{H}}$ (of course when $\mathbb{K}=\mathbb{R},\bar{\mathcal{H}}=\mathcal{H}$). This extends in an obvious way to a functorial automorphism of $\mathbf{Hilb}$ (the identity when $\mathbb{K}=\mathbb{R}$), which is the identity on morphisms[3] (since any linear map is automatically conjugate-linear too). By the Riesz representation theorem ([17, Theorem 2.3.1, p. 98]), $U(\bar{\mathcal{H}})\simeq U(\mathcal{H})^*$ isometrically and canonically under $R_{\mathcal{H}}:x\mapsto\langle\cdot,x\rangle_{\mathcal{H}}$. Now, given $f\in\mathbf{Hilb}(\mathcal{H},\mathcal{K}), f^\dagger\in\mathbf{Hilb}(\mathcal{K},\mathcal{H})$ is the unique map such that the following diagram commutes. In other words, $R_{\mathcal{H}}\circ f^\dagger=f^*\circ R_{\mathcal{K}}$, i.e., $\langle u,f^\dagger(v)\rangle_{\mathcal{H}}=\langle f(u),v\rangle_{\mathcal{K}}, u\in\mathcal{H}, v\in\mathcal{K}$.

$$\begin{array}{ccc} U(\mathcal{K})^* & \xrightarrow{(U(f))^*} & U(\mathcal{H})^* \\ \wr\| & & \wr\| \\ U(\overline{\mathcal{K}}) & \xrightarrow[U(\overline{f^\dagger})]{} & U(\overline{\mathcal{H}}) \end{array} \tag{1}$$

This gives rise to the Hilbert adjoint functor $^\dagger:\mathbf{Hilb}^{\mathrm{op}}\to\mathbf{Hilb}$, which is the identity on objects. A corresponding result holds for $\mathbf{Hilb}_1$ (essentially because $||f||_{op}=||f^\dagger||_{op}$ [17, Theorem 2.4.2, p. 101]). As $(f^\dagger)^\dagger=f$, the dagger functor turns $\mathbf{Hilb}$ and $\mathbf{Hilb}_1$ into *dagger categories* [31]. Record for later the following easy result.

Lemma 1. *Let $f:\mathcal{H}\to\mathcal{K}$ be a bounded linear map between Hilbert spaces. Let $V\subseteq\mathcal{H}$ and $W\subseteq\mathcal{K}$ be closed subspaces. One has $f(V)\subseteq W$ if, and only if, $f^\dagger(W^\perp)\subseteq V^\perp$.*

Hilbert direct sum: Given Hilbert spaces $\mathcal{H},\mathcal{K}$, the algebraic direct sum $\mathcal{H}\oplus\mathcal{K}$ equipped, with the inner product $\langle(x,y),(x',y')\rangle=\langle x,x'\rangle_{\mathcal{H}}+\langle y,y'\rangle_{\mathcal{K}}$, turns out to be a Hilbert space $\mathcal{H}\oplus_2\mathcal{K}$ called the Hilbert (or orthogonal) direct sum. When $(\mathcal{H}_i)_{i\in I}$ is any collection of Hilbert spaces, then the Hilbert direct sum $\bigoplus_{2i\in I}\mathcal{H}_i:=\{(u_i)_{i\in I}:\forall i, u_i\in\mathcal{H}_i, (||u_i||)_{i\in I}\in\ell^2(I)\}$ is a Hilbert space with $\langle(u_i)_i,(v_i)_i\rangle:=\sum_{i\in I}\langle u_i,v_i\rangle$.

Given a subspace V of $\mathcal{H},\mathcal{H}\simeq\mathsf{cl}(V)\oplus_2 V^\perp$, where $\mathsf{cl}(V)$ is the closure of V in the norm topology of $\mathcal{H}$, while $V^\perp$ denotes the orthogonal complement of V. For a closed subspace V, one denotes by $i_V:V\hookrightarrow\mathcal{H}$ (respectively, $\pi_V:\mathcal{H}\to V$) the canonical injection (respectively, projection). Then, $\pi_V^\dagger=i_V$ and $\pi_V\circ i_V=id_V$. The range $ran(i_V)$ of i_V is closed (because of the closed range theorem for Hilbert spaces [22, Proposition 11.12, p. 87], since $ran(i_V^\dagger)=ran(\pi_V)=V$).

2.2. Hilbert tensor product and Hilbertian (co)algebras

The category of Hilbert spaces also has a symmetric monoidal structure, given by the so-called Hilbertian (or Hilbert) tensor product $\hat{\otimes}_2$ (see [7]). $\mathcal{H}\hat{\otimes}_2\mathcal{K}$ is the completion of the pre-Hilbert

[3]To be on the safe side, it should be said that the underlying group homomorphisms of $\mathcal{H}\xrightarrow{f}\mathcal{K}$ and $\bar{\mathcal{H}}\xrightarrow{\bar{f}}\bar{\mathcal{K}}$ are the same. Nevertheless, in what follows such slightly ambiguous formulations will be used.

space $\mathcal{H}\otimes_2\mathcal{K}$ where, for pre-Hilbert spaces H and K, $H\otimes_2 K$ denotes the algebraic tensor product together with the inner product $\langle x\otimes y, x'\otimes y'\rangle = \langle x,x'\rangle_H\langle y,y'\rangle_K$ ([7]).

Let $\mathcal{H},\mathcal{K},\mathcal{L}$ be Hilbert spaces, and $f:\mathcal{H}\times\mathcal{K}\to\mathcal{L}$ be a bounded bilinear map (here as usual bounded means that there is a real number M such that $||f(x,y)||_{\mathcal{L}}\leq M||x||_{\mathcal{H}}||y||_{\mathcal{K}}, x\in\mathcal{H}, y\in\mathcal{K}$ and the least such constant is its (bilinear) operator norm $||f||_{op}$ which is equivalently defined as $||f||_{op}=\sup_{||x||=1=||y||}||f(x,y)||$). It is said to be a *weak Hilbert-Schmidt mapping* if for every $z\in\mathcal{L}$, and every orthonormal basis X of $\mathcal{H}$ and Y of $\mathcal{K}$, the sum $|f_z|_{HS}:=\left(\sum_{x\in X}\sum_{y\in Y}|\langle f(x,y),z\rangle_{\mathcal{L}}|^2\right)^{\frac{1}{2}}$ is finite. By the closed graph theorem, there is a real number d such that $|f_z|_{HS}\leq d||z||$ for each $z\in\mathcal{L}$, and the least possible value of the constant d is denoted by $||f||_{HS}$.

Given such a map f, by [17, Theorem 2.6.4, p. 132], there exists a unique bounded linear map $\hat{f}:\mathcal{H}\hat{\otimes}_2\mathcal{K}\to\mathcal{L}$ such that $\hat{f}(x\otimes y)=f(x,y), x\in\mathcal{H}, y\in\mathcal{K}$ and moreover, $||\hat{f}||_{op}=||f||_{HS}$.

Remark 1. Not all bounded bilinear maps are weak Hilbert-Schmidt mappings. E.g., the inner product $\langle\cdot,\cdot\rangle:\mathcal{H}\times\mathcal{H}\to\mathbb{K}$ is bounded (by the Cauchy-Bunyakovski-Schwarz inequality) but is a weak Hilbert-Schmidt mapping only when $\mathcal{H}$ is finite-dimensional. This incidentally implies that when $\mathcal{H}$ is infinite-dimensional then the unique linear extension of $\langle\cdot,\cdot\rangle$ to $\mathcal{H}\otimes\mathcal{H}$ is not a continuous map $\mathcal{H}\otimes_2\mathcal{H}\to\mathbb{K}$.

Notation 1. Given a bounded linear map $\mathcal{H}\hat{\otimes}_2\mathcal{K}\xrightarrow{f}\mathcal{L}$, in what follows $f_{bil}:\mathcal{H}\times\mathcal{K}\to\mathcal{L}$ stands for the corresponding weak Hilbert-Schmidt mapping and $f_{alg}:\mathcal{H}\otimes\mathcal{K}\to\mathcal{L}$ for the corresponding linear map on the algebraic tensor product, which of course is continuous as a map $f_{alg}:\mathcal{H}\otimes_2\mathcal{K}\to\mathcal{L}$. Obviously, $||f||_{op}=||f_{bil}||_{HS}=||f_{alg}||_{op}$ and $||f_{bil}||_{op}\leq||f||_{op}$ (since $||f_{bil}(x,y)||=||f(x\otimes y)||\leq||f||_{op}||x||||y||$).

Remark 2.

1. The conjugate space functor $\overline{(-)}$ is a (symmetric) strict monoidal functor from $\mathbb{Hilb}:=(\mathbf{Hilb},\hat{\otimes}_2,\mathbb{K})$ to $(\mathbf{Hilb},\hat{\otimes}_2,\bar{\mathbb{K}})$.
2. The Hilbert tensor product still provides a symmetric monoidal structure $\mathbb{Hilb}_1$ for $\mathbf{Hilb}_1$ ([13]) because the coherence constraints are unitary (hence of norm 1), and because $||f\hat{\otimes}_2 g||_{op}=||f||_{op}||g||_{op}$ for every operators f, g ([17, p. 146]).

$\hat{\otimes}_2$ is additive: $\mathcal{H}\hat{\otimes}_2(\mathcal{K}\oplus_2\mathcal{L})\simeq(\mathcal{H}\hat{\otimes}_2\mathcal{K})\oplus_2(\mathcal{H}\hat{\otimes}_2\mathcal{L})$ (canonically, [27]). Moreover, $\hat{\otimes}_2$ preserves closed subspaces (as a consequence of [7, Proposition 3, Chap. V, p. 28]), i.e., if V, W are closed subspaces of $\mathcal{H}$ and $\mathcal{K}$ respectively, then $V\hat{\otimes}_2 W\simeq ran(i_V\hat{\otimes}_2 i_W)$, and in what follows one even treats both subspaces as identical for convenience. Additivity of $\hat{\otimes}_2$ implies the following.

Lemma 2. *Let $\mathcal{H},\mathcal{K}$ be Hilbert spaces with respective subspaces V, W. Then, as subspaces of $\mathcal{H}\hat{\otimes}_2\mathcal{K}$,*

$$(\mathsf{cl}(V)\hat{\otimes}_2\mathsf{cl}(W))^{\perp}=(\mathsf{cl}(V)\hat{\otimes}_2 W^{\perp})\oplus_2(V^{\perp}\hat{\otimes}_2\mathsf{cl}(W))\oplus_2(V^{\perp}\hat{\otimes}_2 W^{\perp}).$$

Let us at last introduce the central object of this study.

Definition 1. By a *Hilbertian algebra* (respectively, *coalgebra*) is meant a semigroup (respectively, cosemigroup) object in $\mathbb{Hilb}$ or $\mathbb{Hilb}_1$. Because from Section 4 we only consider *commutative* Hilbertian algebras *over* $\mathbb{C}$, in order to save space one refers to them simply as "Hilbertian algebras", and in Section 5 the meaning of these words is even restricted to *complex commutative* Hilbertian algebras *with a contractive multiplication*, without further ado.

One refrains from using the name "Hilbert algebras" because this terminology is reserved to another well-established notion related to von Neumann algebras (see [36, Chap. VI]).

Why nonunital algebras rather than unital ones? One recalls [25, Sect. 4.3.5, pp. 4890–4891] that to any semigroup in $\mathbb{Hilb}$ one can freely add a unit so as to obtain a monoid. Whence any semigroup in $\mathbb{Hilb}$ will canonically and functorially provide a monoid object in $\mathbb{Hilb}$. Moreover the fundamental example of Hilbertian algebras $(\ell^2(X), \mu_X)$ (see Example 1 below), essential for the results obtained in Section 5, provides unital algebras only for the finite-dimensional case (i.e., when X is finite). This explains why emphasis is put on semigroups rather than monoids.

Example 1. Let X be a set. For $x \in X$, let $\delta_x : X \to \mathbb{C}$ be given as usually by the Kronecker delta $\delta_x(y) = \delta_{x,y}, y \in X$. $(\ell^2(X), \mu_X)$ is a commutative Hilbertian algebra, with $\mu_X : \ell^2(X) \hat{\otimes}_2 \ell^2(X) \to \ell^2(X), \mu_X(\sum_{x,y\in X} f(x,y)\delta_{(x,y)}) := \sum_{x\in X} f(x,x)\delta_x$ up to the unitary transformation $\ell^2(X) \hat{\otimes}_2 \ell^2(X) \simeq \ell^2(X \times X), \delta_x \otimes \delta_y \mapsto \delta_{(x,y)}, x, y \in X$ ([17, Example 2.6.10, p. 142]). Given $f, g \in \ell^2(X)$, and $x \in X, \mu_X(f \hat{\otimes}_2 g)(x) = f(x)g(x)$ and thus $\mu_X(f \hat{\otimes}_2 g)$ is just the pointwise product fg. One observes that $\mu_X \circ \mu_X^\dagger = id$.

Adjoint Hilbertian coalgebra: The Hilbert adjoint functors †: $\mathbf{Hilb}^{op} \to \mathbf{Hilb}$ and †: $\mathbf{Hilb}_1^{op} \to \mathbf{Hilb}_1$ become strict (symmetric) monoidal functors (in particular $(f \hat{\otimes}_2 g)^\dagger = f^\dagger \hat{\otimes}_2 g^\dagger$ for operators f, g). Consequently, any semigroup in $\mathbb{Hilb}$ shares the same Hilbert space with its "adjoint" cosemigroup. Indeed the Hilbert adjoint functor induces a functor still denoted by † from $\mathbf{Cosem}(\mathbb{Hilb})^{op}$ to $\mathbf{Sem}(\mathbb{Hilb})$, which is actually an isomorphism. (The same is true for $\mathbb{Hilb}_1$.)

Definition 2. Let $(\mathcal{H}, \mu)$ be a Hilbert algebra. Then, $(\mathcal{H}, \mu^\dagger)$ is a Hilbert coalgebra referred to as the *adjoint coalgebra* of $(\mathcal{H}, \mu)$.

2.3. The projective tensor product and Banach algebras

Projective tensor product: A Hilbert space "is" a Banach space but is a Hilbertian (co)algebra a Banach (co)algebra? This question suggests a comparison between (co)semigroup objects which is most naturally discussed in the realm of monoidal categories. Let E, F be Banach spaces. The algebraic tensor product $E \otimes F$ becomes a normed space $E \otimes_\pi F$ under the projective norm $\pi(u) = \inf\{\sum_{i=1}^n ||x_i||_E ||y_i||_F : u = \sum_{i=1}^n x_i \otimes y_i\}$. It is also a normed space $E \otimes_\epsilon F$ for the injective norm $\epsilon(u) = \sup\{|\sum_{i=1}^n \ell_1(x_i)\ell_2(y_i)| : \ell_1 \in E^*, ||\ell_1||_{op} \leq 1, \ell_2 \in F^*, ||\ell_2||_{op} \leq 1, \}$, where $u = \sum_{i=1}^n x_i \otimes y_i$. Let $E \hat{\otimes}_\alpha F$ be the corresponding completion, $\alpha = \pi, \epsilon$.

It is worth mentioning that the projective tensor norm has a universal property: Let E, F, G be Banach spaces, and $f : E \times F \to G$ be a bounded bilinear map, then there is a unique bounded linear map $\hat{f} : E \hat{\otimes}_\pi F \to G$ such that $\hat{f}(x \otimes y) = f(x, y), x \in E, y \in F$.

It is well-known that $(\mathbf{Ban}, \hat{\otimes}_\alpha), \alpha = \pi, \epsilon$, is a symmetric monoidal category, and the coherence constraints are isometries (see e.g., [39, pp. 3–4]). Both tensor products satisfy the following: $||f \hat{\otimes}_\alpha g||_{op} = ||f||_{op}||g||_{op}$, for operators f, g ([29, Proposition 2.4, p. 18, and Proposition 3.2, p. 47]). Thus, $(\mathbf{Ban}_1, \hat{\otimes}_\alpha), \alpha = \pi, \epsilon$, is also a symmetric monoidal category.

We record for later use that the injective tensor product preserves closed subspaces, i.e., if $X \subseteq E$ and $Y \subseteq F$ are closed subspaces, then $X \hat{\otimes}_\epsilon Y$ is a closed subspace of $E \hat{\otimes}_\epsilon F$, see [29]. This is seldom true for the projective tensor product ([29, p. 24]).

"Classical" Banach algebras and semigroup objects: A (not necessarily unital) Banach algebra is ordinarily defined as a pair $((E, ||\cdot||), m)$, where $(E, ||\cdot||)$ is a Banach space, $m : E \times E \to E$ is an associative bounded bilinear map, of norm ≤ 1. With bounded multiplicative[4] linear maps this provides the category $\mathbf{BanAlg}$ of Banach algebras. Letting $\mu : E \hat{\otimes}_\pi E \to E$ be the bounded

[4]By "multiplicative" is meant $f(m(x,y)) = f(x)f(y)$.

linear map obtained from m using the universal property of $\hat{\otimes}_\pi$, $((E, ||\cdot||), \mu)$ provides a semigroup object in $(\mathbf{Ban}_1, \hat{\otimes}_\pi)$. Conversely, any semigroup object $((E, ||\cdot||), \mu)$ in $(\mathbf{Ban}, \hat{\otimes}_\pi)$ provides a $\mathbb{K}$-algebra (E, m), with $m : E \times E \to E$, given by $m(x, y) = \mu(x \otimes y)$, being jointly continuous, but its norm, as a bounded bilinear map, is not necessarily ≤ 1. However the norm $||-||' := ||\mu||_{op}||-||$ ([11, p. 293]) is equivalent to $||-||$ (i.e., the identity map of E is an isomorphism $(E, ||\cdot||) \to (E, ||\cdot||'))$ and $||m(x, y)||' \leq ||x||'||y||', x, y \in E$, so that $BA((E, ||-||), \mu) := ((E, ||-||'), m)$ becomes a Banach algebra. As a matter of fact BA acting as the identity on morphisms, yields an equivalence of categories between $\mathbf{Sem}(\mathbf{Ban}, \hat{\otimes}_\pi)$ and **BanAlg**. Moreover, $\mathbf{Sem}(\mathbf{Ban}_1, \hat{\otimes}_\pi)$ is even isomorphic (concretely over $\mathbf{Ban}_1$) to **BanAlg** (since in this case there is no need to replace the norm by an equivalent one).

Banach coalgebras: To our knowledge there is little literature about Banach coalgebras, i.e., cosemigroup or comonoid objects in $(\mathbf{Ban}, \hat{\otimes}_\epsilon)$. Josef Wichmann 1975 PhD's thesis [39] and Andrew M. Tonge's papers [37, 38] may be considered as important references of the subject. Let us see a fundamental example which makes use of the well-known canonical isomorphism $C(X \times Y) \simeq C(X) \hat{\otimes}_\epsilon C(Y)$, for (Hausdorff) compact spaces X, Y (see [29, p. 50]), where $C(X)$ denotes the Banach spaces of all continuous complex-valued functions on X with the uniform norm $||\cdot||_\infty$. Given a compact monoid $(M, m, 1)$, i.e., a monoid which is also a (Hausdorff) compact topological space, and whose multiplication m is jointly continuous, the Banach space $C(M)$ of all continuous functions on M inherits from $(M, m, 1)$ a structure of a Banach coalgebra. The coproduct $\Delta : C(M) \to C(M \times M) \simeq C(M) \hat{\otimes}_\epsilon C(M)$ is given by $f \mapsto ((x, y) \mapsto f(m(x, y)))$, and the counit $\epsilon : C(M) \to \mathbb{C}$ is obtained as $\epsilon(f) = f(1)$.

The dual of a Banach coalgebra: Given a norm α on the algebraic tensor product $E \otimes F$ of Banach spaces, one denotes by $E \otimes_\alpha F$ the corresponding normed space, and by $E \hat{\otimes}_\alpha F$ its norm completion. To any such norm α, one may associate a number $\alpha^*(\ell)$ for each $\ell \in E^* \otimes F^*$ by setting $\alpha^*(\ell) = \sup\{|\ell(u)| : u \in E \otimes F, \alpha(u) \leq 1\}$, where ℓ is seen as a functional on $E \otimes F$ using the canonical embedding $(E^* \otimes F^*) \hookrightarrow (E^\sharp \otimes E^\sharp) \hookrightarrow (E \otimes F)^\sharp$ with $(-)^\sharp$ the algebraic dual space. This is not always a finite number, but for $\alpha = \pi, \epsilon$, it is; for instance, $\pi^* = \epsilon$ (see [30]). When $\alpha^*(\ell)$ is finite for each ℓ, α^* defines a norm on $E^* \otimes F^*$, and the above canonical embedding co-restricts as $E^* \otimes_{\alpha^*} F^* \hookrightarrow (E \otimes_\alpha F)^*$ (since by definition of $\alpha^*, |\ell(u)| \leq \alpha^*(\ell)\alpha(u), \ell \in E^* \otimes F^*, u \in E \otimes F$). Passing to the completion it uniquely extends to a canonical embedding $E^* \hat{\otimes}_{\alpha^*} F^* \hookrightarrow (E \hat{\otimes}_\alpha F)^* \simeq (E \otimes_\alpha F)^*$. According to the above, for every Banach spaces E, F, one has a canonical map

$$\Theta_{E,F} : E^* \hat{\otimes}_\pi F^* \to E^* \hat{\otimes}_{\epsilon^*} F^* \hookrightarrow (E \hat{\otimes}_\epsilon F)^*. \tag{2}$$

(The first map is obtained using the universal property of $\hat{\otimes}_\pi$, because $\epsilon^*(\ell) \leq \pi(\ell)$ for every $\ell \in E^* \otimes F^*$.) In details, $\Theta_{E,F}(\ell_1 \otimes \ell_2)(x_1 \otimes x_2) = \ell_1(x_1)\ell_2(x_2), \ell_1 \in E^*, x_1 \in E, \ell_2 \in F^*, x_2 \in F^*$.

Following [39, Propositions 7.1 and 7.2, pp. 18–20] in a modern category-theoretic language, Eq. (2) actually implies that the Banach adjoint functor $^* : \mathbf{Ban}^{\mathrm{op}} \to \mathbf{Ban}$ lifts to a (symmetric) monoidal functor from $(\mathbf{Ban}, \hat{\otimes}_\epsilon)^{\mathrm{op}}$ to $(\mathbf{Ban}, \hat{\otimes}_\pi)$. So it induces a functor $^* : \mathbf{Cosem}(\mathbf{Ban}, \hat{\otimes}_\epsilon)^{\mathrm{op}} \to \mathbf{Sem}(\mathbf{Ban}, \hat{\otimes}_\pi)$. In other words, the dual of a Banach coalgebra is a semigroup object of $(\mathbf{Ban}, \hat{\otimes}_\pi)$ that is, up to equivalence a Banach algebra.

Comparisons with $\hat{\otimes}_2$: If $\mathcal{H}, \mathcal{K}$ are Hilbert spaces, then nor $U(\mathcal{H}) \hat{\otimes}_\pi U(\mathcal{K})$ ([13, p. 186] or [10, Corollary 1.1.15, p. 18]) neither $U(\mathcal{H}) \hat{\otimes}_\epsilon U(\mathcal{K})$ are Hilbert spaces in general (e.g., [29, Corollary 4.24, p. 87]). However, there are some nice relations.

Proposition 3. *The forgetful functor $U : \mathbf{Hilb} \to \mathbf{Ban}$ lifts to a (symmetric) monoidal functor from $(\mathbf{Hilb}, \hat{\otimes}_2)$ to $(\mathbf{Ban}, \hat{\otimes}_\pi)$, and to an (symmetric) opmonoidal functor from $(\mathbf{Hilb}, \hat{\otimes}_2)$ to $(\mathbf{Ban}, \hat{\otimes}_\epsilon)$. Corresponding results are true for $U : \mathbf{Hilb}_1 \to \mathbf{Ban}_1$.*

Proof. Let $\mathcal{H}, \mathcal{K}$ be two Hilbert spaces. The canonical bilinear map $-\otimes -: U(\mathcal{H}) \times U(\mathcal{K}) \to U(\mathcal{H} \hat{\otimes}_2 \mathcal{K})$ is continuous. The universal property of the projective tensor product provides the unique bounded linear map $\Phi_{\mathcal{H},\mathcal{K}} : U(\mathcal{H}) \hat{\otimes}_\pi U(\mathcal{K}) \to U(\mathcal{H} \hat{\otimes}_2 \mathcal{K})$ such that $\Phi_{\mathcal{H},\mathcal{K}}(x \otimes y) = x \otimes y$ (observe that $\Phi_{\mathcal{H},\mathcal{K}}$ has norm 1). It is natural by construction. Compatibility with the coherence constraints are easily checked, so that $(U, \Phi, id_{\mathbb{K}})$ is a monoidal functor from $(\mathbf{Hilb}, \hat{\otimes}_2)$ to $(\mathbf{Ban}, \hat{\otimes}_\pi)$. According to [9, p. 351], for each $u \in \mathcal{H} \otimes \mathcal{K}, \epsilon(u) \leq ||u||_2 \leq \pi(u)$, where $|| \cdot ||_2$ is the norm on $\mathcal{H} \otimes \mathcal{K}$ induced by the inner product of $\mathcal{H} \otimes_2 \mathcal{K}$. It follows that the canonical injection $i : \mathcal{H} \otimes_2 \mathcal{K} \hookrightarrow U(\mathcal{H}) \hat{\otimes}_\epsilon U(\mathcal{K})$ is continuous. Since $U(\mathcal{H}) \hat{\otimes}_\epsilon U(\mathcal{K})$ is complete, it has a unique continuous extension $\Psi_{\mathcal{H},\mathcal{K}} : U(\mathcal{H} \hat{\otimes}_2 \mathcal{K}) \to U(\mathcal{H}) \hat{\otimes}_\epsilon U(\mathcal{K})$ (which is of norm 1). One observes that, by definition, $\Psi_{\mathcal{H},\mathcal{K}}(x \otimes y) = x \otimes y, x \in \mathcal{H}, y \in \mathcal{K}$. It is left to the reader to check that $(U, \Psi, id_{\mathbb{K}})$ defines an opmonoidal functor from $(\mathbf{Hilb}, \hat{\otimes}_2)$ to $(\mathbf{Ban}, \hat{\otimes}_\epsilon)$ as desired. □

Proposition 3 implies that there are two induced functors $U : \mathbf{Cosem}(\mathbb{Hilb}) \to \mathbf{Cosem}(\mathbf{Ban}, \hat{\otimes}_\epsilon)$ and $U : \mathbf{Sem}(\mathbb{Hilb}) \to \mathbf{Sem}(\mathbf{Ban}, \hat{\otimes}_\pi)$.

Definition 3. For a Hilbertian algebra $(\mathcal{H}, \mu), BA(U(\mathcal{H}, \mu))$ is its *underlying Banach algebra* and $U(\mathcal{H}, \mu^\dagger)$ its *underlying Banach coalgebra*.

Remark 3. Given Hilbertian algebras $(\mathcal{H}, \mu)$ and $(\mathcal{K}, \gamma)$, and a bounded linear map $f : \mathcal{H} \to \mathcal{K}$, f is a morphism of Hilbertian algebras if, and only if, f is a morphism of the underlying Banach algebras.

Relations with ordinary algebras: Hilbertian algebras "are" ordinary algebras too.

Definition 4. Letting by abuse of notation $\mathcal{H}$ be both a Hilbert space and its underlying vector space, $(\mathcal{H}, \mu_{bil})$ stands for the *underlying* $\mathbb{K}$*-algebra* of the Hilbertian algebra $(\mathcal{H}, \mu)$, where as in Notation 1, $\mu_{bil} : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ is the unique (bounded) bilinear map whose corresponding bounded linear map $\mu_{alg} : \mathcal{H} \otimes_2 \mathcal{H} \to \mathcal{H}$ is the restriction of μ to the algebraic tensor product. This provides a functor from $\mathbf{Sem}(\mathbb{Hilb})$ to the category $\mathbf{Alg}$ of (nonunital) $\mathbb{K}$-algebras (with a bilinear multiplication) and algebra homomorphisms, which acts as the identity on morphisms. By forgetting the norm of $BA(U(\mathcal{H}, \mu))$ one of course recovers $(\mathcal{H}, \mu_{bil})$.

Remark 4. Let $\mathbb{Vect} = (\mathbf{Vect}, \otimes, \mathbb{K})$ be the usual (symmetric) monoidal category of $\mathbb{K}$-vector spaces. Clearly $\mathbf{Alg} \simeq \mathbf{Sem}(\mathbb{Vect})$ concretely over $\mathbf{Vect}$ (bilinear multiplications correspond bi-univocally to their unique linear extension to the algebraic tensor product). The obvious forgetful functor $\mathbf{Hilb} \to \mathbf{Vect}$ is (symmetric) monoidal and thus provides a functor $\mathbf{Sem}(\mathbb{Hilb}) \to \mathbf{Sem}(\mathbb{Vect}) \simeq \mathbf{Alg}$ which corresponds to the one mentioned in Definition 4.

Remark 3 may be completed by the following result.

Lemma 4. *$f \in \mathbf{Sem}(\mathbb{Hilb})((\mathcal{H}, \mu), (\mathcal{K}, \gamma))$ if, and only if, $f : \mathcal{H} \to \mathcal{K}$ is continuous and $f \in \mathbf{Alg}((\mathcal{H}, \mu_{bil}), (\mathcal{K}, \gamma_{bil}))$.*

Proof. The direct implication is clear because the forgetful functor from **Hilb** to **Vect** is monoidal. Let us assume that f is a bounded linear map which is an algebra map from $(\mathcal{H}, \mu_{bil})$ to $(\mathcal{K}, \gamma_{bil})$, that is, $\gamma_{bil}(f(x), f(y)) = f(\mu_{bil}(x, y)), x, y \in \mathcal{H}$. By continuity this provides a bounded algebra map from $(\mathcal{H}, \mu_{alg})$ to $(\mathcal{K}, \gamma_{alg})$, i.e., $\gamma_{alg} \circ (f \otimes f) = f \circ \mu_{alg}$. Then, the extensions by continuity of the maps $\gamma_{alg} \circ (f \otimes f), f \circ \mu_{alg} : \mathcal{H} \otimes_2 \mathcal{H} \to \mathcal{K}$ to the completion of $\mathcal{H} \otimes_2 \mathcal{H}$, remain equal, i.e., $\gamma \circ (f \hat{\otimes}_2 f) = f \circ \mu$. Whence $f \in \mathbf{Sem}(\mathbb{Hilb})((\mathcal{H}, \mu), (\mathcal{K}, \gamma))$. □

The underlying Banach coalgebra of a Hilbertian algebra:

Proposition 5. *For each Hilbertian coalgebra* $(\mathcal{H},\delta), R_{\mathcal{H}} : U(\mathcal{H},\delta^{\dagger}) \simeq U(\mathcal{H},\delta)^{*}$, *conjugate-linear, in* $\mathbf{Sem}(\mathbf{Ban}, \hat{\otimes}_{\pi})$.

Proof. It suffices to prove that the following diagram commutes, where $(\mathcal{H},\delta)$ is a Hilbertian coalgebra.

$$\begin{array}{ccccc}
(U(\mathcal{H})\hat{\otimes}_{\epsilon}U(\mathcal{H}))^{*} & \xrightarrow{\Psi^{*}_{\mathcal{H},\mathcal{H}}} & U(\mathcal{H}\hat{\otimes}_{2}\mathcal{H})^{*} & \xrightarrow{U(\delta)^{*}} & U(\mathcal{H})^{*} \\
\uparrow{\scriptstyle \Theta_{U(\mathcal{H}),U(\mathcal{H})}} & & & & \| {\scriptstyle R_{\mathcal{H}}} \\
U(\mathcal{H})^{*}\hat{\otimes}_{\pi}U(\mathcal{H})^{*} & & & & \\
\| {\scriptstyle R_{\mathcal{H}}\hat{\otimes}_{\pi}R_{\mathcal{H}}} & & & & \\
U(\overline{\mathcal{H}})\hat{\otimes}_{\pi}U(\overline{\mathcal{H}}) & \xrightarrow[\Phi_{\overline{\mathcal{H}},\overline{\mathcal{H}}}]{} & U(\overline{\mathcal{H}}\hat{\otimes}_{2}\overline{\mathcal{H}}) = U(\overline{\mathcal{H}\hat{\otimes}_{2}\mathcal{H}}) & \xrightarrow[U(\overline{\delta^{\dagger}})]{} & U(\overline{\mathcal{H}})
\end{array} \tag{3}$$

Let $x, y, u, v \in \mathcal{H}$. $\Psi^{*}_{\mathcal{H},\mathcal{H}}(\Theta_{U(\mathcal{H}),U(\mathcal{H})}((R_{\mathcal{H}}\hat{\otimes}_{\pi}R_{\mathcal{H}})(x\otimes y)))(u\otimes v) = \langle u,x\rangle\langle v,y\rangle = \langle u\otimes v, \quad x\otimes y\rangle_{\mathcal{H}\hat{\otimes}_{2}\mathcal{H}} = R_{\mathcal{H}\hat{\otimes}_{2}\mathcal{H}}(\Phi_{\mathcal{H},\mathcal{H}}(x\otimes y))(u\otimes v)$. By definition of the Riesz isomorphism, $R_{\mathcal{H}}\circ U(\overline{\delta^{\dagger}}) = U(\delta)^{*}\circ R_{\mathcal{H}\hat{\otimes}_{2}\mathcal{H}}$. Therefore, the leftmost and rightmost diagrams below commute (by linearity and density), so commutes Diag. (3).

$$\begin{array}{ccccc}
(U(\mathcal{H})\hat{\otimes}_{\epsilon}U(\mathcal{H}))^{*} & \xrightarrow{\Psi^{*}_{\mathcal{H},\mathcal{H}}} & U(\mathcal{H}\hat{\otimes}_{2}\mathcal{H})^{*} & \xrightarrow{U(\delta)^{*}} & U(\mathcal{H})^{*} \\
\uparrow{\scriptstyle \Theta_{U(\mathcal{H}),U(\mathcal{H})}} & & \| {\scriptstyle R_{\mathcal{H}\hat{\otimes}_{2}\mathcal{H}}} & & \| {\scriptstyle R_{\mathcal{H}}} \\
U(\mathcal{H})^{*}\hat{\otimes}_{\pi}U(\mathcal{H})^{*} & & & & \\
\| {\scriptstyle R_{\mathcal{H}}\hat{\otimes}_{\pi}R_{\mathcal{H}}} & & & & \\
U(\overline{\mathcal{H}})\hat{\otimes}_{\pi}U(\overline{\mathcal{H}}) & \xrightarrow[\Phi_{\overline{\mathcal{H}},\overline{\mathcal{H}}}]{} & U(\overline{\mathcal{H}}\hat{\otimes}_{2}\overline{\mathcal{H}}) = U(\overline{\mathcal{H}\hat{\otimes}_{2}\mathcal{H}}) & \xrightarrow[U(\overline{\delta^{\dagger}})]{} & U(\overline{\mathcal{H}})
\end{array} \tag{4}$$

□

Proposition 5 is not used in what follows but one notwithstanding provides the following consequences.

Because of Proposition 3, Theorem 7 below is essentially due to [37, Lemma 2.10, p. 14] which states that any Banach coalgebra in $(\mathbf{Ban}_1, \hat{\otimes}_{\epsilon})$ embeds into a Banach coalgebra of functions on a compact semigroup. Nonetheless one provides a direct proof in our Hilbertian setting.

Lemma 6. *Let* $(\mathcal{H},\mu)$ *be a complex Hilbertian algebra. One furthermore assumes that* $||\mu||_{op} = ||\mu^{\dagger}||_{op} \leq 1$. *Equivalently one requires* $(\mathcal{H},\mu)$ *to be a semigroup object in the monoidal category* $\mathbb{Hilb}_1$ *of Hilbert spaces with linear contractions. Let* $B_{\mathcal{H}}(0,1]$ *be the closed unit ball of* $\mathcal{H}$. *Then,* $B_{\mathcal{H}}(0,1]$ *is a compact semigroup (in the weak topology).*

Proof. Since $||\mu||_{op} \leq 1$, it is clear that $B_{\mathcal{H}}(0,1]$ is a semigroup under the restriction say m, of μ_{bil}. According to Banach-Alaoglu theorem it is also compact in the weak topology, i.e., the weakest topology on $\mathcal{H}$ that makes continuous all functionals $\langle\cdot,x\rangle, x\in\mathcal{H}$. It remains to prove that the multiplication in $B_{\mathcal{H}}(0,1]$ is jointly continuous (in the weak topology), i.e., that for each $x\in\mathcal{H}, R_{\mathcal{H}}(x)\circ m : B_{\mathcal{H}}(0,1]\times B_{\mathcal{H}}(0,1] \to \mathbb{C}$ is continuous. Since m is the restriction of μ, it follows that $R_{\mathcal{H}}(x)(m(y,z)) = R_{\mathcal{H}}(x)(\mu(y\otimes z)) = \langle\mu(y\otimes z),x\rangle_{\mathcal{H}} = \langle y\otimes z, \mu^{\dagger}(x)\rangle_{\mathcal{H}\hat{\otimes}_{2}\mathcal{H}} = R_{\mathcal{H}\hat{\otimes}_{2}\mathcal{H}}(\mu^{\dagger}(x))(y\otimes z), y,z\in B_{\mathcal{H}}(0,1]$. But $R_{\mathcal{H}\hat{\otimes}_{2}\mathcal{H}}(\mu^{\dagger}(x))$ is continuous for the weak topology on $\mathcal{H}\hat{\otimes}_{2}\mathcal{H}$, so we are done. □

Let $\mathcal{H}$ be a Hilbert space. For $x \in \mathcal{H}$, let $\hat{x} \in (\bar{\mathcal{H}})^*$ be defined by $\hat{x}(y) := \langle x, y \rangle_{\mathcal{H}}$, i.e., $\hat{x} = R_{\bar{\mathcal{H}}}(x)$.

Theorem 7. Under the same assumptions as in Lemma 6, the underlying Banach coalgebra of $(\mathcal{H}, \mu)$ embeds as a closed subcoalgebra[5] of $C(B_{\mathsf{H}}(0, 1])$.

Proof. For each $x \in \mathcal{H}, \hat{x}$ induces by restriction a continuous map from $B_{\mathcal{H}}(0, 1] = B_{\bar{\mathcal{H}}}(0, 1]$ to $\mathbb{C}$, i.e., $\hat{x} \in C(B_{\mathcal{H}}(0, 1])$, since $R_{\bar{\mathcal{H}}}(x)$ is continuous. Now, $R_{\bar{\mathcal{H}}} : x \mapsto \hat{x}$, considered as a linear map R from $\mathcal{H}$ to $C(B_{\mathcal{H}}(0, 1])$, is clearly one-to-one because $B_{\mathcal{H}}(0, 1]$ contains every orthonormal basis. Therefore, as vector spaces, $\mathcal{H}$ embeds into $C(B_{\mathcal{H}}(0, 1])$. R is continuous since, by the Cauchy-Bunyakovski-Schwarz inequality, $||\hat{x}||_\infty \le ||x||_{\mathcal{H}}$. Moreover, for $x \neq 0, |\hat{x}\left(\frac{x}{||x||}\right)| = ||x||$, whence R is even an isometry, and thus the image of $\mathcal{H}$ under R is a closed subspace. R is a coalgebra map. Indeed for $y, z \in B_{\mathcal{H}}(0, 1], (\Delta(\mathsf{R}(x)))(y, z) = \hat{x}(m(y, z)) = \langle x, \mu(y \otimes z) \rangle_{\mathcal{H}} = \langle \mu^\dagger(x), \ y \otimes z \rangle_{\mathcal{H} \hat{\otimes}_2 \mathcal{H}} = (\mathsf{R} \hat{\otimes}_\epsilon \mathsf{R})(\Psi(\mu^\dagger(x)))(y, z)$, where $\Psi = \Psi_{\mathcal{H},\mathcal{H}}$ is the coherence constraint for the opmonoidal functor from $\mathbb{Hilb}_1$ to $(\mathbf{Ban}_1, \hat{\otimes}_\epsilon)$ from (the proof of) Proposition 3. Concerning the last equality, first of all given bounded linear maps $f : E \to C(X)$ and $g : F \to C(Y)$ for Banach spaces E, F and compact spaces X, Y, under $C(X \times X) \simeq C(X) \hat{\otimes}_\epsilon C(X)$, one has $(f \hat{\otimes}_\epsilon g)(u \otimes v)(x, y) = f(u)(x)g(v)(y), \quad u \in E, v \in F, x \in X, y \in Y$. Therefore, $(\mathsf{R} \hat{\otimes}_\epsilon \mathsf{R})(\Psi(u \otimes v))(y, z) = \langle u \otimes v, \ y \otimes z \rangle_{\mathcal{H} \hat{\otimes}_2 \mathcal{H}}$, since Ψ is the identity map on the algebraic tensor product $\mathcal{H} \otimes \mathcal{H}$. By continuity, this equality between maps on a dense subset extends to an equality on the completion. □

Given a compact set X, $\mathcal{M}(X)$ denotes the Banach space of all *Radon measures* of X, that is, complex regular Borel measures of X. Under Riesz representation theorem (e.g., [32, Theorem 1.1, p. 325]), $\mathcal{M}(X)$ identifies with the dual space $C(X)^*$. Given a compact semigroup S with jointly continuous multiplication m, as is well-known ([12]) $(\mathcal{M}(S), \star)$ is a Banach algebra under convolution $\nu_1 \star \nu_2 = (\nu_1 \times \nu_2) \circ m^{-1}$, where $\nu_1 \times \nu_2$ is the product measure.

Corollary 8. *The underlying Banach algebra of the conjugate of a Hilbertian algebra $(\mathcal{H}, \mu)$ with $||\mu||_{op} \le 1$, is the image of $\mathcal{M}(S)$ for some compact semigroup S.*

Proof. Since $(C(S), \Delta)$ is a Banach coalgebra by [39, Propositions 7.1, p. 18] its dual $C(S)^*$ is a Banach algebra. Its multiplication is explicitly given by $(\ell_1 \star \ell_2)(f) = \int_S \int_S f(m(x, y)) d\nu_1(x) d\nu_2(y), f \in C(S)$, where under Riesz representation theorem, ν_i is the Radon measure on S corresponding to ℓ_i. The observation that $(\ell_1 \star \ell_2)(f) = \int_S f d(\nu_1 \star \nu_2)$ ([12, p. 53] with other notations, or [34, Theorem 2, p. 350] for locally compact semigroups) does not mean anything other than $(\mathcal{M}(S), \star) \simeq (C(S)^*, \star)$ under Riesz representation.

Now, the map $U(\mathcal{H}, \mu^\dagger) \xrightarrow{\mathsf{R}} C(B_{\mathcal{H}}(0, 1])$ from the proof of Theorem 7 is a coalgebra map and an isometry. Therefore, $C(B_{\mathcal{H}}(0, 1])^* \xrightarrow{\mathsf{R}^*} U(\mathcal{H}, \mu^\dagger)^*$ is a morphism in $\mathbf{Sem}(\mathbf{Ban}, \hat{\otimes}_\pi)$ and is a *coisometry* which means that it maps the open unit ball of $C(B_{\mathcal{H}}(0, 1])^*$ onto that of $U(\mathcal{H}, \mu^\dagger)^*$ (this is essentially due to the Hahn-Banach theorem). So of course, it is onto, and it is so as a morphism of Banach algebras $BA(C(B_{\mathcal{H}}(0, 1])^*)) \xrightarrow{\mathsf{R}^*} BA(U(\mathcal{H}, \mu^\dagger)^*)$. The conclusion is obtained by the composition $(\mathcal{M}(B_{\mathcal{H}}(0, 1]), \star) \simeq (C(B_{\mathcal{H}}(0, 1])^*, \star) = BA(C(B_{\mathcal{H}}(0, 1])^*)) \xrightarrow{\mathsf{R}^*} BA(U(\mathcal{H}, \mu^\dagger)^*) \simeq BA(U(\bar{\mathcal{H}}, \bar{\mu}))$, where the last isomorphism is the result of Proposition 5. □

[5]C.f. Definition 6 and the discussion that follows.

2.4. H^*-algebras, in-between Banach and Hilbertian algebras

In this section all algebras are over the field of complex numbers.

Let us consider a H^*-algebra [5] $(U(\mathcal{H}), m)$, that is a complex Banach algebra with $\mathcal{H}$ a Hilbert space such that for each $x \in \mathcal{H}$ there is a (not necessarily unique) $x^* \in \mathcal{H}$, called an *adjoint* of x, such that for all $y, z \in \mathcal{H}, \langle m(x,y), z\rangle = \langle y, m(x^*, z)\rangle$ and $\langle m(y,x), z\rangle = \langle y, m(z, x^*)\rangle$. It is said to be *proper* whenever every element has a unique adjoint ([5, Theorem 2.1, p. 370]).[6]

According to the proof of [5, Corollary 4.1, p. 382] a proper commutative H^*-algebra is isomorphic to one of the form $\ell^2_\alpha(X) := (U(\ell^2_\alpha(X), \langle\cdot,\cdot\rangle_\alpha), m_X)$ where $\ell^2_\alpha(X) := \{f \in \mathbb{C}^X : \sum_{x\in X} \alpha(x)|f(x)|^2 < +\infty\}$ for some fixed set X and map $X \xrightarrow{\alpha} [1, +\infty[, \langle f, g\rangle_\alpha := \sum_{x\in X} \alpha(x) f(x)\overline{g(x)}$, $m_X(f,g)(x) := f(x)g(x)$, and $f^*(x) := \overline{f(x)}, x \in X, f, g \in \ell^2_\alpha(X)$. Observe that $||m_X||_{op} \leq 1$.

More precisely (see e.g. [20]) for any proper commutative H^*-algebra $(U(\mathcal{H}), m), \mathcal{H}$ is the Hilbert direct sum of its (necessarily closed) minimal ideals (these are pairwise orthogonal) which, as $\mathbb{C}$-algebras, are isomorphic to $\mathbb{C}$. So $(U(\mathcal{H}), m) \simeq \ell^2_{\alpha_{(\mathcal{H},m)}}(Min(\mathcal{H}, m))$ (isometrically) under $\Omega_{(\mathcal{H},m)} : e_I \mapsto \delta_I, I \in Min(\mathcal{H}, m)$, where $Min(\mathcal{H}, m)$ stands for the set of all minimal ideals and $\alpha_{(\mathcal{H},m)}(I) := ||e_I||^2$ with e_I the identity of $I \in Min(\mathcal{H}, m)$.

Such proper commutative H^*-algebras will play a fundamental rôle in what follows (see below Section 5). For now on let us see how they are related to Hilbertian algebras. The multiplication m_X of the most general model $\ell^2_\alpha(X)$, is a weak Hilbert-Schmidt mapping as $\sum_{x,y\in X} |\langle m_X(u_x, u_y), f\rangle_\alpha|^2 = \sum_{x\in X} |\overline{f(x)}|^2 \leq ||f||^2_\alpha$ (because $\alpha(x) \geq 1, x \in X$) with $||f||_\alpha := \sqrt{\langle f, f\rangle_\alpha}$, where $u_x := \frac{1}{\sqrt{\alpha(x)}}\delta_x, x \in X$. Whence $((\ell^2_\alpha(X), \langle\cdot,\cdot\rangle_\alpha), \mu_X)$ is a commutative Hilbertian algebra, with $\mu_X : \ell^2_\alpha(X) \hat{\otimes}_2 \ell^2_\alpha(X) \to \ell^2_\alpha(X)$ the unique bounded linear map such that $(\mu_X)_{bil} = m_X$. One has $||\mu_X||_{op} = ||m_X||_{HS} \leq 1$.

Remark 5. Of course, $(\ell^2_\alpha(X), \langle\cdot,\cdot\rangle_\alpha)$ as above, is unitarily isomorphic to $\ell^2(X)$ under $\Lambda_X : u_x \mapsto \delta_x$ $(x \in X)$, i.e., Λ_X is the multiplication operator $f \mapsto \alpha^{\frac{1}{2}} f$ by $\alpha^{\frac{1}{2}} : x \mapsto \sqrt{\alpha(x)}$. This is not an algebra isomorphism (under pointwise product) whenever for at least one x, $\alpha(x) > 1$.

The situation is quite different for noncommutative H^*-algebras. E.g., a full matrix algebra with an infinite set of indices never is (the underlying Banach algebra of) a Hilbertian algebra, because its multiplication fails to be a weak Hilbert-Schmidt mapping. However the convolution L^2-algebra $L^2(G)$ of any compact group G turns to be a semigroup in $\mathbb{Hilb}$ (by the Peter-Weyl theorem).

Theorem 9. *Let $\alpha: X \to [1, +\infty[$. $((\ell^2_\alpha(X), \langle\cdot,\cdot\rangle_\alpha), \mu_X) \simeq (\ell^2(Y), \mu_Y)$ (topological isomorphism in ${}_c\mathbf{Sem}(\mathbb{Hilb})$) for some set Y if, and only if, α is bounded above. In this case Y may be chosen equal to X. Moreover, if for some x, $\alpha(x) > 1$, then no topological isomorphism as above is unitary. In particular, a proper commutative H^*-algebra $(U(\mathcal{H}), m)$ is topologically isomorphic to $BA(U(\ell^2(X), \mu_X)) = ((\ell^2(X), ||\cdot||_2), m_X)$ if, and only if, $\alpha_{(\mathcal{H},m)}$ is bounded above. Moreover, if $\alpha_{(\mathcal{H},m)}(I) > 1$ for some closed minimal ideal I, then no topological isomorphism as above is an isometry.*

Proof. One first observes that $\ell^2_\alpha(X) \subseteq \ell^2(X)$ since $1 \leq \alpha(x), x \in X$, and the inclusion is continuous because $||f||_2 \leq ||f||_\alpha$. Secondly, when α is bounded above, then $\ell^2_\alpha(X) = \ell^2(X)$ (qua vector spaces). In this case one notices that $||f||_\alpha \leq ||\alpha||^{\frac{1}{2}}_\infty ||f||_2$ so that $id : ((\ell^2_\alpha(X), \langle\cdot,\cdot\rangle_\alpha), \mu_X) \simeq (\ell^2(X), \mu_X)$ is a topological isomorphism in ${}_c\mathbf{Sem}(\mathbb{Hilb})$ (albeit not unitary in general).

[6] More rigorously, a H^*-algebra is a Banach algebra (E, m) such that there exists a Hilbert space $\mathcal{H}$ with $U(\mathcal{H}) = E$ and admitting adjoints. But actually the Hilbert space $\mathcal{H}$ is entirely determined by E since U is injective on objects (the norm of E satisfies the parallelogram law and thus uniquely determines the inner product of $\mathcal{H}$).

Conversely let us assume that $((\ell^2_\alpha(X), \langle\cdot,\cdot\rangle_\alpha), \mu_X) \simeq (\ell^2(Y), \mu_Y)$ for some set Y. In particular the Hilbert spaces $(\ell^2_\alpha(X), \langle\cdot,\cdot\rangle_\alpha)$ and $\ell^2(Y)$ are topologically isomorphic, and the polar decomposition of such an isomorphism (see [19, Corollary 5.91, p. 405]) shows that they are actually even unitarily isomorphic. So, without loss of generality one may assume that $X = Y$. In this situation, it is possible to transport the pointwise product μ_X of $(\ell^2_\alpha(X), \langle\cdot,\cdot\rangle_\alpha)$ on $\ell^2(X)$ using the unitary transformation Λ_X from Remark 5. Let μ_α be the resulting multiplication: $\mu_\alpha(f \otimes g) = \alpha^{-\frac{1}{2}} fg, f, g \in \ell^2(X)$. Of course, $\Lambda_X : ((\ell^2_\alpha(X), \langle\cdot,\cdot\rangle_\alpha), \mu_X) \simeq (\ell^2(X), \mu_\alpha)$ (unitarily so).

One now has a topological isomorphism $(\ell^2(X), \mu_\alpha) \simeq (\ell^2(X), \mu_X)$. Call it Π. If $X = \emptyset$, then α is of course bounded above. So let us assume that $X \neq \emptyset$. As a matter of fact, $\Pi\left(\alpha^{-\frac{1}{2}} fg\right) = \Pi(f)\Pi(g)$ for each $f, g \in \ell^2(X)$. In particular, for $f = \delta_x = g, x \in X$, $\frac{1}{\sqrt{\alpha(x)}}\Pi(\delta_x) = \Pi(\delta_x)^2$. Observe that by continuity and linearity, Π is entirely determined by its values on δ_x, $x \in X$. Let $S_x := \{z \in X : \Pi(\delta_x)(z) \neq 0\} \neq \emptyset$ (since Π is one-to-one). Then, $\Pi(\delta_x) = \frac{1}{\sqrt{\alpha(x)}} \sum_{z \in S_x} \delta_z$ as the sum of a summable family in $\ell^2(X)$, so that S_x is necessarily finite.

Now let $x, y \in X$ with $x \neq y$. Then, $0 = \Pi(\delta_x)\Pi(\delta_y) = \frac{1}{\sqrt{\alpha(x)\alpha(y)}} \sum_{z \in S_x \cap S_y} \delta_z$ from what it follows that $S_x \cap S_y = \emptyset$. Let $\cup_{x \in X} S_x \xrightarrow{\pi} X$ be defined by $\pi(z) = x$ if, and only if, $z \in S_x$. It is onto.

Since Π is onto, it follows that $X = \cup_{x \in X} S_x$. Indeed, let $y \in X \setminus \cup_{x \in X} S_x$. Since $\delta_y \in ran(\Pi)$, there exists $f \in \ell^2(X)$ such that $\delta_y = \Pi(f) = \sum_{x \in X} \frac{f(x)}{\sqrt{\alpha(x)}} \sum_{z \in S_x} \delta_z = \sum_{z \in \cup_{x \in X} S_x} \frac{f(\pi(z))}{\sqrt{\alpha(\pi(z))}} \delta_z$ which contradicts $\delta_y \notin \cup_{x \in X} S_x$.

Now it follows that $\Pi(f) = \sum_{z \in X} \frac{f(\pi(z))}{\sqrt{\alpha(\pi(z))}} \delta_z = \left(\alpha^{-\frac{1}{2}} f\right) \circ \pi, f \in \ell^2(X)$. Let us assume that $X \xrightarrow{\pi} X$ is not one-to-one. Let $x, y \in X, x \neq y$ such that $\pi(x) = \pi(y)$. Since Π is onto, there exists $f \in \ell^2(X)$ such that $\delta_x = \Pi(f) = \left(\alpha^{-\frac{1}{2}} f\right) \circ \pi$. Then, $0 = \delta_x(y) = \frac{f(\pi(y))}{\sqrt{\alpha(\pi(y))}} = \frac{f(\pi(x))}{\sqrt{\alpha(\pi(x))}} = \delta_x(x) = 1$, which is impossible.

Consequently, π is a permutation of X, and $\Pi^{-1}(f) = \alpha^{\frac{1}{2}}(f \circ \pi^{-1})$. Since Π^{-1} is also bounded, α is necessarily bounded, and thus bounded above. So the first equivalence of the statement is proved. (Observe that $\Pi^\dagger = \alpha^{-\frac{1}{2}}(f \circ \pi^{-1})$ and thus Π is unitary if, and only if, $\alpha(x) = 1$ for each $x \in X$.)

One may check directly that for $f \in \ell^2_\alpha(X), (\mu_X(\mu_X^\dagger(f)) = \alpha^{-1} f$ while one knows that for $f \in \ell^2(X), \mu_X \circ \mu_X^\dagger = id$. Thus, even if $((\ell^2_\alpha(X), \langle\cdot,\cdot\rangle_\alpha), \mu_X) \simeq (\ell^2(Y), \mu_Y)$, it cannot be unitary so when $\alpha(x) > 1$ for some x.

Finally for a proper commutative H^*-algebra $(U(\mathcal{H}, m), (U(\mathcal{H}), m) \simeq ((\ell^2(X), ||\cdot||_2), m_X)$ if, and only if, $\ell^2_{\alpha_{(\mathcal{H},m)}}(Min(\mathcal{H}, m)) \simeq ((\ell^2(X), ||\cdot||_2), m_X)$ if, and only if, $((\ell^2_{\alpha_{(\mathcal{H},m)}}(Min(\mathcal{H}, m)), \langle\cdot,\cdot\rangle_{\alpha_{(\mathcal{H},m)}}), \mu_{Min(\mathcal{H},m)}) \simeq (\ell^2(X), \mu_X)$ if, and only if, $\alpha_{(\mathcal{H},m)}$ is bounded above. With $X = Min(\mathcal{H}, m)$, one may choose $\Pi \circ \Lambda_{Min(\mathcal{H},m)} \circ \Omega_{(\mathcal{H},m)}$ as an isomorphism (which is not an isometry in general because of the presence of Π).

By the above if $(U(\mathcal{H}), m) \simeq ((\ell^2(X), ||\cdot||_2), m_X)$ but $\alpha_{(\mathcal{H},m)} \not\equiv 1$, then the isomorphism cannot be an isometry. □

3. Ideals and subcoalgebras

In this section is described how the closed ideals of a Hilbertian algebra and the closed subcoalgebras of its adjoint coalgebra are related (see Theorem 18 below).

In what follows $\mathcal{H}$ denotes a Hilbert space. Given two subspaces V, W, their sum $V+W$ is the subspace $\{v+w : v\in V, w\in W\}$. Of course, $(V+W)^\perp = V^\perp\cap W^\perp$ and $(\mathsf{cl}(V)\cap\mathsf{cl}(W))^\perp = \mathsf{cl}(V^\perp + W^\perp)$. More generally, given a family $(V_i)_{i\in I}$ of subspaces of $\mathcal{H}$, $\sum_{i\in I} V_i$ is the subspace consisting of all finite sums of elements of $\cup_{i\in I} V_i$. The following result is clear.

Lemma 10. *Let $(V_i)_{i\in I}$ be a family of subspaces of $\mathcal{H}$. Then, $(\sum_{i\in I} V_i)^\perp = \cap_{i\in I} V_i^\perp$ and $(\cap_{i\in I}\mathsf{cl}(V_i))^\perp = \mathsf{cl}(\sum_{i\in I} V_i^\perp)$.*

Lemma 11. *For a closed subspace V of $\mathcal{H}$, $\mathsf{cl}(\mathcal{H}\hat{\otimes}_2 V^\perp + V^\perp\hat{\otimes}_2\mathcal{H}) = \ker(\pi_V\hat{\otimes}_2\pi_V) = (V\hat{\otimes}_2 V)^\perp = (V^\perp\hat{\otimes}_2 V)\oplus_2(V^\perp\hat{\otimes}_2 V^\perp)\oplus_2(V\hat{\otimes}_2 V^\perp)$.*

Proof. The last equality is due to Lemma 2, while the penultimate one is obvious. Clearly $\mathcal{H}\otimes V^\perp + V^\perp\otimes\mathcal{H}\subseteq\ker(\pi_V\hat{\otimes}_2\pi_V)$, so $\mathsf{cl}(\mathcal{H}\hat{\otimes}_2 V^\perp + V^\perp\hat{\otimes}_2\mathcal{H})\subseteq\ker(\pi_{I^\perp}\hat{\otimes}_2\pi_{I^\perp})$, and $(V^\perp\hat{\otimes}_2 V)\oplus_2(V^\perp\hat{\otimes}_2 V^\perp)\oplus_2(V\hat{\otimes}_2 V^\perp)\subseteq\mathsf{cl}(\mathcal{H}\hat{\otimes}_2 V^\perp + V^\perp\hat{\otimes}_2\mathcal{H})$. □

Now, let us assume that $(\mathcal{H},\mu)$ is a Hilbertian algebra and let $(\mathcal{H},\mu_{bil})$ be its underlying $\mathbb{K}$-algebra (Definition 4).

Definition 5. A *two-sided ideal* I of $(\mathcal{H},\mu)$ is defined as a two-sided ideal of $(\mathcal{H},\mu_{bil})$, i.e., a subspace such that $\mu_{alg}(\mathcal{H}\otimes I + I\otimes\mathcal{H})\subseteq I$.

Lemma 12. *Let I be a closed subspace of $\mathcal{H}$. I is a two-sided ideal if, and only if, $\mu(\mathsf{cl}(\mathcal{H}\hat{\otimes}_2 I + I\hat{\otimes}_2\mathcal{H}))\subseteq I$ if, and only if, $\mu((I^\perp\hat{\otimes}_2 I)\oplus_2(I\hat{\otimes}_2 I)\oplus_2(I\hat{\otimes}_2 I^\perp))\subseteq I$.*

Proof. Let us assume that $\mu(\mathsf{cl}(\mathcal{H}\hat{\otimes}_2 I + I\hat{\otimes}_2\mathcal{H}))\subseteq I$. Then I is a two-sided ideal of $(\mathcal{H},\mu_{bil})$ since $\mu_{alg}(\mathcal{H}\otimes I + I\otimes\mathcal{H}) = \mu(\mathcal{H}\otimes I + I\otimes\mathcal{H})\subseteq\mu(\mathsf{cl}(\mathcal{H}\hat{\otimes}_2 I + I\hat{\otimes}_2\mathcal{H}))\subseteq I$ follows from $\mathcal{H}\otimes I + I\otimes\mathcal{H}\subseteq\mathsf{cl}(\mathcal{H}\hat{\otimes}_2 I + I\hat{\otimes}_2\mathcal{H})$. Conversely, let us assume that I is a two-sided ideal of $(\mathcal{H},\mu_{bil})$. By continuity of μ, $\mu(\mathsf{cl}(\mathcal{H}\otimes I + I\otimes\mathcal{H}))\subseteq\mathsf{cl}(\mu(\mathcal{H}\otimes I + I\otimes\mathcal{H})) = \mathsf{cl}(\mu_{alg}(\mathcal{H}\otimes I + I\otimes\mathcal{H}))\subseteq\mathsf{cl}(I) = I$ (I being closed). But $\mathsf{cl}(\mathcal{H}\otimes I + I\otimes\mathcal{H}) = \mathsf{cl}(\mathcal{H}\hat{\otimes}_2 I + I\hat{\otimes}_2\mathcal{H})$ and we are done. The second equivalence follows from Lemma 11. □

Definition 6. Let $(\mathcal{H},\mu)$ be a Hilbertian algebra and $(\mathcal{K},\delta)$ a Hilbertian coalgebra.

1. A *closed subalgebra* of $(\mathcal{H},\mu)$ is a closed subspace C of $\mathcal{H}$ such that $\mu(C\hat{\otimes}_2 C)\subseteq C$. Let $C\hat{\otimes}C\overset{\mu_{|C}}{\to}C$ be the corresponding co-restriction of μ.
2. A *closed subcoalgebra* of $(\mathcal{K},\delta)$ is a closed subspace C of $\mathcal{K}$ with $\delta(C)\subseteq C\hat{\otimes}_2 C$. The corresponding co-restriction of δ is denoted $C\overset{\delta_{|C}}{\to}C\hat{\otimes}_2 C$.

It is clear from the preservation of closed subspaces by the Hilbert tensor product, that a closed sub(co)algebra is a (co)algebra on its own and the canonical inclusion is a morphism of Hilbertian (co)algebras. Moreover by Lemma 12 a closed ideal is also a closed subalgebra. Since the injective tensor product also preserves closed subspaces, the notion of a *closed subcoalgebra* makes sense as well for objects of $\mathbf{Cosem}(\mathbf{Ban},\hat{\otimes}_\epsilon)$.

Lemma 13. *Let $(\mathcal{H},\mu)$ be a Hilbertian algebra, and let V be a closed subspace of $\mathcal{H}$. If V is both a closed subalgebra of $(\mathcal{H},\mu)$ and a closed subcoalgebra of $(\mathcal{H},\mu^\dagger)$, then $(\mu_{|_V})^\dagger = (\mu^\dagger)_{|_V}$. In this situation, both maps $\pi_V:\mathcal{H}\to V$ and $i_V:V\hookrightarrow\mathcal{H}$ are morphisms of Hilbertian algebras and coalgebras. Moreover, if $\mu\circ\mu^\dagger = id_{\mathcal{H}}$, then $\mu_{|_V}\circ(\mu_{|_{V^\perp}})^\dagger = id_V$.*

Proof. By construction $(\mu^\dagger)_{|_V} = (\pi_V \hat{\otimes}_2 \pi_V) \circ \mu^\dagger \circ i_V$ and $\mu_{|_V} = \pi_V \circ \mu \circ (i_V \hat{\otimes}_2 i_V)$. Whence $(\mu_{|_V})^\dagger = (\mu^\dagger)_{|_V}$. Since V is both a closed subalgebra and a closed subcoalgebra, $i_V : (V, (\mu^\dagger)_{|_V}) \to (\mathcal{H}, \mu^\dagger)$ and $i_V : (V, \mu_{|_V}) \to (\mathcal{H}, \mu)$ are respectively morphisms of coalgebras and algebras. Passing to the adjoint, in view of the equality $(\mu_{|_V})^\dagger = (\mu^\dagger)_{|_V}, \pi_V : (\mathcal{H}, \mu) \to (V, \mu_{|_V})$ and $\pi_V : (\mathcal{H}, \mu^\dagger) \to (V, (\mu^\dagger)_{|_V})$ are respectively morphisms of algebras and coalgebras.

As a consequence of $(\mu_{|_V})^\dagger = (\mu^\dagger)_{|_V}$, when $\mu \circ \mu^\dagger = id_{\mathcal{H}}, i_V \circ \mu_{|_V} \circ (\mu_{|_V})^\dagger = \mu \circ (i_V \hat{\otimes}_2 i_V) \circ (\mu_{|_V})^\dagger = \mu \circ \mu^\dagger \circ i_V = i_V$. Since i_V is a monomorphism, $\mu_{|_V} \circ (\mu_{|_V})^\dagger = id_V$. □

Now let us establish a series of lemmas so as to provide a relation between closed ideals and coalgebras (Theorem 18 below).

Lemma 14. *If C is a closed subcoalgebra of $(\mathcal{H}, \mu^\dagger)$, then $C^\perp$ is a closed two-sided ideal of $(\mathcal{H}, \mu)$.*

Proof. $\mu^\dagger(C) \subseteq C \hat{\otimes}_2 C$ if, and only if, $\mu((C \hat{\otimes}_2 C)^\perp) \subseteq C^\perp$ (by Lemma 1). According to Lemma 2, $(C \hat{\otimes}_2 C)^\perp = (C \hat{\otimes}_2 C^\perp) \oplus_2 (C^\perp \hat{\otimes}_2 C) \oplus_2 (C^\perp \hat{\otimes}_2 C^\perp) \supseteq \mathcal{H} \otimes C^\perp + C^\perp \otimes \mathcal{H}$. Whence it follows that $\mu(\mathcal{H} \otimes C^\perp + C^\perp \otimes \mathcal{H}) \subseteq C^\perp$, and thus $C^\perp$ is a closed two-sided ideal of $(\mathcal{H}, \mu)$. □

Lemma 15. *Let I be a closed two-sided ideal of $(\mathcal{H}, \mu)$. Then, $I^\perp$ is a closed subcoalgebra of $(\mathcal{H}, \mu^\dagger)$.*

Proof. First $(I^\perp \hat{\otimes}_2 I) \oplus_2 (I \hat{\otimes}_2 I^\perp) \oplus_2 (I \hat{\otimes}_2 I) \subseteq \mathcal{H} \hat{\otimes}_2 I + I \hat{\otimes}_2 \mathcal{H} \subseteq \mathsf{cl}(\mathcal{H} \hat{\otimes}_2 I + I \hat{\otimes}_2 \mathcal{H})$. Whence $\mu((I^\perp \hat{\otimes}_2 I) \oplus_2 (I \hat{\otimes}_2 I^\perp) \oplus_2 (I \hat{\otimes}_2 I)) \subseteq \mu(\mathsf{cl}(\mathcal{H} \hat{\otimes}_2 I + I \hat{\otimes}_2 \mathcal{H})) \subseteq I$. But $(I^\perp \hat{\otimes}_2 I) \oplus_2 (I \hat{\otimes}_2 I^\perp) \oplus_2 (I \hat{\otimes}_2 I) = (I^\perp \hat{\otimes}_2 I^\perp)^\perp$. So $\mu^\dagger(I^\perp) \subseteq I^\perp \hat{\otimes}_2 I^\perp$. □

Definition 7. *Let $(\mathcal{H}, \mu)$ be a Hilbertian algebra. Let $V \subseteq \mathcal{H}$ be a closed subspace. Let us assume that there exists a bounded linear map $\mu_V : V \hat{\otimes}_2 V \to V$. (V, μ_V) is said to be a quotient Hilbertian algebra of $(\mathcal{H}, \mu)$ if the following diagram commutes. In particular, $\mu_V = \pi_V \circ \mu \circ (i_V \hat{\otimes}_2 i_V)$ and it is called the quotient multiplication.*

$$\begin{array}{ccc} \mathcal{H}\hat{\otimes}_2\mathcal{H} & \xrightarrow{\mu} & \mathcal{H} \\ {\scriptstyle \pi_V\hat{\otimes}_2\pi_V}\downarrow & & \downarrow{\scriptstyle \pi_V} \\ V\hat{\otimes}_2 V & \xrightarrow[\mu_V]{} & V \end{array} \tag{5}$$

Lemma 16. *(V, μ_V) is a quotient Hilbertian algebra of $(\mathcal{H}, \mu)$ if, and only if, $(V, \mu_V^\dagger)$ is a closed subcoalgebra of the adjoint coalgebra $(\mathcal{H}, \mu^\dagger)$. In particular every quotient Hilbertian algebra is a Hilbertian algebra on its own right and π_V is a morphism of Hilbertian algebras.*

Proof. Taking the adjoint on Diagram (5) provides the following commutative diagram which shows that $(V, \mu_V^\dagger)$ is a closed subcoalgebra of $(\mathcal{H}, \mu^\dagger)$. Associativity of μ_V follows from coassociativity of $\mu_V^\dagger$.

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\mu^\dagger} & \mathcal{H}\hat{\otimes}_2\mathcal{H} \\ {\scriptstyle i_V}\uparrow & & \uparrow{\scriptstyle i_V\hat{\otimes}_2 i_V} \\ V & \xrightarrow[\mu_V^\dagger]{} & V\hat{\otimes}_2 V \end{array} \tag{6}$$

From a closed subcoalgebra, by passing to the adjoint, one of course gets a quotient Hilbertian algebra. □

Lemma 17. *Let V be a closed subspace of $(\mathcal{H}, \mu)$.*

1. *If I is a closed two-sided ideal of $(\mathcal{H}, \mu)$, then $(I^{\perp}, \pi_{I^{\perp}} \circ \mu \circ (i_{I^{\perp}} \hat{\otimes}_2 i_{I^{\perp}}))$ is a quotient Hilbertian algebra of $(\mathcal{H}, \mu)$.*
2. *If (V, μ_V) is a quotient Hilbertian algebra of $(\mathcal{H}, \mu)$, then $V^{\perp}$ is a closed two-sided ideal of $(\mathcal{H}, \mu)$.*
3. *Let (V, μ_V) be a quotient Hilbertian algebra of $(\mathcal{H}, \mu)$. Let $\mathcal{H}/V^{\perp}$ be the quotient Hilbert space, and let $can_{V^{\perp}} : \mathcal{H} \to \mathcal{H}/V^{\perp}$ be the canonical epimorphism (which is bounded). Then, $(V, \mu_V) \simeq (\mathcal{H}/V^{\perp}, \tilde{\mu})$ (as Hilbertian algebras) under the canonical isomorphism $V \simeq \mathcal{H}/V^{\perp}$, where $\tilde{\mu} : \mathcal{H} \hat{\otimes}_2 \mathcal{H} \to \mathcal{H}$ is defined by the following diagram.*

$$\begin{array}{ccc} V\hat{\otimes}_2 V & \xrightarrow{\mu_V} & V \\ \wr\| & & \wr\| \\ \mathcal{H}/V^{\perp}\hat{\otimes}_2\mathcal{H}/V^{\perp} & \xrightarrow[\tilde{\mu}]{} & \mathcal{H}/V^{\perp} \end{array} \tag{7}$$

Proof. Let us prove the first point. Let I be a closed two-sided ideal of $(\mathcal{H}, \mu)$. Thus, $\mu((I^{\perp} \hat{\otimes}_2 I^{\perp})^{\perp}) \subseteq I$. According to Lemma 1, $\mu^{\dagger}(I^{\perp}) \subseteq I^{\perp} \hat{\otimes}_2 I^{\perp}$. Diagrammatically the situation is depicted as follows (i.e., $(\mu^{\dagger})_{|_{I^{\perp}}}$ is given by co-restricting $\mu^{\dagger}$).

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\mu^{\dagger}} & \mathcal{H}\hat{\otimes}_2\mathcal{H} \\ {\scriptstyle i_{I^{\perp}}}\uparrow & & \uparrow{\scriptstyle i_{I^{\perp}}\hat{\otimes}_2 i_{I^{\perp}}} \\ I^{\perp} & \xrightarrow[(\mu^{\dagger})_{|_{I^{\perp}}}]{} & I^{\perp}\hat{\otimes}_2 I^{\perp} \end{array} \tag{8}$$

Taking the adjoint provides the following commutative diagram which shows that $(I^{\perp}, (\mu^{\dagger})^{\dagger}_{|_{I^{\perp}}})$ is a quotient Hilbertian algebra.

$$\begin{array}{ccc} \mathcal{H}\hat{\otimes}_2\mathcal{H} & \xrightarrow{\mu} & \mathcal{H} \\ {\scriptstyle \pi_{I^{\perp}}\hat{\otimes}_2\pi_{I^{\perp}}}\downarrow & & \downarrow{\scriptstyle \pi_{I^{\perp}}} \\ I^{\perp}\hat{\otimes}_2 I^{\perp} & \xrightarrow[(\mu^{\dagger})^{\dagger}_{|_{I^{\perp}}}]{} & I^{\perp} \end{array} \tag{9}$$

Multiplying on the right by $i_{I^{\perp}} \hat{\otimes}_2 i_{I^{\perp}}$ both equal maps from Diagram (9) one obtains $(\mu^{\dagger})^{\dagger}_{|_{I^{\perp}}} = \pi_{I^{\perp}} \circ \mu \circ (i_{I^{\perp}} \hat{\otimes}_2 i_{I^{\perp}})$. The second point follows from Lemmas 14 and 16. The third point is merely obvious since $V \simeq \mathcal{H}/V^{\perp}$ (canonically as Hilbert spaces). □

Lemmas 14, 15, 16 and 17 together establish a nice connection between closed (two-sided) ideals of $(\mathcal{H}, \mu)$, closed subcoalgebras of $(\mathcal{H}, \mu^{\dagger})$ and quotient Hilbertian algebras of $(\mathcal{H}, \mu)$ as stated by the following result. Recall that a maximal ideal is assumed to be a *proper* ideal (i.e., different from $\mathcal{H}$).

Definition 8. Let us call *simple* a closed subcoalgebra C of $(\mathcal{H}, \mu^{\dagger})$ different from (0) and such that for any closed subcoalgebra $D \subseteq C$, either $D = (0)$ or $D = C$.

Theorem 18. *$(-)^{\perp}$ is an order-decreasing one-one correspondence from the set of closed two-sided ideals of $(\mathcal{H}, \mu)$ onto the set of closed subcoalgebras of $(\mathcal{H}, \mu^{\dagger})$ (or the set of quotient Hilbertian algebras of $(\mathcal{H}, \mu)$). In particular, maximal closed two-sided ideals of $(\mathcal{H}, \mu)$ correspond to simple closed subcoalgebras of $(\mathcal{H}, \mu^{\dagger})$.*

Remark 6. Letting I be the set of all closed two-sided ideals of $(\mathcal{H},\mu)$ and S be the set of all closed subcoalgebras of $(\mathcal{H},\mu^\dagger)$, $(I \overset{(-)^\perp}{\to} S, S \overset{(-)^\perp}{\to} T)$ is a Galois correspondence.

4. The underlying Banach algebra of a commutative Hilbertian algebra

In this section is proved (Theorem 22 below) that the orthogonal complement of the Jacobson radical of a complex commutative Hilbertian algebra is the closure of the linear span of its group-like elements (Definition 10), i.e., its members which correspond by the Riesz representation, to its nonzero multiplicative functionals (Lemma 19). Moreover, the correspondences sending a Hilbertian algebra to its radical, and to the orthogonal complement of its radical, are shown to be functorial and to be part of adjunctions (Theorem 24).

4.1. A structure theorem

One recalls that a two-sided ideal I of an algebra A is said to be *modular* (also called *regular*) if there exists some element $e \in A$ such that for all $x \in A, x-ex \in I$. e is called a *modular unit* of I because being modular is equivalent to the fact that A/I is a unital algebra (with unit $e+I$). Of course, if A is itself unital, then every two-sided ideal is modular. Also, if e is a modular unit for I which belongs to I, then $I=A$.

As announced in Definition 1, from now on to the end of Section 4, we only consider commutative Hilbertian algebras over $\mathbb{K}=\mathbb{C}$

Their underlying (complex) Banach algebras thus are commutative too and the next discussion applies as well to them. Following e.g., [18], maximal closed modular ideals of a (commutative) complex Banach algebra A are in one-one correspondence with its nonzero multiplicative functionals. Note also that any maximal modular ideal is actually closed ([18, Lemma 1.4.5, p. 23]).

The *Jacobson radical*, or simply *radical*, J of a Banach algebra A is defined as $\cap ModMax = \cap_\ell \ker \ell$, where $ModMax$ is the set of all modular maximal (closed) ideals of A, and in the second intersection ℓ runs over the set of all nonzero multiplicative functionals. By definition, if there are no maximal modular closed ideals, then $J=A$. In any case, J is a closed ideal of A.

Definition 9. It is clear from the definition that the set all modular maximal closed ideals of $(\mathcal{H},\mu)$ is equal to that of its underlying Banach algebra. Therefore it makes sense to consider the *radical*, denoted $J(\mathcal{H},\mu)$ or simply J, of a Hilbertian algebra $(\mathcal{H},\mu)$ as the intersection of all its maximal modular ideals, i.e., as the radical of its underlying Banach algebra. As a matter of fact, a Hilbertian algebra is said to be *semisimple* (resp. *radical*) when its underlying Banach algebra is so (see e.g., [18, Def. 2.1.9, p. 50]).

Applying Lemma 4 with $\mathbb{C}$ for $(\mathcal{K},\gamma)$, i.e., for bounded multiplicative functionals on $(\mathcal{H},\mu)$, gives that $f \in \mathbf{Sem}(\mathbb{Hilb})((\mathcal{H},\mu),\mathbb{C})$ if, and only if, f is an algebra map from $(\mathcal{H},\mu_{bil})$ to $\mathbb{C}$. (This follows from automatic continuity of algebra maps from the underlying algebra of a Banach algebra to $\mathbb{C}$ (see [26, Corollary 3.1.7, p. 112] or [18, Corollary 2.1.10, p. 50]).) Therefore, the bounded multiplicative functionals on a complex commutative Hilbertian algebra are exactly the *characters* of its underlying algebra. The *trivial* character, identically null, is denoted by 0. Because the topological dual of a Hilbert space is (conjugate-linear) isomorphic to it, one obtains an intrinsic characterization of these characters as explained below.

Definition 10. Let us call *group-like element* (following the usual terminology from the theory of ordinary coalgebras, see for instance [35]; such elements are sometimes called *copyable* in the context of Frobenius algebras [2]) a nonzero member x of a Hilbertian algebra $(\mathcal{H},\mu)$ such that $\mu^\dagger(x)=x\otimes x$. The set of all such elements is denoted by $\mathcal{G}(\mathcal{H},\mu)$.

Observe that for a group-like element x of $(\mathcal{H},\mu), \langle\cdot,x\rangle$ is a nonzero character as $\langle\mu(u\otimes v),x\rangle = \langle u\otimes v,\mu^\dagger(x)\rangle = \langle u\otimes v, x\otimes x\rangle = \langle u,x\rangle\langle v,x\rangle, u,v\in\mathcal{H}$.

Lemma 19. *The Riesz representation $R_\mathcal{H}:\bar{\mathcal{H}}\to\mathcal{H}^*, x\mapsto R_\mathcal{H}(x) := \langle\cdot,x\rangle_\mathcal{H}$, provides a bijection from $\mathcal{G}(\mathcal{H},\mu)$ onto ${}_c\mathbf{Sem}(\mathbb{Hilb})((\mathcal{H},\mu),\mathbb{C})\setminus\{0\}$.*

Lemma 19 also provides a nice characterization of maximal modular closed ideals. Given $x\in\mathcal{G}(\mathcal{H},\mu)$, then by the above $\ker R_\mathcal{H}(x) = \{x\}^\perp = (\mathbb{C}x)^\perp$ is a maximal modular closed ideal of $(\mathcal{H},\mu)$, and every maximal modular closed ideal is of this form for some unique group-like element x. (A modular unit for $(\mathbb{C}x)^\perp$ is given by $\frac{x}{||x||^2}$ since $R_\mathcal{H}(x)\left(\frac{x}{||x||^2}\right) = 1$.)

Dedekind's lemma (see [8, Lemma 7.5.1 and its proof, pp. 206–207]) may be extended without difficulty.

Lemma 20. (Dedekind's lemma) *Let $E\subseteq{}_c\mathbf{Sem}(\mathbb{Hilb})((\mathcal{H},\mu),\mathbb{C})\setminus\{0\}$. Then E is linearly independent.*

Corollary 21. *The set of group-like elements of $(\mathcal{H},\mu)$ is linearly independent.*

Lemma 19 implies that the radical J of $(\mathcal{H},\mu)$ is equal to $\cap_{x\in\mathcal{G}(\mathcal{H},\mu)}(\mathbb{C}x)^\perp = (\sum_{x\in\mathcal{G}(\mathcal{H},\mu)}\mathbb{C}x)^\perp$ (by Lemma 10) $= (\bigoplus_{x\in\mathcal{G}(\mathcal{H},\mu)}\mathbb{C}x)^\perp$ (Corollary 21). Because the radical of a Banach algebra is a closed ideal, an application of Lemma 15 shows that $J^\perp = \mathsf{cl}(\bigoplus_{x\in\mathcal{G}(\mathcal{H},\mu)}\mathbb{C}x)$ is a closed subcoalgebra of $(\mathcal{H},\mu^\dagger)$. This discussion makes it possible to state the following structure theorem for commutative Hilbertian algebras.

Theorem 22. *The carrier Hilbert space of a commutative complex Hilbertian algebra $(\mathcal{H},\mu)$ splits as $\mathcal{H} = \mathsf{cl}(\bigoplus_{x\in\mathcal{G}(\mathcal{H},\mu)}\mathbb{C}x)\oplus_2 J$, and the first member of the Hilbert direct sum is a closed subcoalgebra.*

Corollary 23. *$(\mathcal{H},\mu)$ is semisimple (i.e., $J=(0)$) if, and only if, the set of all finite linear combinations of group-like elements is dense in $\mathcal{H}$. $(\mathcal{H},\mu)$ is radical (i.e., $J=\mathcal{H}$) if, and only if, $\mathcal{G}(\mathcal{H},\mu)=\emptyset$.*

Example 2. The Hilbertian algebra $(\ell^2(X),\mu_X)$ from Example 1, is semisimple since $\mathcal{G}(\ell^2(X),\mu_X) = \{\delta_x : x\in X\}$.

One may even describe the quotient multiplication on $J^\perp$ using Lemma 16. Since $J^\perp$ is a closed subcoalgebra, its coproduct is given by $(\pi_{J^\perp}\hat{\otimes}_2\pi_{J^\perp})\circ\mu^\dagger\circ i_{J^\perp}$ (the co-restriction of $\mu^\dagger$) and thus the quotient multiplication is the adjoint $\mu_{J^\perp} = \pi_{J^\perp}\circ\mu\circ(i_{J^\perp}\hat{\otimes}_2 i_{J^\perp})$.

Remark 7. Of course, $J(J(\mathcal{H},\mu)) = J(\mathcal{H},\mu)$ and $J^\perp$ is semisimple, see [23, Theorem 4.3.2, p. 474].

Definition 11. Let ${}_{\mathsf{radical},c}\mathbf{Sem}(\mathbb{Hilb})$ (resp. ${}_{\mathsf{semisimple},c}\mathbf{Sem}(\mathbb{Hilb})$) be the full subcategory of ${}_c\mathbf{Sem}(\mathbb{Hilb})$ spanned by radical (resp. semisimple) (complex) commutative Hilbertian algebras.

For complex commutative Hilbertian algebras, the radical and its orthogonal complement have nice category-theoretic features.

Theorem 24. *The canonical embedding functors ${}_{\mathsf{radical},c}\mathbf{Sem}(\mathbb{Hilb})\hookrightarrow{}_c\mathbf{Sem}(\mathbb{Hilb})$ and ${}_{\mathsf{semisimple},c}\mathbf{Sem}(\mathbb{Hilb})\hookrightarrow{}_c\mathbf{Sem}(\mathbb{Hilb})$ have a right resp. left adjoint given by $J(-):(\mathcal{H},\mu)\mapsto J$ and $J(-)^\perp:(\mathcal{H},\mu)\mapsto J^\perp$.*

Proof. Let us first check that $(\mathcal{H},\mu)\mapsto J$ is the object map of a functor. Let $f:(\mathcal{H},\mu)\to(\mathcal{K},\gamma)$ be a morphism of Hilbertian algebras, so that $f^\dagger:(\mathcal{K},\gamma^\dagger)\to(\mathcal{H},\mu^\dagger)$ is a morphism of Hilbertian coalgebras, and according to Lemma 25 (below) $f^\dagger(\mathcal{G}(\mathcal{K},\gamma))\subseteq\mathcal{G}(\mathcal{H},\mu)\cup\{0\}$. By continuity it follows that $f^\dagger(J(\mathcal{K},\gamma)^\perp)\subseteq J(\mathcal{H},\mu)^\perp$. According to Lemma 1, $f(J(\mathcal{H},\mu))\subseteq J(\mathcal{K},\gamma)$. The co-restriction $J(\mathcal{H},\mu)\xrightarrow{J(f)}J(\mathcal{K},\gamma)$ of f provides, with $(\mathcal{H},\mu)\mapsto J$, the expected functor from ${}_c\mathbf{Sem}(\mathbb{Hilb})$ to ${}_{\text{radical},c}\mathbf{Sem}(\mathbb{Hilb})$.

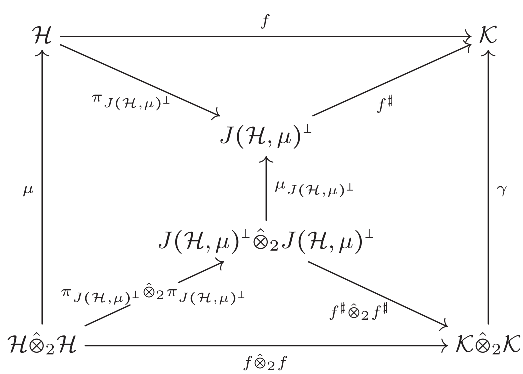
Let also $J(\mathcal{K},\gamma)^\perp\xrightarrow{g}J(\mathcal{H},\mu)^\perp$ be the co-restriction of $f^\dagger$ obtained from the above discussion. Passing to the adjoint one gets the commutative diagram below.

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\quad f\quad} & \mathcal{K} \\ {\scriptstyle \pi_{J(\mathcal{H},\mu)^\perp}}\downarrow & & \downarrow{\scriptstyle \pi_{J(\mathcal{K},\gamma)^\perp}} \\ J(\mathcal{H},\mu)^\perp & \xrightarrow[g^\dagger]{} & J(\mathcal{K},\gamma)^\perp \end{array} \tag{10}$$

That $g^\dagger=\pi_{J(\mathcal{K},\gamma)^\perp}\circ f\circ i_{J(\mathcal{H},\mu)^\perp}$ is a morphism of Hilbertian algebras is clear from the definition of the quotient multiplications of the orthogonal complement of the radicals. The correspondence $f\mapsto g^\dagger$ provides with $(\mathcal{H},\mu)\mapsto J^\perp$ a functor $J(-)^\perp$ from ${}_c\mathbf{Sem}(\mathbb{Hilb})$ to ${}_{\text{semisimple},c}\mathbf{Sem}(\mathbb{Hilb})$. (Functoriality is not immediate at first sight as for $(\mathcal{H},\mu)\xrightarrow{f}(\mathcal{K},\gamma)$ and $(\mathcal{K},\gamma)\xrightarrow{g}(\mathcal{L},\rho)$ one has $J(g\circ f)^\perp=\pi_{J(\ell,\rho)^\perp}\circ g\circ f\circ i_{J(\mathcal{H},\mu)^\perp}$ while $J(g)^\perp\circ J(f)^\perp=\pi_{J(\ell,\rho)^\perp}\circ g\circ i_{J(\mathcal{K},\gamma)^\perp}\circ\pi_{J(\mathcal{K},\gamma)^\perp}\circ f\circ i_{J(\mathcal{H},\mu)^\perp}$ and in general $i_{J(\mathcal{K},\gamma)^\perp}\circ\pi_{J(\mathcal{K},\gamma)^\perp}$ is a nonidentity orthogonal projection. Nevertheless from the first equation, by definition of $J(g)^\perp$, and then by that of $J(f)^\perp$ one obtains $J(g\circ f)^\perp=J(g)^\perp\circ\pi_{J(\mathcal{K},\gamma)^\perp}\circ f\circ i_{J(\mathcal{H},\mu)^\perp}=J(g)^\perp\circ J(f)^\perp\circ\pi_{J(\mathcal{H},\mu)^\perp}\circ i_{J(\mathcal{H},\mu)^\perp}=J(g)^\perp\circ J(f)^\perp$.)

Now let us assume first that $(\mathcal{H},\mu)$ is radical (and $(\mathcal{K},\gamma)$ is arbitrary). Since $f(\mathcal{H})=f(J(\mathcal{H},\mu))\subseteq J(\mathcal{K},\gamma)$, there is a unique bounded linear map $(\mathcal{H},\mu)\xrightarrow{\tilde{f}}J(\mathcal{K},\gamma)$ such that $i_{J(\mathcal{K},\gamma)}\circ\tilde{f}=f$, and which is of course a morphism of Hilbertian algebras. Therefore, $J(-)$ is a right adjoint of the embedding functor ${}_{\text{radical},c}\mathbf{Sem}(\mathbb{Hilb})\hookrightarrow{}_c\mathbf{Sem}(\mathbb{Hilb})$.

Secondly, let us assume that $(\mathcal{K},\gamma)$ is semisimple (and $(\mathcal{H},\mu)$ is arbitrary). Whence $f(J(\mathcal{H},\mu))\subseteq J(\mathcal{K},\gamma)=(0)$ so that $\ker\pi_{J(\mathcal{H},\mu)^\perp}\subseteq\ker(f)$. Therefore, there is a unique bounded linear map $J(\mathcal{H},\mu)^\perp\xrightarrow{f^\sharp}(\mathcal{K},\gamma)$ such that $f^\sharp\circ\pi_{J(\mathcal{H},\mu)^\perp}=f$. $f^\sharp$ is a morphism of Hilbertian algebras as is shown by the following commutative diagram (of course $\pi_{J(\mathcal{H},\mu)^\perp}\hat{\otimes}_2\pi_{J(\mathcal{H},\mu)^\perp}$ is an epimorphism). Consequently $J(-)^\perp$ is a left adjoint of the embedding functor ${}_{\text{semisimple},c}\mathbf{Sem}(\mathbb{Hilb})\hookrightarrow{}_c\mathbf{Sem}(\mathbb{Hilb})$.



(11)

□

Lemma 25. *Let $f : (\mathcal{H}, \mu^\dagger) \to (\mathcal{K}, \gamma^\dagger)$ be a morphism of Hilbertian coalgebras. Then, $f(\mathcal{G}(\mathcal{H}, \mu)) \subseteq \mathcal{G}(\mathcal{K}, \gamma) \cup \{0\}$.*

Proof. Let $x \in \mathcal{G}(\mathcal{H}, \mu)$. Then,

$$\begin{array}{rcl} \gamma^\dagger(f(x)) & = & (f \hat{\otimes}_2 f)(\mu^\dagger(x)) \\ & = & (f \hat{\otimes}_2 f)(x \otimes x) \\ & = & f(x) \otimes f(x). \end{array} \tag{12}$$

□

4.2. Interpretation in terms of the Gelfand transform

Since the group-like elements of a complex commutative Hilbertian algebra correspond to the nontrivial characters of its underlying Banach algebra (Lemma 19), it seems instructive to treat some notions from the previous section in a way similar to the Gelfand theory of Banach algebras. See e.g. [18, Chap. 2] for basic facts and results about the Gelfand theory.

One assumes that $(\mathcal{H}, \mu)$ is a commutative complex Hilbertian algebra with $||\mu||_{op} = ||\mu^\dagger||_{op} \leq 1$, i.e., $(\mathcal{H}, \mu)$ is a semigroup object in $(\mathbf{Hilb}_1, \hat{\otimes}_2, \mathbb{C})$.

The set $\mathcal{G}(\mathcal{H}, \mu) \subseteq \mathcal{H}$ inherits the weak topology from $\mathcal{H}$, i.e., the weakest topology that makes continuous every functional $R_{\mathcal{H}}(x), x \in \mathcal{H}$. A neighborhood basis at $x \in \mathcal{G}(\mathcal{H}, \mu)$ is given by the sets

$$V_{x, u_1, \ldots, u_n, \epsilon} := \{z \in \mathcal{G}(\mathcal{H}, \mu) : |\langle u_i, z\rangle - \langle u_i, x\rangle| < \epsilon, 1 \leq i \leq n\}, \tag{13}$$

$u_1, \ldots, u_n \in \mathcal{H}, \epsilon > 0$.

Under this topology $\mathcal{G}(\mathcal{H}, \mu)$ becomes a Hausdorff space (as the $R_{\mathcal{H}}(x)$'s separate the points of $\mathcal{G}(\mathcal{H}, \mu)$). It is not difficult to check that $\mathcal{G}(\mathcal{H}, \mu) \cup \{0\}$ is closed in the closed unit ball $B_{\mathcal{H}}(0, 1]$, which is compact under the weak topology, according to the Banach-Alaoglu theorem (since $||\mu^\dagger||_{op} \leq 1$, for each $x \in \mathcal{G}(\mathcal{H}, \mu), ||x||^2 = ||x \otimes x|| \leq ||x||$, so that $\mathcal{G}(\mathcal{H}, \mu) \subseteq B_{\mathcal{H}}(0, 1]$). Therefore, $\mathcal{G}(\mathcal{H}, \mu)$ is a locally compact space.

For each $z \in \mathcal{H}$, let us define $\hat{z} : \mathcal{G}(\mathcal{H}, \mu) \to \mathbb{C}$ by $\hat{z}(x) = \langle z, x\rangle = R_{\bar{\mathcal{H}}}(z)(x)$, i.e., $\hat{z}$ is the restriction of $R_{\bar{\mathcal{H}}}(z)$ to $\mathcal{G}(\mathcal{H}, \mu)$. This provides a continuous map from $\mathcal{G}(\mathcal{H}, \mu)$ to $\mathbb{C}$ which vanishes at infinity. (Indeed, for $z \in \mathcal{H}$ and $\epsilon > 0$, the set of all group-like elements x such that $|\hat{z}(x)| = |\langle z, x\rangle_{\mathcal{H}}| \geq \epsilon$ is closed in the weak topology, whence is compact, and thus $\hat{z}$ vanishes at infinity.) Therefore $\hat{z} \in C_0(\mathcal{G}(\mathcal{H}, \mu)), z \in \mathcal{H}$, where as usual $C_0(X)$ stands for the Banach space of complex-valued continuous functions on a locally compact space, vanishing at infinity, with the uniform norm.

Moreover the following co-restriction of the Riesz representation $R_{\bar{\mathcal{H}}} : \mathcal{H} \simeq (\bar{\mathcal{H}})^*$ namely the map $\mathcal{R}_{(\mathcal{H}, \mu)} : \mathcal{H} \to C_0(\mathcal{G}(\mathcal{H}, \mu)), z \mapsto \hat{z}$, is a linear contraction (by the Cauchy-Bunyakovski-Schwarz inequality one has $||\mathcal{R}_{(\mathcal{H}, \mu)}(z)||_\infty \leq \sup_{x \in \mathcal{G}(\mathcal{H}, \mu)} ||x|| ||z|| \leq ||z||$). Besides $\mathcal{R}_{(\mathcal{H}, \mu)}$ is an algebra map from $(\mathcal{H}, \mu_{bil})$ to $C_0(\mathcal{G}(\mathcal{H}, \mu))$ since

$$\begin{array}{rcl} \mathcal{R}_{(\mathcal{H}, \mu)}(\mu(y \otimes z))(x) & = & \langle x, \mu(y \otimes z)\rangle_{\mathcal{H}} \\ & = & \langle \mu^\dagger(x), y \otimes z\rangle_{\mathcal{H} \hat{\otimes}_2 \mathcal{H}} \\ & = & \langle x \otimes x, y \otimes z\rangle_{\mathcal{H} \hat{\otimes}_2 \mathcal{H}} \\ & = & \mathcal{R}_{(\mathcal{H}, \mu)}(y)(x) \mathcal{R}_{(\mathcal{H}, \mu)}(z)(x). \end{array} \tag{14}$$

Remark 8. Up to the Riesz isomorphism $R_{\mathcal{H}} : \bar{\mathcal{H}} \simeq \mathcal{H}^*, \mathcal{R}_{(\mathcal{H}, \mu)}$ thus corresponds to the usual Gelfand transform $G_{(\mathcal{H}, \mu_{bil})}$ of the underlying Banach algebra of $(\mathcal{H}, \mu)$; more precisely $G_{(\mathcal{H}, \mu_{bil})}(z)(R_{\mathcal{H}}(x)) = \mathcal{R}_{(\mathcal{H}, \mu)}(z)(x), x \in \mathcal{G}(\mathcal{H}, \mu)$ and $z \in \mathcal{H}$.

Finally, $\ker \mathcal{R}_{(\mathcal{H},\mu)} = \{z \in \mathcal{H} : \langle z, x\rangle_{\mathcal{H}} = 0, x \in \mathcal{G}(\mathcal{H},\mu)\} = \mathcal{G}(\mathcal{H},\mu)^{\perp} = J$ (as above J denotes the radical), and $\mathcal{H}/J \simeq J^{\perp} = \overline{(\bigoplus_{x\in\mathcal{G}(\mathcal{H},\mu)} \mathbb{C}x)}$ is (algebraically) isomorphic to the subalgebra $ran(\mathcal{R}_{(\mathcal{H},\mu)})$ of $C_0(\mathcal{G}(\mathcal{H},\mu))$. Using the notation from the end of Section 4.1 this means that $\langle \mu_{J^{\perp}}(x\otimes y), z\rangle = \langle x,z\rangle\langle y,z\rangle = \langle x\otimes y, z\otimes z\rangle = \langle x\otimes y, \mu^{\dagger}(z)\rangle, x,y \in J^{\perp}, z \in \mathcal{G}(\mathcal{H},\mu)$.

5. Hilbertian function algebras

In this section, in accordance with Definition 1, by *Hilbertian algebra* is meant a **complex commutative Hilbertian algebra with** $||\mu||_{op} \leq 1$ (the morphisms however are not assumed contractive).

Let us briefly disclose the content of this section. As any complex commutative (nonunital) C^*-algebra is isomorphic to the model C^*-algebra $C_0(X,\tau)$ with (X,τ) its structure space, under the Gelfand transform, one may wonder at which conditions is a Hilbertian algebra isomorphic to the model $(\ell^2(X), \mu_X)$ (Example 1) with X its set of group-like elements, under the "Riesz transform" $\mathcal{R}$. Observe immediately that this underlies the fact that $\mathcal{R}$ factors through the inclusion $\ell^2(X) \hookrightarrow C_0(X)$. For this to hold it is *sufficient*[7] to consider Hilbertian algebras with a coisometric multiplication (or an isometric comultiplication; see Definition 12) in such a way the group-like elements form an orthonormal family (see below Lemma 27). Next, with the assumption of an isometric comultiplication, the condition found to the above question is that the Hilbertian algebra has adjoints in the sense of Ambrose's H^*-algebras (Theorem 32). Under the assumption of an isometric comultiplication this is a characterization of semisimplicity (Corollary 33).

Afterward this characterization is used to provide a dual equivalence between Hilbertian algebras with H^*-adjoints and isometric coproduct and a category of pointed sets (see Theorem 40) which is similar to the folklore duality between commutative C^*-algebras and pointed compact spaces.

5.1. Special Hilbertian algebras

Let us begin with some preliminary observations which will be found useful hereafter. Let (X, d) be a discrete space. Then, $\ell^2(X) \subseteq C_0(X,\mathsf{d})$ and the inclusion is continuous, i.e., $||f||_{\infty} \leq ||f||_2$, $f \in \ell^2(X)$.

Now let us consider again the question of semisimplicity but for some particular algebras. Let $(\mathcal{H},\mu)$ be a Hilbertian algebra in $\mathbb{Hilb}_1$ (i.e., $||\mu||_{op} \leq 1$), and let us assume for a while that its set $\mathcal{G}(\mathcal{H},\mu)$ of group-like elements forms an orthonormal set. So according to Theorem 22, it is an orthonormal basis of $J^{\perp}$ and thus $J^{\perp} = \bigoplus_{2_{x\in\mathcal{G}(\mathcal{H},\mu)}} \mathbb{C}x \simeq \ell^2(\mathcal{G}(\mathcal{H},\mu))$ under the unitary transformation $\delta_{\mathcal{G}(\mathcal{H},\mu)} : x \mapsto \delta_x$ (see Example 1). In this situation $\mathcal{G}(\mathcal{H},\mu)$ is discrete under the weak topology because the open neighborhood $V_{x,x,1}$ (see Eq. (12)) of x reduces to $\{x\}$ for each group-like element x.

One furthermore notices that $\mathcal{R}_{(\mathcal{H},\mu)}(u) \in \ell^2(\mathcal{G}(\mathcal{H},\mu))$ for each $u \in \mathcal{H}$ since one has $\mathcal{R}_{(\mathcal{H},\mu)}(u)(x) = \langle u, x\rangle, x \in \mathcal{G}(\mathcal{H},\mu)$, and thus $(\langle u,x\rangle)_{x\in\mathcal{G}(\mathcal{H},\mu)}$ is square-summable (since $\mathcal{G}(\mathcal{H},\mu)$ is an orthonormal set), so $\mathcal{R}_{(\mathcal{H},\mu)}(u) = \sum_{x\in\mathcal{G}(\mathcal{H},\mu)} \langle u,x\rangle\delta_x$. This means that $\mathcal{R}_{(\mathcal{H},\mu)}$ uniquely factors as indicated by the following diagram, where d denotes the discrete topology on $\mathcal{G}(\mathcal{H},\mu)$.

[7]Whether or not this condition is also necessary is not studied in this paper.

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\mathcal{R}_{(\mathcal{H},\mu)}} & C_0(\mathcal{G}(\mathcal{H},\mu),\mathsf{d}) \\ {\scriptstyle \pi_{J^\perp}}\downarrow & \searrow & \hookuparrow \\ J^\perp & \underset{\delta_{\mathcal{G}(\mathcal{H},\mu)}}{\simeq} & \ell^2(\mathcal{G}(\mathcal{H},\mu)) \end{array} \tag{15}$$

The co-restriction thus obtained is still denoted by $\mathcal{R}_{(\mathcal{H},\mu)}$ and is called the *Gelfand-Riesz transform* of $(\mathcal{H},\mu)$.

It is a coisometry: $\mathcal{R}^\dagger_{(\mathcal{H},\mu)}(f) = \sum_{x\in\mathcal{G}(\mathcal{H},\mu)} f(x)x$, and thus of course $\mathcal{R}_{(\mathcal{H},\mu)} \circ \mathcal{R}^\dagger_{(\mathcal{H},\mu)} = id_{\ell^2(\mathcal{G}(\mathcal{H},\mu))}$. In particular, $||\mathcal{R}_{(\mathcal{H},\mu)}||_{op} = 1$ (if $J^\perp \neq (0)$) or 0 (if $J^\perp = (0)$). As a matter of fact $||\mathcal{R}_{(\mathcal{H},\mu)}||_{op} \leq 1$, and $ran(\mathcal{R}_{(\mathcal{H},\mu)}) = \ell^2(\mathcal{G}(\mathcal{H},\mu))$.

One knows from Example 1 that $(\ell^2(\mathcal{G}(\mathcal{H},\mu)), \mu_{\mathcal{G}(\mathcal{H},\mu)})$ is a Hilbertian algebra and it is rather apparent that $\mathcal{R}_{(\mathcal{H},\mu)}$ is an epimorphism of Hilbertian algebras from $(\mathcal{H},\mu)$ to $(\ell^2(\mathcal{G}(\mathcal{H},\mu)), \mu_{\mathcal{G}(\mathcal{H},\mu)})$.

By the above discussion the following result is clear ($\mathcal{R}_{(\mathcal{H},\mu)}$ is an isomorphism if, and only if, it is one-to-one if, and only if, it is a unitary transformation).

Proposition 26. *Let $(\mathcal{H},\mu)$ be a Hilbertian algebra in $\mathbb{H}\mathrm{ilb}_1$, and let us assume that $\mathcal{G}(\mathcal{H},\mu)$ is an orthonormal set. $(\mathcal{H},\mu)$ is semisimple if, and only if, $(\mathcal{H},\mu) \simeq (\ell^2(\mathcal{G}(\mathcal{H},\mu)), \mu_{\mathcal{G}(\mathcal{H},\mu)})$ under the Gelfand-Riesz transform. In particular, when $(\mathcal{H},\mu)$ is semisimple, $\mu\circ\mu^\dagger = id$.*

Definition 12. In accordance with the notion of speciality in the context of Frobenius algebras, one says that $(\mathcal{H},\mu)$ is a *special* Hilbertian algebra when the coproduct $\mu^\dagger$ is an isometry or equivalently when the product μ is a coisometry, that is, when $\mu\circ\mu^\dagger = id_{\mathcal{H}}$. Let ${}_c^\dagger\mathbf{Sem}(\mathbb{H}\mathrm{ilb})$ be the full subcategory of ${}_c\mathbf{Sem}(\mathbb{H}\mathrm{ilb})$ spanned by these algebras[8].

Lemma 27. (Compare to [2, Prop. 14 and 15, pp. 11–12].) *If $(\mathcal{H},\mu)$ is a special Hilbertian algebra, then $||\mu||_{op} \leq 1$ and $\mathcal{G}(\mathcal{H},\mu)$ is an orthonormal set. The converse assertion holds when $(\mathcal{H},\mu)$ is semisimple.*

Proof. Let $(\mathcal{H},\mu)$ be a special Hilbertian algebra. That $||\mu||_{op} \leq 1$ is obvious. Let x be a group-like element of $(\mathcal{H},\mu)$ (in particular $x\neq 0$). Since $\mu^\dagger$ is an isometry, $||x|| = ||\mu^\dagger(x)|| = \langle \mu^\dagger(x), \mu^\dagger(x)\rangle^{\frac{1}{2}} = \langle x\otimes x, x\otimes x\rangle^{\frac{1}{2}} = (\langle x,x\rangle\langle x,x\rangle)^{\frac{1}{2}} = ||x||^2$, so that $||x|| = 1$. Let y be also a group-like element of $(\mathcal{H},\mu)$ such that $x\neq y$. Then, $\langle x,y\rangle = \langle \mu^\dagger(x), \mu^\dagger(y)\rangle = \langle x\otimes x, y\otimes y\rangle = \langle x,y\rangle^2$. Whence $\langle x,y\rangle \in \{0,1\}$. In particular, $\langle x,y\rangle = \langle y,x\rangle$. Moreover, $||x-y||^2 = \langle x-y, x-y\rangle = ||x||^2 + ||y||^2 - 2\langle x,y\rangle = 2 - 2\langle x,y\rangle$. If $\langle x,y\rangle = 1$, then $x=y$, which is a contradiction, whence $\langle x,y\rangle = 0$.

Let us assume that $(\mathcal{H},\mu)$ is a semisimple Hilbertian algebra, and that $\mathcal{G}(\mathcal{H},\mu)$ is an orthonormal family. According to Theorem 22, $\mathcal{G}(\mathcal{H},\mu)$ is a Hilbertian basis of $\mathcal{H}$. Let x, y, z be group-like elements. Then, $\langle \mu(x\otimes y), z\rangle = \langle x\otimes y, z\otimes z\rangle = \langle x,z\rangle\langle y,z\rangle = \delta_{x,z}\delta_{y,z}$. Therefore $\mu(\mu^\dagger(x)) = \mu(x\otimes x) = x$ for each group-like element x, from which it follows that $\mu\circ\mu^\dagger = id$, that is, $(\mathcal{H},\mu)$ is special. □

[8]Since special Hilbertian algebras are in particular objects of ${}_c\mathbf{Sem}(\mathbb{H}\mathrm{ilb}_1)$ at first glance it seems more natural to define their category as the full subcategory ${}_c^\dagger\mathbf{Sem}(\mathbb{H}\mathrm{ilb}_1)$ of ${}_c\mathbf{Sem}(\mathbb{H}\mathrm{ilb}_1)$ they span. But doing so one forces the isomorphisms to be unitary transformations and more generally the morphisms to be contractive which can be too strong in some cases (see e.g., Remark 15 below).

Remark 9. In view of Theorem 22, the first assertion of Lemma 27 implies that $\mathcal{G}(\mathcal{H},\mu)$ is a Hilbertian basis of $\mathcal{H}$ for a semisimple special Hilbertian algebra $(\mathcal{H},\mu)$. Besides the proof of this first assertion does not make use of commutativity nor of associativity of μ.

Corollary 28. *Let $(\mathcal{H},\mu)$ be a special Hilbertian algebra. Then, for each $x,y\in\mathcal{G}(\mathcal{H},\mu), \mu(x\otimes y)\equiv \delta_{x,y}x$ modulo J. In particular, the quotient multiplication on $J^{\perp}$ is defined by $\mu_{J^{\perp}}(x\otimes y)=\delta_{x,y}x, x,\ y\in\mathcal{G}(\mathcal{H},\mu)$.*

Proof. Let x, y, z be group-like elements of $(\mathcal{H},\mu)$. Then,

$$\begin{array}{rcl}\langle\mu(x\otimes y),z\rangle & = & \langle x\otimes y,\mu^{\dagger}(z)\rangle\\ & = & \langle x,z\rangle\langle y,z\rangle\\ & = & \delta_{x,z}\delta_{y,z}.\\ & & \text{(according to Lemma 27)}\end{array}\tag{16}$$

□

Remark 10. Let $(\mathcal{H},\mu)$ be a special Hilbertian algebra. Then for each $x\in\mathcal{G}(\mathcal{H},\mu), \mathbb{C}x$ is both a closed subcoalgebra (since $(\mathbb{C}x)^{\perp}$ is a closed ideal) and a closed subcoalgebra (since $\mu(x\otimes x)=\mu(\mu^{\dagger}(x))=x$) unitarily isomorphic to $\mathbb{C}$. By Lemma 13, $(\mathbb{C}x,\mu_{|_{\mathbb{C}x}})$ is a special Hilbertian algebra.

5.2. Special Hilbertian algebras with H*-adjoints

Definition 13. Let $(\mathcal{H},\mu)$ be a Hilbertian algebra. It is said to have H*-adjoints when there exists a set-theoretic map $(-)^{\sharp}:\mathcal{H}\to\mathcal{H}$, called a map of H*-adjoints, such that $\langle\mu(x\otimes y),z\rangle=\langle x,\mu(y^{\sharp}\otimes z)\rangle$ for all $x,y,z\in\mathcal{H}$.

Remark 11.

1. Let us assume that $(\mathcal{H},\mu)$ is a Hilbertian algebra with the property that each $y\in\mathcal{H}$ has a *H^*-adjoint*, that is, there exists $w\in\mathcal{H}$ such that $\langle\mu(x\otimes y),z\rangle=\langle x,\mu(w\otimes z)\rangle$ for all $x,z\in\mathcal{H}$. Then, $(\mathcal{H},\mu)$ has H^*-adjoints. Indeed a map of H^*-adjoints is just a section of the projection onto the first factor $\cup_{y\in\mathcal{H}}\{y\}\times A(y)\to\mathcal{H}$, where for each y, $A(y)$ is the set of all H^*-adjoints of y (the existence of such a section is guaranteed by the axiom of choice). Conversely, the existence of a map of H^*-adjoints implies that each member of the algebra has a H^*-adjoint. Therefore, the two notions are identical.
2. The above definition of a (commutative) Hilbertian algebra with H^*-adjoints is slightly different from Ambrose's definition [5] of H^*-algebras (see Section 2.4) since the former are Hilbertian algebras while the latter are Banach algebras. But in both cases, being an algebra of one or the other kind is *to have a defining property*, that is, the requirement of the existence of H^*-adjoints or a map of H^*-adjoints. Moreover, the underlying Banach algebra of a Hilbertian algebra with H^*-adjoints is an Ambrose's H^*-algebra.

Remark 12. The annihilator $\mathcal{A}nn(E,m)$ of a (commutative) Banach algebra $(E,\ m)$ is the ideal $\{x\in E: m(x,y)=0\forall y\in E\}$. Let $(\mathcal{H},\mu)$ be a Hilbertian algebra with map of H^*-adjoints $(-)^{\sharp}$. Then, the following equations hold modulo $\mathcal{A}nn(\mathcal{H},\mu_{bil})$. For all $x,y\in\mathcal{H}$,

$$(x+y)^{\sharp}\equiv x^{\sharp}+y^{\sharp},(\alpha x)^{\sharp}\equiv\bar{\alpha}x^{\sharp},\mu_{bil}(x,y)^{\sharp}\equiv\mu_{bil}(x^{\sharp},y^{\sharp}),(x^{\sharp})^{\sharp}\equiv x,\tag{17}$$

which equivalently reads as, for all w, $\mu_{bil}(u,w)=\mu_{bil}(v,w)$, where $u\equiv v$ is any of the above relations.

Example 3. $(\ell^2(X), \mu_X)$ with map of H^*-adjoints $f \mapsto f^* := \sum_{x\in X} \overline{f(x)}\delta_x$ corresponds to $\ell_1^2(X)$ from Section 2.4, with $1(x) = 1, x \in X$.

The operation of orthogonal complementation inherits from the defining property of H^*-adjoints a pleasant symmetry between closed ideals.

Lemma 29. *Let $(\mathcal{H}, \mu)$ be a Hilbertian algebra with H^*-adjoints. If I is a closed ideal of $(\mathcal{H}, \mu)$, then $I^\perp$ is a closed ideal too. Moreover, I is maximal if, and only if, $I^\perp$ is minimal.*

Proof. Let $(-)^\sharp$ be a map of H^*-adjoints for $(\mathcal{H}, \mu)$. Let I be a closed ideal. Let $u \in I, v \in \mathcal{H}$ and $w \in I^\perp$. Then, $\langle u, \mu_{bil}(v, w)\rangle = \langle u, \mu_{bil}((v^\sharp)^\sharp, w)\rangle = \langle \mu_{bil}(v^\sharp, u), w\rangle = 0$. Therefore, $\mu_{bil}(v, w) \in I^\perp$. Now, let us assume that I is maximal. Let I_1 be a closed ideal such that $I_1 \subseteq I^\perp$. Then, $I_1^\perp \supseteq I$ so that, by maximality, $I_1^\perp = \mathcal{H}$ or $I_1^\perp = I$, which is equivalent to $I_1 = (0)$ or $I_1 = I^\perp$. Whence $I^\perp$ is minimal. □

Lemma 30. ([24, Theorem 11.6.12, p. 1210]) *For a Hilbertian algebra $(\mathcal{H}, \mu)$ with H^*-adjoints, $\mathcal{A}nn(\mathcal{H}, \mu_{bil}) = J$.*

Corollary 31. *Let $(\mathcal{H}, \mu)$ be a Hilbertian algebra with H^*-adjoints. If $(\mathcal{H}, \mu)$ is semisimple, then there is a unique map of H^*-adjoints $(-)^\sharp$ for $(\mathcal{H}, \mu)$. Moreover $(-)^\sharp : \overline{(\mathcal{H}, \mu)} = (\bar{\mathcal{H}}, \bar{\mu}) \to (\mathcal{H}, \mu)$ is a morphism of Hilbertian algebras (and in particular is bounded), $(-)^\sharp \circ \overline{(-)^\sharp} = id_{\mathcal{H}}$ and $((-)^\sharp)^\dagger = \overline{(-)^\sharp}$ (so $(-)^\sharp$ is both a conjugate-linear involution and a unitary transformation).*

Proof. Uniqueness of the map of H^*-adjoints follows from [5, Theorem 2.1, p. 370]. That $(-)^\sharp : (\bar{\mathcal{H}}, \bar{\mu}_{bil}) \to (\mathcal{H}, \mu_{bil})$ is a homomorphism of $\mathbb{C}$-algebras and $(-)^\sharp \circ \overline{(-)^\sharp} = id_{\mathcal{H}}$ is clear from Remark 12 since by Lemma 30 the annihilator corresponds to the Jacobson radical. By [18, Corollary 2.1.10, p. 50], $(-)^\sharp$ is bounded. It remains to prove that $((-)^\sharp)^\dagger = \overline{(-)^\sharp}$. First of all the property $\langle uv, w\rangle = \langle v, u^\sharp w\rangle$ implies that $\mathcal{A}nn(\mathcal{H}, \mu_{bil}) = (\mathcal{H}^2)^\perp$ where $\mathcal{H}^2 := \{\mu_{bil}(u, v) : u, v \in \mathcal{H}\}$. By semisimplicity it follows that $\langle \mathcal{H}^2\rangle := \{\sum_{i=1}^n \mu_{bil}(u_i, v_i) : n \in \mathbb{N}, u_i, v_i \in \mathcal{H}\}$, being the subspace generated by $\mathcal{H}^2$, is dense in $\mathcal{H}$. Now, let $u, v, w \in \mathcal{H}$. Then, $\langle \mu_{bil}(v, w), u^\sharp\rangle = \langle w, \mu_{bil}(u^\sharp, v^\sharp)\rangle = \langle \mu_{bil}(u, w), v^\sharp\rangle = \langle u, (\mu_{bil}(v, w))^\sharp\rangle$. Whence $((-)^\sharp)^\dagger = \overline{(-)^\sharp}$ on $\langle \mathcal{H}^2\rangle$ so that by density and continuity, they are equal on $\mathcal{H}$ as desired. □

Remark 13. Let $(\mathcal{H}, \mu)$ be a semisimple Hilbertian algebra with H^*-adjoints. The uniqueness of the map of H^*-adjoints provided by Corollary 31 reads in the notation from Remark 11.1 as the fact that $A(y)$ is reduced to a one-point set, namely $y^\sharp$, for each $y \in \mathcal{H}$.

Theorem 32. *Let $(\mathcal{H}, \mu)$ be a special Hilbertian algebra with H^*-adjoints. Then, $(\mathcal{H}, \mu)$ is semisimple, that is $(\mathcal{H}, \mu) \simeq (\ell^2(\mathcal{G}(\mathcal{H}, \mu)), \mu_{\mathcal{G}(\mathcal{H}, \mu)})$ under the Gelfand-Riesz transform. Moreover denoting by $(-)^\sharp$ the (unique) map of adjoints for $(\mathcal{H}, \mu)$, one has $\mathcal{R}_{(\mathcal{H}, \mu)}(u^\sharp) = (\mathcal{R}_{(\mathcal{H}, \mu)}(u))^*$, or in other words, $\langle u^\sharp, x\rangle = \overline{\langle u, x\rangle}, u \in \mathcal{H}, x \in \mathcal{G}(\mathcal{H}, \mu)$.*

Proof. According to Lemma 29, since J is an ideal, $J^\perp$ also is an ideal. Whence, $J^{\perp\perp} = J$ is a closed subcoalgebra by Lemma 15. Thus, J is both a closed subalgebra (since it is an ideal) under the co-restriction $\mu_{|_J} : J \hat{\otimes}_2 J \to J$ of μ, and a subcoalgebra under the co-restriction $(\mu^\dagger)_{|_J} : J \to J \hat{\otimes}_2 J$ of $\mu^\dagger$. According to Lemma 13, $\mu_{|_J} \circ (\mu_{|_J})^\dagger = id_J$. But $\mu(u \otimes v) = 0$ for each $u, v \in J$ by

Lemma 30, and thus $\mu(p) = 0$ for each $p \in J \otimes J$. By density of $J \otimes J$ and by continuity of μ, $\mu_{|_J}(f) = \mu(f) = 0$ for each $f \in J \hat{\otimes}_2 J$. Because for each $u \in J, \mu^\dagger(u) = (\mu_{|_J})^\dagger(u) \in J \hat{\otimes}_2 J, u = \mu_{|_J}((\mu_{|_J})^\dagger(u)) = 0$. Whence $J = (0)$, and $(\mathcal{H}, \mu)$ is semisimple.

By Lemma 31, the unique map of H^*-adjoints $(-)^\sharp : \overline{(\mathcal{H}, \mu)} \to (\mathcal{H}, \mu)$ is a morphism of Hilbertian algebras, so $((-)^\sharp)^\dagger = \overline{(-)^\sharp} : (\mathcal{H}, \mu^\dagger) \to (\bar{\mathcal{H}}, (\bar{\mu})^\dagger)$ is a morphism of coalgebras. As a matter of fact $((-)^\sharp)^\dagger(\mathcal{G}(\mathcal{H}, \mu)) \subseteq \mathcal{G}(\bar{\mathcal{H}}, \bar{\mu})$ (since $((-)^\sharp)^\dagger$ is also an isomorphism) by Lemma 25. But $\mathcal{G}(\bar{\mathcal{H}}, \bar{\mu}) = \mathcal{G}(\mathcal{H}, \mu)$. Let $x, y \in \mathcal{G}(\mathcal{H}, \mu)$. Then, since $x^\sharp \in \mathcal{G}(\mathcal{H}, \mu), \delta_{x^\sharp, y} = \langle x^\sharp, y \rangle = \langle x^\sharp, y \rangle \langle y, y \rangle = \langle x^\sharp \otimes y, y \otimes y \rangle = \langle x^\sharp \otimes y, \mu^\dagger(y) \rangle = \langle \mu(x^\sharp \otimes y), y \rangle = \langle y, \mu(x \otimes y) \rangle = \langle y \otimes y, x \otimes y \rangle = \langle y, x \rangle ||y||^2 = \langle y, x \rangle = \delta_{x,y}$ so that $x^\sharp = x$. Now, given $u \in \mathcal{H} = \mathsf{cl}(\bigoplus_2 x \in \mathcal{G}(\mathcal{H}, \mu) \mathbb{C} x)$, since $(-)^\sharp$ is conjugate-linear, $u^\sharp = \sum_{x \in \mathcal{G}(\mathcal{H}, \mu)} \overline{\langle u, x \rangle} x^\sharp = \sum_{x \in \mathcal{G}(\mathcal{H}, \mu)} \overline{\langle u, x \rangle} x$, and thus $\mathcal{R}_{(\mathcal{H}, \mu)}(u^\sharp) = \sum_{x \in \mathcal{G}(\mathcal{H}, \mu)} \overline{\langle u, x \rangle} \delta_x = (\mathcal{R}_{(\mathcal{H}, \mu)}(u))^*$. □

Corollary 33. *Let $(\mathcal{H}, \mu)$ be a special Hilbertian algebra. It is semisimple if, and only if, it has H^*-adjoints.*

Proof. The reverse implication is provided by Theorem 32. Regarding the direct implication, by Proposition 26, a semisimple and special Hilbertian algebra $(\mathcal{H}, \mu)$ is isomorphic to $(\ell^2(\mathcal{G}(\mathcal{H}, \mu), \mu_{\mathcal{G}(\mathcal{H}, \mu)})$ under the Gelfand-Riesz transform and the definition $u^\sharp := \sum_{x \in \mathcal{G}(\mathcal{H}, \mu)} \overline{\langle u, x \rangle} x$ provides a map of H^*-adjoints for $(\mathcal{H}, \mu)$. □

Let ${}_{\text{semisimple},}{}^{\dagger}_{c}\mathbf{Sem}(\mathbb{Hilb})$ be the full subcategory of ${}^{\dagger}_{c}\mathbf{Sem}(\mathbb{Hilb})$ (Definition 12) spanned by the semisimple special Hilbertian algebras (or equivalently by special Hilbertian algebras with H^*-adjoints).

Let ${}_{\text{proper}}H^*\!-_c\mathbf{BanAlg}$ be the full subcategory of $\mathbf{BanAlg}$ spanned by Ambrose's proper commutative H^*-algebras (see Section 2.4).

Let ${}_{\text{bnd, proper}}H^*\!-_c\mathbf{BanAlg}$ (resp. ${}_{\text{unbnd, proper}}H^*\!-_c\mathbf{BanAlg}$) be the full subcategory of ${}_{\text{proper}}H^*\!-_c\mathbf{BanAlg}$ spanned by the *bounded* (resp. *unbounded*) proper commutative H^*-algebras, i.e., those H^*-algebras $(U(\mathcal{H}), m)$ such that $\alpha_{(\mathcal{H}, m)}$ is bounded (resp. unbounded) above (see Section 2.4).

Semisimple special Hilbertian algebras are essentially the same as bounded proper commutative H^*-algebras since by Theorem 9 together with Corollary 33, and using the fact that the underlying Banach algebra functor is injective on objects (assuming that the multiplication is contractive) and fully faithful, one obtains the following.

Corollary 34. *The restriction of the underlying Banach algebra functor provides an equivalence between ${}_{\text{semisimple},}{}^{\dagger}_{c}\mathbf{Sem}(\mathbb{Hilb})$ and ${}_{\text{bnd, proper}}H^*\!-_c\mathbf{BanAlg}$. Moreover, no object of ${}_{\text{semisimple},}{}^{\dagger}_{c}\mathbf{Sem}(\mathbb{Hilb})$ has its underlying Banach algebra isomorphic to some object of ${}_{\text{unbnd, proper}}H^*\!-_c\mathbf{BanAlg}$.*

5.3. Category-theoretic recasting

It is well known and easily checked, that the correspondence $X \mapsto \ell^2(X)$ does not extend to a functor from **Set** to **Hilb**, however it does so when the domain category is $\mathbf{Set}_{\bullet,<+\infty}$ as described below.

Given a set-theoretic map $f : X \to Y$, the *Banach indicatrix* $\mathsf{B}_f : Y \to \mathbb{N} \sqcup \{+\infty\}$ is given by $\mathsf{B}_f(y) := |f^{-1}(\{y\})|$ when the cardinal number into consideration is finite, and $+\infty$ otherwise (cf. [6, p. 226]).

The objects of $\mathbf{Set}_{\bullet,<+\infty}$ are pointed sets $(X, x_0), x_0 \in X$, and a morphism $(X, x_0) \xrightarrow{f} (Y, y_0)$ is a base-point preserving map ($f(x_0) = y_0$) such that for all $y \neq y_0, \mathsf{B}_f(y) < +\infty$ and furthermore $||\mathsf{B}_f||_\infty := \sup_{y \neq y_0} \mathsf{B}_f(y) < +\infty$ (observe that $|f^{-1}(\{y_0\})|$ is allowed to be infinite).

Given a pointed set (X, x_0), let $\ell^2_\bullet(X, x_0) := \{u \in \ell^2(X) : u(x_0) = 0\} = \{\delta_{x_0}\}^\perp$.

Remark 14. $\ell^2(X \setminus \{x_0\}) \simeq \ell^2_\bullet(X, x_0)$ unitarily. In details the space $\ell^2(X \setminus \{x_0\})$ embeds into $\ell^2(X)$ under $\imath_{(X,x_0)}$ given by $\imath_{(X,x_0)}(u)(x) = u(x)$, for $x \neq x_0$, and $\imath_{(X,x_0)}(u)(x_0) = 0$, and $\ell^2(X)$ projects onto $\ell^2(X \setminus \{x_0\})$ under $r_{(X,x_0)}$ given by $r_{(X,x_0)}(u) := u_{|_{X \setminus \{x_0\}}} = u \circ incl$, where $incl : X \setminus \{x_0\} \hookrightarrow X$ is the canonical inclusion. The co-restriction $\imath_{(X,x_0)} : \ell^2(X \setminus \{x_0\}) \to \ell^2_\bullet(X, x_0)$ and the restriction $r_{(X,x_0)} : \ell^2_\bullet(X, x_0) \to \ell^2(X \setminus \{x_0\})$ are unitary transformations, inverse one from the other.

Let $f : (Y, y_0) \to (X, x_0)$ be a morphism in $\mathbf{Set}_{\bullet,<+\infty}$, and let $u \in \ell^2_\bullet(X, x_0)$. Then $u \circ f \in \ell^2_\bullet(Y, y_0)$. Indeed, of course $u(f(y_0)) = u(x_0) = 0$. Now let $A \subseteq Y \setminus \{y_0\}$ be a finite set. Then, $\sum_{y \in A} |u(f(y))|^2 = \sum_{x \in f(A) \setminus \{x_0\}} |f^{-1}(\{x\})| |u(x)|^2$ (since $u(x_0) = 0$ and $f(A)$ is finite) $\leq ||\mathsf{B}_f||_\infty \sum_{x \in f(A) \setminus \{x_0\}} |u(x)|^2 \leq ||\mathsf{B}_f||_\infty ||u||^2$.

The correspondence $u \mapsto u \circ f$ is manifestly linear in u, and since $||u \circ f|| \leq ||\mathsf{B}_f||_\infty^{\frac{1}{2}} ||u||$ a functor $\ell^2_\bullet : \mathbf{Set}^{\mathsf{op}}_{\bullet,<+\infty} \to \mathbf{Hilb}$ is provided by $\ell^2_\bullet(f)(u) := u \circ f$. In particular, for each $x \neq x_0, \ell^2_\bullet(f)(\delta_x) = \sum_{y \in f^{-1}(\{x\})} \delta_y$ and $||\ell^2_\bullet(f)||_{op} \leq ||\mathsf{B}_f||_\infty^{\frac{1}{2}}$.

For each pointed set $(X, x_0), \ell^2_\bullet(X, x_0)$ is both a closed subalgebra of $(\ell^2(X), \mu_X)$ (it is even the maximal ideal $(\mathbb{C}\delta_{x_0})^\perp$) and a closed subcoalgebra of $(\ell^2(X), \mu_X^\dagger)$ (indeed $\mathbb{C}\delta_{x_0}$ is also a closed ideal), and by Lemma 13, it is a special Hilbertian algebra on its own right, denoted by $(\ell^2_\bullet(X, x_0), \mu_{(X,x_0)})$, (unitarily) isomorphic to $(\ell^2(X \setminus \{x_0\}), \mu_{X \setminus \{x_0\}})$ (under the isomorphism from Remark 14), and thus it is semisimple.

Moreover given a morphism $f : (Y, y_0) \to (X, x_0)$ in $\mathbf{Set}_{\bullet,<+\infty}, (uv) \circ f = (u \circ f)(v \circ f)$, where uv is the pointwise product of u and v (i.e., $uv = (\mu_{(X,x_0)})_{bil}(u, v)$), whence $\ell^2_\bullet(f)$ is a morphism of Hilbertian algebras from $(\ell^2_\bullet(X, x_0), \mu_{(X,x_0)})$ to $(\ell^2_\bullet(Y, y_0), \mu_{(Y,y_0)})$, and $\ell^2_\bullet$ actually provides a functor from $\mathbf{Set}^{\mathsf{op}}_{\bullet,<+\infty}$ to ${}^\dagger_c\mathbf{Sem}(\mathbb{Hilb})$ (Definition 12).

Remark 15. One observes that $\ell^2_\bullet$ does not provide a functor from $\mathbf{Set}^{\mathsf{op}}_{\bullet,<+\infty}$ to ${}^\dagger_c\mathbf{Sem}(\mathbb{Hilb}_1)$ (see Footnote 8). Let $f : (Y, y_0) \to (X, x_0)$. Let $x \neq x_0$. Then, $\sqrt{|f^{-1}(\{x\})|} = || \sum_{y \in f^{-1}(\{x\})} \delta_y || = ||\ell^2_\bullet(f)(\delta_x)|| \leq ||\ell^2_\bullet(f)||_{op}$. Therefore, $||\mathsf{B}_f||_\infty^{\frac{1}{2}} = ||\ell^2_\bullet(f)||_{op}$. Moreover, $||\mathsf{B}_f||_\infty \leq 1$ if, and only if, $dom(f) \xrightarrow{f} Y$ is one-to-one, where $dom(f) := \cup_{x \in R_f} f^{-1}(\{x\}) \subseteq Y \setminus \{y_0\}, R_f := \{x \in X \setminus \{x_0\} : \mathsf{B}_f(x) \neq 0\}$.

Remark 16. Let $\mathbf{Set}_{<+\infty}$ be the subcategory of sets with maps $X \xrightarrow{f} Y$ such that $\sup_{y \in Y} \mathsf{B}_f(y) < +\infty$. Let $\mathbf{Set}^{\mathsf{op}}_{<+\infty} \xrightarrow{\ell^2} {}^\dagger_c\mathbf{Sem}(\mathbb{Hilb})$ be the functor given by $X \mapsto (\ell^2(X), \mu_X)$ and for $f \in \mathbf{Set}_{<+\infty}(Y, X), \ell^2(f) : u \mapsto u \circ f, u \in \ell^2(X)$. Let $\mathbf{Set}_{<+\infty} \xrightarrow{(-)^+} \mathbf{Set}_{\bullet,<+\infty}$ be the functor $X \mapsto (X \sqcup \{0\}, 0)$, and $(f : X \to Y) \mapsto (f^+ : X^+ \to Y^+)$ with $f^+(x) = f(x), x \in X, f^+(0) = 0$. This functor is injective on objects and faithful but not full. Moreover $\ell^2_\bullet \circ (-)^+$ and ℓ^2 are naturally isomorphic functors (as in Remark 14).

In the reverse direction one has the following.

Proposition 35. *There is a functor* $\mathcal{G}_\bullet :_c^\dagger \mathbf{Sem}(\mathbb{Hilb}) \to \mathbf{Set}^{\mathrm{op}}_{\bullet,<+\infty}$ *whose object component is given by* $\mathcal{G}_\bullet(\mathcal{H},\mu) := (\mathcal{G}(\mathcal{H},\mu)\cup\{0\},0)$.

Proof. Let $(\mathcal{H},\mu),(\mathcal{K},\gamma)$ be special Hilbertian algebras, and let $f:(\mathcal{H},\mu)\to(\mathcal{K},\gamma)$ be a morphism of Hilbertian algebras. By Lemma 25, $f^\dagger(\mathcal{G}(\mathcal{K},\gamma)\cup\{0\})\subseteq\mathcal{G}(\mathcal{H},\mu)\cup\{0\}$, and one obtains by co-restriction of $f^\dagger$, a base-point preserving map $(\mathcal{G}(\mathcal{K},\gamma)\cup\{0\},0)\overset{\mathcal{G}_\bullet(f)}{\to}(\mathcal{G}(\mathcal{H},\mu)\cup\{0\},0)$. Let $g:=\mathcal{G}_\bullet(f)$.

1. $|g^{-1}(\{x\})|$ is finite for each $x\in\mathcal{G}(\mathcal{H},\mu)$: Assume to the contrary that for some $x\in\mathcal{G}(\mathcal{H},\mu)$ (in particular $x\neq 0$), there are infinitely many y's in $\mathcal{G}(\mathcal{K},\gamma)$ such that $g(y)=x$. Let a choose pairwise distinct $y_n\in\mathcal{G}(\mathcal{K},\gamma)$ such that $g(y_n)=x, n\geq 1$. Since $\{y_n : n\geq 1\}\subseteq\mathcal{G}(\mathcal{K},\gamma)$, by Lemma 27 it is an orthonormal set, thus one may define the following member u of $\mathcal{K}$ by $u:=\sum_{n\geq 1}\frac{1}{n}y_n$. Whence $f^\dagger(u)\in\mathcal{K}$. By linearity and continuity of $f^\dagger$,$f^\dagger(u)$ is the sum of the summable family $(\frac{1}{n}f^\dagger(y_n))_{n\geq 1}$. But for each n, $f^\dagger(y_n)=x$ and the family $(\frac{1}{n}x)_{n\geq 1}$ is not summable in $\mathcal{H}$.
2. B_g is bounded on $\mathcal{G}(\mathcal{H},\mu)$: Since f is assumed bounded, one can consider the associated bounded sesquilinear form $q_{f^\dagger}:\mathcal{K}\times\mathcal{H}\to\mathbb{C}$, given by $q_{f^\dagger}(u,v)=\langle f^\dagger(u),v\rangle$ ([4, p. 21]). Boundedness of $q_{f^\dagger}$ means that there exists some positive real number M such that for each $u\in\mathcal{K}$ and $v\in\mathcal{H}$,
$$|\langle f^\dagger(u),v\rangle|\leq M||u||||v||.$$
Let $x\in\mathcal{G}(\mathcal{H},\mu)$, and let $u:=\sum_{y\in g^{-1}(\{x\})}y\in\mathcal{K}$ (recall that $g^{-1}(\{x\})$ is finite). Replacing v by x in the above inequality one gets
$$|g^{-1}(\{x\})|=|\langle f^\dagger(u),x\rangle|\leq M||u||||x||=M\sqrt{|g^{-1}(\{x\})|}.$$
If $g^{-1}(\{x\})\neq\emptyset$, then $|g^{-1}(\{x\})|\leq M^2$, so that $\sup_{x\in\mathcal{G}(\mathcal{H},\mu)}|g^{-1}(\{x\})|\leq M^2$.

The above discussion shows that indeed $\mathcal{G}_\bullet(f):\mathcal{G}_\bullet(\mathcal{K},\gamma)\to\mathcal{G}_\bullet(\mathcal{H},\mu)$ is a morphism in $\mathbf{Set}_{\bullet,<+\infty}$. Functoriality of this construction is straightforward. □

Proposition 36. $\ell^2_\bullet:\mathbf{Set}^{\mathrm{op}}_{\bullet,<+\infty}\to_c^\dagger\mathbf{Sem}(\mathbb{Hilb})$ *is a right adjoint of* $\mathcal{G}_\bullet :_c^\dagger \mathbf{Sem}(\mathbb{Hilb})\to\mathbf{Set}^{\mathrm{op}}_{\bullet,<+\infty}$.

Proof. Let $(X,x_0)\overset{f}{\to}\mathcal{G}_\bullet(\mathcal{H},\mu)$ be a morphism in $\mathbf{Set}_{\bullet,<+\infty}$, where $(\mathcal{H},\mu)$ is a special Hilbertian algebra. Let $\mathcal{H}\overset{f^\sharp}{\to}\ell^2_\bullet(X,x_0)$ be the bounded linear map given by the composition $\mathcal{H}\overset{\pi_{J(\mathcal{H},\mu)^\perp}}{\longrightarrow}J(\mathcal{H},\mu)^\perp\overset{U}{\to}\ell^2_\bullet(\mathcal{G}_\bullet(\mathcal{H},\mu))\overset{\ell^2_\bullet(f)}{\longrightarrow}\ell^2_\bullet(X,x_0)$, where U is the unitary transformation given by $U(x):=\delta_x, x\in\mathcal{G}(\mathcal{H},\mu)$. Since each of the component maps is a morphism of Hilbertian algebras (concerning U it is a consequence of Corollary 28 and for $\pi_{J^\perp}$ this follows from Lemmas 16 and 17), $f^\sharp:(\mathcal{H},\mu)\to(\ell^2_\bullet(X,x_0),\mu_{(X,x_0)})$ is a morphism of Hilbertian algebras too. For $u\in\ell^2_\bullet(X,x_0)$, $(f^\sharp)^\dagger(u)=\sum_{y\in\mathcal{G}(\mathcal{H},\mu)}(\sum_{x\in f^{-1}(\{y\})}u(x))y$, so that $(f^\sharp)^\dagger(\delta_x)=f(x), x\in X\setminus\{x_0\}$. Let $g:(\mathcal{H},\mu)\to(\ell^2_\bullet(X,x_0),\mu_{(X,x_0)})$ be a morphism of Hilbertian algebras such that $g^\dagger(\delta_x)=f(x)$ for each $x\in X\setminus\{x_0\}$. Let $u\in\mathcal{H}, x\in X\setminus\{x_0\}$. Then, $\langle g(u),\delta_x\rangle=\langle u,f(x)\rangle=\langle f^\sharp(u),\delta_x\rangle$. Whence $f^\sharp=g$. □

Remark 17. The unit of the adjunction from Proposition 36 is given by $\eta_{(\mathcal{H},\mu)}=(\mathcal{H},\mu)\overset{\pi_{J^\perp}}{\to}J^\perp\overset{U}{\to}\ell^2_\bullet(\mathcal{G}_\bullet(\mathcal{H},\mu))$ using the notation from the proof of the aforementioned result. Up to the canonical isomorphism $\ell^2_\bullet(\mathcal{G}_\bullet(\mathcal{H},\mu)) \simeq \ell^2(\mathcal{G}(\mathcal{H},\mu))$ (Remark 14) $\eta_{(\mathcal{H},\mu)}$ is nothing else that the Gelfand-Riesz transformation $\mathcal{R}_{(\mathcal{H},\mu)}$ at $(\mathcal{H},\mu)$. The counit is given by the obvious bijection $\epsilon_{(X,x_0)} : \mathcal{G}_\bullet(\ell^2_\bullet(X,x_0)) = (\{\delta_x : x \in X \setminus \{x_0\}\} \cup \{0\}, 0) \to (X, x_0)$.

Let us call *special Hilbertian coalgebra* a (complex) cocommutative Hilbertian coalgebra $(\mathcal{H},\delta)$ with a coisometric multiplication, that is, $\delta^\dagger \circ \delta = id_{\mathcal{H}}$. These structures spanned the full subcategory ${}^\dagger_{coc}\mathbf{Cosem}(\mathbb{Hilb})$ of ${}_{coc}\mathbf{Cosem}(\mathbb{Hilb})$ and the Hilbert adjoint functor co-restricts as an isomorphism ${}^\dagger_{coc}\mathbf{Cosem}(\mathbb{Hilb})^{\mathsf{op}} \xrightarrow{(-)^\dagger} {}^\dagger_c\mathbf{Sem}(\mathbb{Hilb})$, whose inverse is denoted by ${}^\dagger_c\mathbf{Sem}(\mathbb{Hilb}) \xrightarrow{(-)^\dagger} {}^\dagger_{coc}\mathbf{Cosem}(\mathbb{Hilb})^{\mathsf{op}}$.

Consider the composite functors

$$\mathcal{G}^\dagger_\bullet := {}^\dagger_{coc}\mathbf{Cosem}(\mathbb{Hilb}) \xrightarrow{((-)^\dagger)^{\mathsf{op}}} {}^\dagger_c\mathbf{Sem}(\mathbb{Hilb})^{\mathsf{op}} \xrightarrow{\mathcal{G}^{\mathsf{op}}_\bullet} \mathbf{Set}_{\bullet,<+\infty}$$

and

$$\ell^{2\dagger}_\bullet := \mathbf{Set}_{\bullet,<+\infty} \xrightarrow{(\ell^2_\bullet)^{\mathsf{op}}} {}^\dagger_c\mathbf{Sem}(\mathbb{Hilb})^{\mathsf{op}} \xrightarrow{((-)^\dagger)^{\mathsf{op}}} {}^\dagger_{coc}\mathbf{Cosem}(\mathbb{Hilb}).$$

In details, for $f : (X,x_0) \to (Y,y_0)$ in $\mathbf{Set}_{\bullet,<+\infty}$, $\ell^{2\dagger}_\bullet(f) = \ell^2_\bullet(f)^\dagger$ is given for $u \in \ell^2_\bullet(X,x_0)$ by $\ell^2_\bullet(f)^\dagger(u) = \sum_{y\neq y_0}(\sum_{x\in f^{-1}(\{y\})} u(x))\delta_y$. In particular, for $x \neq x_0, \ell^2_\bullet(f)^\dagger(\delta_x) = 0$ if $f(x) = y_0$ and $\ell^2_\bullet(f)^\dagger(\delta_x) = \delta_{f(x)}$ if $f(x) \neq y_0$.

Remark 18. Let $(X,x_0) \xrightarrow{f} (Y,y_0)$ be an isomorphism in $\mathbf{Set}_{\bullet,<+\infty}$. Then, it induces a usual bijection $X \setminus \{x_0\} \xrightarrow{f} Y \setminus \{y_0\}$. For each $x \in X \setminus \{x_0\}, \ell^2_\bullet(f^{-1})(\delta_x) = \delta_x \circ f^{-1} = \delta_{f(x)} = \ell^2_\bullet(f)^\dagger(\delta_x)$. It follows that $\ell^2_\bullet(f)$ is a unitary transformation.

As a direct consequence of Proposition 36 one has

Corollary 37. *$\ell^{2\dagger}_\bullet$ is a left adjoint of $\mathcal{G}^\dagger_\bullet$.*

Remark 19. Let **PInj** be the category of sets with partial injections ([15]), i.e., the partially defined functions $f : X \to Y$ which are one-to-one when considered as maps $f : dom(f) \subseteq X \to Y$. **PInj** embeds into $\mathbf{Set}_{\bullet,<+\infty}$ under the functor $(-)^+$ that acts on objects as $X \mapsto X^+ := X \sqcup \{0\}$, and on partial injections $(f : X \to Y) \mapsto (f^+ : X^+ \to Y^+)$ with $f^+(x) = f(x), x \in dom(f), f^+(x) = 0, x \in X^+ \setminus dom(f)$. By inspection one checks that the ℓ^2-functor from [15] is naturally isomorphic to $\mathbf{PInj} \xrightarrow{U_s \circ \circ \ell^{2\dagger}_\bullet \circ (-)^+} \mathbf{Hilb}$, where ${}^\dagger_{coc}\mathbf{Cosem}(\mathbb{Hilb}) \xrightarrow{U_s} \mathbf{Hilb}$ is the obvious forgetful functor.

Lemma 38. *Let $(\mathcal{H},\mu) \xrightarrow{f} (\mathcal{K},\gamma)$ be a morphism of Hilbertian algebras where $(\mathcal{H},\mu)$ and $(\mathcal{K},\gamma)$ both are special and have H^*-adjoints. Then, for each $u \in \mathcal{H}, f(u^\sharp) = f(u)^\sharp$, where $(-)^\sharp$ denotes the (unique) maps of H^*-adjoints of both algebras.*

Proof. Let $u \in \mathcal{H}$ and let $y \in \mathcal{G}(\mathcal{K},\gamma)$. Then, $\langle f(u^\sharp), y\rangle = \langle u^\sharp, f^\dagger(y)\rangle = \overline{\langle u, f^\dagger(y)\rangle}$ (by Theorem 32 since $f^\dagger(\mathcal{G}(\mathcal{K},\gamma)) \subseteq \mathcal{G}(\mathcal{H},\mu) \cup \{0\}) = \overline{\langle f(u), y\rangle} = \langle f(u)^\sharp, y\rangle$ (by Theorem 32). □

Rather than having a property, one may prefer to define "Hilbertian H^*-algebras" as algebras *with a structure*. We show below that the result is almost the same.

Definition 14. Let $I : \overline{(\mathcal{H},\mu)} \to (\mathcal{H},\mu)$ be a morphism of Hilbertian algebras such that

1. $I \circ \bar{I} = id_{\mathcal{H}}, I^\dagger = \bar{I}$,
2. $\langle \mu(u \otimes v), w\rangle = \langle v, \mu(I(u) \otimes w)\rangle, u, v, w \in \mathcal{H}$.

The triple $(\mathcal{H},\mu,I)$ is referred to as a *special Hilbertian H^*-algebra* when furthermore $\mu\circ\mu^\dagger = id_{\mathcal{H}}$. A morphism $f:(\mathcal{H},\mu,I)\to(\mathcal{K},\gamma,J)$ of special Hilbertian H^*-algebras is a morphism f: $(\mathcal{H},\mu)\to(\mathcal{K},\gamma)$ of Hilbertian algebras such that $f\circ I = J\circ f$. All of this forms the category $H^* -_c^\dagger \mathbf{Sem}(\mathbb{Hilb})$.

Lemma 39. *Let $P: H^* -_c^\dagger \mathbf{Sem}(\mathbb{Hilb}) \to_c^\dagger \mathbf{Sem}(\mathbb{Hilb})$ be the obvious forgetful functor. It factors through the canonical embedding ${}_{\text{semisimple},}{}_c^\dagger\mathbf{Sem}(\mathbb{Hilb}), \to_c^\dagger \mathbf{Sem}(\mathbb{Hilb})$, and the categories $H^* -_c^\dagger$ $\mathbf{Sem}(\mathbb{Hilb})$ and ${}_{\text{semisimple},}{}_c^\dagger\mathbf{Sem}(\mathbb{Hilb})$ are isomorphic under the corresponding co-restriction.*

Proof. By definition $(\mathcal{H},\mu)$ is a special Hilbertian algebra with H^*-adjoints (given by $I(u)$, $u\in\mathcal{H}$), whence is semisimple (by Theorem 32) for each special Hilbertian H^*-algebra $(\mathcal{H},\mu,I)$. This provides the co-restricted functor $P: H^* -_c^\dagger \mathbf{Sem}(\mathbb{Hilb}) \to {}_{\text{semisimple},}{}_c^\dagger\mathbf{Sem}(\mathbb{Hilb})$. That P is an isomorphism is a direct consequence of the uniqueness of the map I of H^*-adjoints. In more details: let $(\mathcal{H},\mu)$ be a semisimple special Hilbertian algebra. By Corollary 33 it is a Hilbertian algebra with H^*-adjoints, and thus there is a unique map $(-)^\star:\mathcal{H}\to\mathcal{H}$ of H^*-adjoints. It is clear that $V(\mathcal{H},\mu) := (\mathcal{H},\mu,(-)^*)$ is a special Hilbertian H^*-algebra. Moreover given a morphism of Hilbertian algebras $(\mathcal{H},\mu)\xrightarrow{f}(\mathcal{K},\gamma)$ between semisimple special Hilbertian algebras, by Lemma 38, $V(f) := V(\mathcal{H},\mu)\xrightarrow{f}V(\mathcal{K},\gamma)$ is a morphism of special Hilbertian H^*-algebras. Functoriality of V:${}_{\text{semisimple},}{}_c^\dagger\mathbf{Sem}(\mathbb{Hilb}) \to H^* -_c^\dagger \mathbf{Sem}(\mathbb{Hilb})$ is clear so is clear that $P\circ V = id$ and $V\circ P = id$. □

Let us denote ${}_{\text{cosemisimple},}{}_{coc}^\dagger\mathbf{Cosem}(\mathbb{Hilb})$ the full subcategory of ${}_{coc}^\dagger\mathbf{Cosem}(\mathbb{Hilb})$ spanned by the coalgebras $(\mathcal{H},\delta)$ with $(\mathcal{H},\delta^\dagger)$ semisimple. There is of course the isomorphism $(-)^\dagger$:${}_{\text{cosemisimple},}{}_{coc}^\dagger\mathbf{Cosem}(\mathbb{Hilb})^{\text{op}} \simeq {}_{\text{semisimple},}{}_c^\dagger\mathbf{Sem}(\mathbb{Hilb})$.

The unit of the adjunction from Proposition 36 (see Remark 17) is an isomorphism if, and only if, the Hilbertian algebra is semisimple. Consequently,

Theorem 40. *The category $\mathbf{Set}^{\text{op}}_{\bullet,<+\infty}$ is equivalent to the isomorphic categories ${}_{\text{semisimple},}{}_c^\dagger\mathbf{Sem}(\mathbb{Hilb}), H^* -_c^\dagger \mathbf{Sem}(\mathbb{Hilb})$ and ${}_{\text{cosemisimple},}{}_{coc}^\dagger\mathbf{Cosem}(\mathbb{Hilb})^{\text{op}}$, under either of the adjunctions from Proposition 36 or Corollary 37. $\mathbf{Set}^{\text{op}}_{\bullet,<+\infty}$ is also equivalent to ${}_{\text{bnd, proper}}H^* -_c\mathbf{BanAlg}$.*

Corollary 41. *Let ϕ: $(\mathcal{H},\mu)\to(\mathcal{K},\gamma)$ be a morphism between semisimple special Hilbertian algebras. If ϕ is an isomorphism, then $\phi:\mathcal{H}\to\mathcal{K}$ is a unitary transformation and $\phi:(\mathcal{H},\mu^\dagger)\to(\mathcal{K},\gamma^\dagger)$ is also a morphism of Hilbertian coalgebras. In particular the only topological isomorphisms in ${}_{\text{semisimple},}{}_c^\dagger\mathbf{Sem}(\mathbb{Hilb})$ are unitary and they are automatically also isomorphisms of coalgebras between the adjoint coalgebras.*

Proof. Let $(\ell^2_\bullet(X,x_0),\mu_{(X,x_0)})\xrightarrow{\psi}(\ell^2_\bullet(Y,y_0),\mu_{(Y,y_0)})$ be an isomorphism in the category ${}_c^\dagger\mathbf{Sem}(\mathbb{Hilb})$. According to Theorem 40 there exists a unique isomorphism $(Y,y_0)\xrightarrow{f}(X,x_0)$ in $\mathbf{Set}_{\bullet,<+\infty}$ such that $\psi = \ell^2_\bullet(f)$. By Remark 18, ψ is a unitary transformation. Since $(\ell^2_\bullet(Y,y_0),\mu^\dagger_{(Y,y_0)})\xrightarrow{\psi^\dagger=\ell^2_\bullet(f)^\dagger=\psi^{-1}}(\ell^2_\bullet(X,x_0),\mu^\dagger_{(X,x_0)})$ is an isomorphism in ${}_{coc}\mathbf{Cosem}(\mathbb{Hilb})$, it follows that its inverse ψ also is an isomorphism of Hilbertian coalgebras.

Now let $(\mathcal{H},\mu)\xrightarrow{\phi}(\mathcal{K},\gamma)$ be an isomorphism of semisimple special algebras. There is a unique isomorphism $(\ell^2_\bullet(\mathcal{G}_\bullet(\mathcal{H},\mu)),\mu_{\mathcal{G}_\bullet(\mathcal{H},\mu)})\xrightarrow{\psi}(\ell^2_\bullet(\mathcal{G}_\bullet(\mathcal{K},\gamma)),\mu_{\mathcal{G}_\bullet(\mathcal{K},\gamma)})$ such that the following diagram commutes in ${}_c^\dagger\mathbf{Sem}(\mathbb{Hilb})$.

$$\begin{array}{ccc} (\mathcal{H},\mu) & \xrightarrow{\phi} & (\mathcal{K},\gamma) \\ \mathcal{R}_{(\mathcal{H},\mu)} \Big\| \wr & & \Big\| \wr \mathcal{R}_{(\mathcal{K},\gamma)} \\ (\ell^2_\bullet(\mathcal{G}_\bullet(\mathcal{H},\mu)),\mu_{\mathcal{G}_\bullet(\mathcal{H},\mu)}) & \xrightarrow[\psi]{} & (\ell^2_\bullet(\mathcal{G}_\bullet(\mathcal{K},\gamma)),\mu_{\mathcal{G}_\bullet(\mathcal{K},\gamma)}) \end{array} \tag{18}$$

By semisimplicity, the components of the Gelfand-Riesz transform are both unitary transformations and morphisms of Hilbertian coalgebras (e.g., for $\mathcal{R}_{(\mathcal{H},\mu)}$: given $u \in \mathcal{H}$, one has $(\mathcal{R}_{(\mathcal{H},\mu)} \hat{\otimes}_2 \mathcal{R}_{(\mathcal{H},\mu)})(\mu^\dagger(u)) = \sum_{x\in\mathcal{G}(\mathcal{H},\mu)} \langle u, x\rangle \mathcal{R}_{(\mathcal{H},\mu)}(x) \otimes \mathcal{R}_{(\mathcal{H},\mu)}(x) = \sum_{x\in\mathcal{G}(\mathcal{H},\mu)} \langle u, x\rangle \delta_x \otimes \delta_x = \mu^\dagger_{\mathcal{G}_\bullet(\mathcal{H},\mu)}(\mathcal{R}_{(\mathcal{H},\mu)}(u))$), and by the above so is ψ, and thus also ϕ. □

Remark 20. The category ${}^\dagger_c\mathbf{Sem}(\mathbb{Hilb})$ (respectively ${}_{\text{semisimple},}{}^\dagger_c\mathbf{Sem}(\mathbb{Hilb})$) is closed under unitary isomorphisms from ${}_c\mathbf{Sem}(\mathbb{Hilb})$, i.e., if $(\mathcal{H},\mu) \simeq (\mathcal{K},\gamma)$ under such an isomorphism, then either both algebras are special (respectively, semisimple and special) or none of them are. But they are not closed under topological isomorphisms from ${}_c\mathbf{Sem}(\mathbb{Hilb})$. For example, for a bounded map $\alpha : X \to [1, +\infty[, \alpha \not\equiv 1, ((\ell^2_\alpha(X), \langle\cdot,\cdot\rangle_\alpha), \mu_X) \simeq (\ell^2(X), \mu_X)$ but never unitarily so (by Theorem 9).

Corollary 42. *The structures of semisimple special Hilbertian algebras on a given Hilbert space $\mathcal{H}$, that is, the maps $\mathcal{H} \hat{\otimes}_2 \mathcal{H} \xrightarrow{\mu} \mathcal{H}$ that turn $(\mathcal{H},\mu)$ into an object of ${}_{\text{semisimple},}{}^\dagger_c\mathbf{Sem}(\mathbb{Hilb})$, are in one-one correspondence with the Hilbertian bases of $\mathcal{H}$ and the corresponding Hilbertian algebras $(\mathcal{H},\mu)$ are pairwise (unitarily) isomorphic.*

Proof. Given a Hilbertian basis X of $\mathcal{H}, (\mathcal{H},\mu_X)$ is a semisimple special Hilbertian algebra, when μ_X is given on the basis elements $x\otimes y, x, y \in X$, by $\mu_X(x\otimes y) = \delta_{x,y}x$. The map $X \mapsto (\mathcal{H},\mu_X)$ between Hilbertian bases of $\mathcal{H}$ and structures of semisimple special Hilbertian algebras on $\mathcal{H}$ is onto since $\mathcal{G}(\mathcal{H},\mu)$ is a Hilbert basis of $\mathcal{H}$ (Remark 9) and $\mu(g\otimes h) = \delta_{g,h}g, g, h \in \mathcal{G}(\mathcal{H},\mu)$ (Corollary 28). It is also one-to-one as $\mathcal{G}(\mathcal{H},\mu_X) = X$. The operator on $\mathcal{H}, \Phi : u \mapsto \sum_{y\in Y}\langle u, \phi^{-1}(y)\rangle y$ induced by a bijection $\phi : X \to Y$ between Hilbertian bases X, Y of $\mathcal{H}$, is a unitary isomorphism from $(\mathcal{H},\mu_X)$ onto $(\mathcal{H},\mu_Y)$. □

One finally provides a dual equivalence of categories between the category $\mathbf{Set}_{<+\infty}$ from Remark 16 and a subcategory of semisimple special Hilbertian algebras, which is similar to the Gelfand duality between the opposite of the category of locally compact spaces and proper maps and that of commutative (nonunital) C^*-algebras and proper morphisms ([16, p. 33]).

Let $(\mathcal{H},\mu) \xrightarrow{f} (\mathcal{K},\gamma)$ be a morphism in ${}_c\mathbf{Sem}(\mathbb{Hilb})$ (here the multiplications are not assumed contractive). It is said to be *proper* (see e.g., [16, Theorem 6.6, p. 33]) when *ran*(*f*) is not included in any maximal modular (closed) ideal of $(\mathcal{K},\gamma)$ or alternatively for each $y \in \mathcal{G}(\mathcal{K},\gamma)$, there exists $u \in \mathcal{H}$ such that $\langle f(u), y\rangle \neq 0$. In other words, $\mathcal{G}(\mathcal{K},\gamma) \not\subseteq \ker f^\dagger$. Observe that any isomorphism of Hilbertian algebras is proper. Since *id* and the composition of proper morphisms are proper too, the category ${}_c\mathbf{Sem}(\mathbb{Hilb})_{\text{proper}}$ of commutative Hilbertian algebras and proper morphisms is available to us, so is its full subcategory ${}^\dagger_c\mathbf{Sem}(\mathbb{Hilb})_{\text{proper}}$ (resp., ${}_{\text{semisimple},}{}^\dagger_c\mathbf{Sem}(\mathbb{Hilb})_{\text{proper}}$) spanned by special (resp. semisimple special) Hilbertian algebras.

Given a proper morphism $(\mathcal{H},\mu) \xrightarrow{f} (\mathcal{K},\gamma)$, since $f^\dagger$ is a morphism of coalgebras, $f^\dagger(\mathcal{G}(\mathcal{K},\gamma)) \subseteq \mathcal{G}(\mathcal{H},\mu)$ according to Lemma 25. Let $\mathcal{G}(\mathcal{K},\gamma) \xrightarrow{\mathcal{G}(f)} \mathcal{G}(\mathcal{H},\mu)$ be the co-restriction of $f^\dagger$ thus obtained. If one furthermore assumes that $(\mathcal{K},\gamma)$ is special, in a way similar to what was proved for $\mathcal{G}_\bullet(f)$ in the proof of Proposition 35, it can be shown that $\mathcal{G}(f) \in$

$\mathbf{Set}_{<+\infty}(\mathcal{G}(\mathcal{K},\gamma),\mathcal{G}(\mathcal{H},\mu))$ (see Remark 16). All of this provides a functor ${}^{\dagger}_{c}\mathbf{Sem}(\mathbb{Hilb})_{\mathsf{proper}} \xrightarrow{\mathcal{G}} \mathbf{Set}^{\mathsf{op}}_{<+\infty}$.

Given $f \in \mathbf{Set}_{<+\infty}(Y,X)$, $\langle \ell^2(f)(u),\delta_y\rangle = \langle u,\ell^2(f)^{\dagger}(\delta_y)\rangle = \langle u,\delta_{f(y)}\rangle$ for each $y \in Y$ and $u \in \ell^2(X)$, so that $(\ell^2(X),\mu_X) \xrightarrow{\ell^2(f)} (\ell^2(Y),\mu_Y)$ is a proper morphism. Whence the functor from Remark 16 co-restricts as $\mathbf{Set}^{\mathsf{op}}_{<+\infty} \xrightarrow{\ell^2} {}^{\dagger}_{c}\mathbf{Sem}(\mathbb{Hilb})_{\mathsf{proper}}$.

Proposition 43. *ℓ^2 is a right adjoint of $\mathcal{G}$, and this adjunction restricts to an equivalence of categories between $\mathbf{Set}^{\mathsf{op}}_{<+\infty}$ and ${}_{\mathsf{semisimple},}{}^{\dagger}_{c}\mathbf{Sem}(\mathbb{Hilb})_{\mathsf{proper}}$.*

Proof. Let $X \xrightarrow{f} \mathcal{G}(\mathcal{H},\mu)$ be a morphism in $\mathbf{Set}_{<+\infty}$ where $(\mathcal{H},\mu)$ is a special Hilbertian algebra. Let us define the morphism of Hilbertian algebras $(\mathcal{H},\mu) \xrightarrow{f^{\sharp}} (\ell^2(X),\mu_X) = (\mathcal{H},\mu) \xrightarrow{\pi_{J^{\perp}}} (J^{\perp},\mu_{J^{\perp}}) \xrightarrow{U} (\ell^2(\mathcal{G}(\mathcal{H},\mu)),\mu_{\mathcal{G}(\mathcal{H},\mu)}) \xrightarrow{\ell^2(f)} (\ell^2(X),\mu_X)$, where U is the unitary isomorphism given by $U(x) = \delta_x, x \in \mathcal{G}(\mathcal{H},\mu)$. Since $(J^{\perp},(\mu_{J^{\perp}})^{\dagger}) \xrightarrow{i_{J^{\perp}}=(\pi_{J^{\perp}})^{\dagger}} (\mathcal{H},\mu^{\dagger})$ is a one-to-one coalgebra morphism, it follows that $i_{J^{\perp}}(\mathcal{G}(J^{\perp},\mu_{J^{\perp}})) \subseteq \mathcal{G}(\mathcal{H},\mu)$, so that $\pi_{J^{\perp}}$ is a proper morphism, so is U because it is an isomorphism. Therefore $f^{\sharp}$ itself is a proper morphism and it satisfies $(f^{\sharp})^{\dagger}(\delta_x) = f(x), x \in X$. Let $g : (\mathcal{H},\mu) \to (\ell^2(X),\mu_X)$ be a proper morphism such that $g^{\dagger}(\delta_x) = f(x), x \in X$. Then it follows easily that $g = f^{\sharp}$. So the adjunction is proved.

The unit of the above adjunction is $\eta_{(\mathcal{H},\mu)} = (\mathcal{H},\mu) \xrightarrow{\pi_{J^{\perp}}} J^{\perp} \xrightarrow{U} \ell^2(\mathcal{G}(\mathcal{H},\mu))$ and the counit is the obvious bijection $\epsilon_X : \mathcal{G}(\ell^2(X),\mu_X) = \{\delta_x : x \in X\} \to X$. The adjunction restricts to an equivalence exactly when the unit is an isomorphism, i.e., when $(\mathcal{H},\mu)$ is semisimple. □

The following diagram commutes (this completes Remark 16) up to natural isomorphisms, where $\simeq$ stands for one of the above equivalences of categories, and the rightmost hook shaped arrow is the obvious (nonfull) embedding functor.

$$\begin{array}{ccc}
\mathbf{Set}_{\bullet,<+\infty} & \simeq & {}_{\mathsf{semisimple},}{}^{\dagger}_{c}\mathbf{Sem}(\mathbb{Hilb}) \\
{\scriptstyle (-)^{+}}\uparrow & & \uparrow \\
\mathbf{Set}_{<+\infty} & \simeq & {}_{\mathsf{semisimple},}{}^{\dagger}_{c}\mathbf{Sem}(\mathbb{Hilb})_{\mathsf{proper}}
\end{array} \qquad (19)$$

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References

[1] Abramsky, S., Coecke, B. (2004). A categorical semantics of quantum protocols. In: Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science, 2004. Washington, DC, USA: IEEE, pp. 415–425.

[2] Abramsky, S., Heunen, C. (2012). H^*-algebras and nonunital Frobenius algebras: First steps in infinite-dimensional categorical quantum mechanics. *Cliff. Lect.* 71:1–24.

[3] Aguiar, M., Mahajan, S. (2010). Monoidal *Functors*, *Species* and Hopf *Algebras*. Providence, RI, USA: American Mathematical Society.

[4] Akhiezer, N. I., Glazman, I. M. (1993). *Theory of Linear Operators in Hilbert Space – Two Volumes Bound as One*. New York, USA: Dover Publications, Inc.

[5] Ambrose, W. (1945). Structure theorems for a special class of Banach algebras. *Trans. Amer. Math. Soc.* 57(3): 364–386. DOI: 10.2307/1990182.

[6] Banach, S. (1925). Sur les lignes rectifiables et les surfaces dont l'aire Est finie. *Fund. Math.* 7: 225–236. DOI: 10.4064/fm-7-1-225-236.
[7] Bourbaki, N. (2003). *Topological Vector Spaces*, Chapters 1–5. Berlin Heidelberg, Germany: Springer.
[8] Cohn, P. M. (2004). *Basic Algebra: Groups, Rings and Fields*. Berlin Heidelberg, Germany: Springer.
[9] Defant, A., Floret, K. (1992). *Tensor Norms and Operator Ideals*. Amsterdam, The Netherlands: Elsevier.
[10] Diestel, J., Fourie, J. H., Swart, J. (2008). The Metric Theory of Tensor Products: Grothendieck's Résumé Revisited. Providence, RI, USA: American Mathematical Society.
[11] Dieudonné, J. (1983). *Éléments d'analyse – Tome II*. Paris, France: Éditions Jacques Gabay.
[12] Glicksberg, I. (1959). Convolution semigroups of measures. *Pacific J. Math.* 9(1): 51–67. DOI: 10.2140/pjm.1959.9.51.
[13] Helemskii, Y. A. (2006). Lectures and Exercices on Functional Analysis. Providence, RI, USA: American Mathematical Society.
[14] Heunen, C. (2012). Complementarity in categorical quantum mechanics. *Found. Phys.* 42(7): 856–873. DOI: 10.1007/s10701-011-9585-9.
[15] Heunen, C. (2013). On the functor ℓ^2, In: *Computation, Logic, Games, and Quantum Foundations*. The Many Facets of Samson Abramsky. *Vol. 7860 of Lecture Notes in Computer Science*. Berlin Heidelberg, Germany: Springer. pp. 107–121.
[16] Hofmann, K. H. (1970). *The Duality of Compact Semigroups and C*-Bigebras*. Lecture Notes in Mathematics, Vol. 129, Berlin Heidelberg, Germany: Springer.
[17] Kadison, R. V., Ringrose, J. R. (1983). *Fundamentals of the Theory of Operator Algebras – Volume 1 Elementary Theory*. Orlando, USA: Academic Press.
[18] Kaniuth, E. (2008). *A Course in Commutative Banach Algebras*. New York, USA: Springer Science & Business Media.
[19] Kubrusly, C. S. (2011). *The Elements of Operator Theory*, 2nd ed. Boston, USA: Birkhäuser.
[20] Loomis, L. H. (1953). *An Introduction to Abstract Harmonic Analysis*. Princeton, USA: D. Van Nostrand Company, Inc.
[21] Mac Lane, S. (1978). *Categories for the Working Mathematician*. New York, USA: Springer Science & Business Media.
[22] Meise, R., Vogt, D. (1997). *Introduction to Functional Analysis*. Oxford, UK: Clarendon Press.
[23] Palmer, T. W. (1994). *Banach Algebras and the General Theory of *-Algebras, Volume I: Algebras and Banach Algebras*, *Vol.* 49 *of Encyclopedia of Mathematics and Its Applications*. Cambridge, UK: Cambridge University Press.
[24] Palmer, T. W. (2001). *Banach Algebras and the General Theory of *-Algebras, Volume II: *–Algebras*, *Vol. 79 of Encyclopedia of Mathematics and its Applications*. Cambridge, UK: Cambridge University Press.
[25] Poinsot, L., Porst, H.-E. (2015). Free monoids over semigroups in a monoidal category: Construction and applications. *Commun. Algebra* 43(11): 4873–4899. DOI: 10.1080/00927872.2014.955575.
[26] Rickart, C. E. (1974). *General Theory of Banach Algebras*. Huntington, USA: Krieger Pub Co.
[27] Rieffel, M. A. (1972). Unitary representations induced from compact subgroups. *Studia Math.* 42(2): 145–175. DOI: 10.4064/sm-42-2-145-175.
[28] Rudin, W. (1991). *Functional Analysis*, 2nd ed. New York, USA: McGraw-Hill, Inc.
[29] Ryan, R. A. (2013). *Introduction to Tensor Products of Banach Spaces*. New York, USA: Springer Science & Business Media.
[30] Schatten, R. (1950). *A Theory of Cross-Spaces*. Princeton, USA: Princeton University Press.
[31] Selinger, P. (2007). Dagger compact closed categories and completely positive maps. In: Third *International Workshop on Quantum Programming Languages*, Vol. 170 of *Electronic Notes in Theoretical Computer Science*. Chicago, USA. pp. 139–163, DOI: 10.1016/j.entcs.2006.12.018.
[32] Semadeni, Z. (1965). Spaces of continuous functions on compact sets. *Adv. Math.* 1(3): 319–382. DOI: 10.1016/0001-8708(65)90041-1.
[33] Street, R. (2007). *Quantum Groups: A Path to Current Algebra*. Cambridge, UK: Cambridge University Press.
[34] Stromberg, K. (1959). A note on the convolution of regular measures. *Math. Scand.* 7: 347–352. DOI: 10.7146/math.scand.a-10580.
[35] Sweedler, M. E. (1969). *Hopf Algebras*. New York, USA: W.A. Benjamin.
[36] Takesaki, M. (2013). *Theory of Operator Algebras II*. New York, USA: Springer Science & Business Media.
[37] Tonge, A. M. (1976). Translation invariance, compact semigroups and injective coalgebras. *Publ. Math. d'Orsay* 185: 1–22.
[38] Tonge, A. M. (1980). La presque-périodicité et les coalgèbres injectives. *Studia Math.* 67(2): 103–118. DOI: 10.4064/sm-67-2-103-118.
[39] Wichmann, J. (1975). The theory of Banach coalgebras. PhD dissertation. Texas Christian University, Fort Worth, TX.