Hilbertian Frobenius algebras

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Abstract

Commutative Hilbertian Frobenius algebras are those commutative semigroup objects in the monoidal category of Hilbert spaces, for which the Hilbert adjoint of the multiplication satisfies the Frobenius compatibility relation, that is, this adjoint "comultiplication" is a bimodule map. In this note we show that the Frobenius relation forces the multiplication operators to be normal. We then prove that these algebras have a strong Wedderburn decomposition where the (ortho)complement of the Jacobson radical or equivalently of the annihilator, is the closure of the linear span of elements which essentially are the non-trivial characters. As a consequence such an algebra is semisimple if, and only if, its multiplication has a dense range. In particular every commutative special Hilbertian Frobenius algebra, that is, with a coisometric multiplication, is semisimple. Moreover we characterize from a setting *a priori* free of an involution, Ambrose's commutative H^*-algebras as the underlying algebras of Hilbertian Frobenius algebras. Extending a known result in the finite-dimensional situation, we prove that the structures of such Frobenius algebras on a given Hilbert space are in one-to-one correspondence with its bounded above orthogonal sets. We show, moreover, that the category of commutative Hilbertian Frobenius algebras is dually equivalent to a category of pointed sets.

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Introduction

As is well-known the Hilbert space $\ell^2(X)$ of square-summable functions on X is by no means free over X in any obvious ways. However in this note we prove that X –

or more precisely X plus a new point adjoined – freely generates a (non-unital, when X is not finite) commutative Frobenius algebra, whose underlying space is (unitarily isomorphic to) $\ell^2(X)$. More generally we show that every commutative Frobenius algebra (H, μ), where (H, μ) is a semigroup in the category of Hilbert spaces and for which $\mu^\dagger$ satisfies the Frobenius compatibility relation, splits into an orthogonal direct sum of two ideals $\ell^2(X) \oplus_2 A(H, \mu)$, where $A(H, \mu)$ is the annihilator of (H, μ) and X is a set equipotent to that of minimal ideals of (H, μ). The free Frobenius algebras, that is, those of the form $\ell^2(X)$, are precisely the semisimple ones. Before describing in more detail the content of this note, let us put it into perspective.

A Frobenius algebra may be described in several equivalent ways ([7, Theorem 61.3, p. 414]), for instance it is a unital algebra over some base field which as a left module over itself is isomorphic to its algebraic (right) dual. This definition implies directly that the algebra must be finite-dimensional or at least finitely generated projective if base rings are allowed [10].

Not all equivalent characterizations are equally suitable for the applications or generalizations we have in mind, for instance if one expects to extend this notion to not necessarily finite-dimensional algebras. Thus alternatively a Frobenius algebra is a finite-dimensional unital algebra together with a non-degenerate and "associative" bilinear form. Dropping the finiteness condition leads to infinite-dimensional Frobenius algebras studied in [14]. At this point one may add that under this form the Frobenius algebras are substantially similar to a class of (non unital) algebras combining Banach algebras and Hilbert spaces, called H^*-algebras [3], where the Hilbert adjoint of the operators of left multiplication still are operators of left multiplication.

Another way to characterize Frobenius algebras appears in [1, Theorem 1, p. 572] in a commutative situation. A Frobenius algebra is a finite-dimensional unital algebra which at the same time is a counital coalgebra such that both structures interact nicely. More precisely the compatibility condition between the algebra and the coalgebra of a Frobenius algebra – the so-called *Frobenius relation* – asserts that the comultiplication of the latter is a morphism of bimodules over the former.

Because the above compatibility relation is stated entirely using only tensor products and linear maps, the former description has the advantage over others to allow for talking about Frobenius algebras in the realm of monoidal categories. This is precisely the point of view adopted in [17] and in [13, Chap. V]. This approach is used with success in [6, 2] where the authors take advantage of the presence of the Hilbertian adjoint to define a comultiplication from a multiplication. More precisely they consider commutative *†-Frobenius monoids*, that is, Frobenius algebras (H, μ, η) over a finite-dimensional Hilbert space H, with corresponding coalgebra $(H, \mu^\dagger, \eta^\dagger)$.

In what follows "*Hilbertian Frobenius*" stands for "*†-Frobenius*", to recall the fact that the comultiplication is the Hilbert adjoint of the multiplication, because no other kinds of Frobenius semigroups in the category of Hilbert spaces are considered here. To summarize the coalgebra structure of a Hilbertian Frobenius algebra is the Hilbert adjoint of its algebra structure. We stress here that *Hilbertian* algebras, that is, roughly speaking Banach algebras over a Hilbert spaces, shouldn't be confused with the well-known *Hilbert algebras* [8] since the former don't have an involution *a priori.*

Most notably the main result in [6] is the statement that orthogonal bases on a given finite-dimensional Hilbert space and its structures of commutative (unital and counital) Hilbertian Frobenius algebras are in a one-to-one correspondence.

In an effort to extend this result to arbitrary Hilbert spaces, a notion of non-unital Frobenius algebra is proposed in [2], referred to as (commutative) *Hilbertian Frobenius algebras* in what follows, obtained by dropping the unital (and thus also counital) assumption. More precisely, these are Hilbertian Frobenius semigroups (H,μ) (in the monoidal category $(\mathbf{Hilb},\hat{\otimes}_2,\mathbb{C})$ of Hilbert spaces and bounded linear maps), that is, $H\hat{\otimes}_2 H \xrightarrow{\mu} H$ is commutative and associative, and its adjoint $H \xrightarrow{\mu^\dagger} H\hat{\otimes}_2 H$ satisfies the Frobenius condition.

While the authors only partially achieve one of their goals, namely the characterization of arbitrary orthonormal bases by means of Frobenius structures, one of their merit is a clarification of the relation between commutative *special* Hilbertian Frobenius algebras, that is, those Hilbertian Frobenius algebras (H,μ) with an isometric comultiplication ($\mu\circ\mu^\dagger = \mathrm{id}$), and Ambrose's H^*-algebras.

It is precisely the intention of this note to rigorously explain how orthogonal bases or better orthogonal sets and structures of commutative Hilbertian Frobenius algebras over not necessarily finite-dimensional Hilbert spaces are related. We, hence, provide a structure theorem for commutative Hilbertian Frobenius algebras (Theorem 2.14) which states that they have a strong Wedderburn decomposition [4]. Of course the underlying Hilbert space of such algebras splits as an orthogononal direct sum of the Jacobson radical and the topological closure of the linear span of the *group-like* elements, that is, those non zero elements x sent to $x\otimes x$ by the comultiplication $\mu^\dagger$. But what is less immediate is that, in fact, the orthogonal complement of the Jacobson radical is a subalgebra, a fact that may be reduced to the closure of the set of group-like elements under multiplication. Actually it is not difficult to notice that the product of two *distinct* group-like elements is equal to zero, but what is not as immediate is that the square of a group-like element belongs to the one-dimensional space spanned by this element (to be fair this observation is free when the comultiplication is assumed isometric as in [2], which is not the case in what follows), which by the way turns this space into a minimal ideal.

A similar result is discussed in [2] for the particular case of commutative Hilbertian Frobenius semigroups with an isometric comultiplication, but that preliminary treatment was not completely accurate (see the introduction of [22] for more details). Our structure theorem is completed by the observation that the radical is precisely the annihilator of the algebra, which is a direct consequence of the Frobenius condition (see Proposition 2.13) which forces the multiplication operators to be normal, that is, to commute with their own adjoint.

Thus, clearly, a commutative Hilbertian Frobenius algebra (H,μ) not only splits into an orthogonal direct sum of a subalgebra and an ideal, but actually of two (closed) ideals, one being radical while the other is semisimple. As becomes clear, the question of semisimplicity of such an algebra is completely governed by this structure theorem: a necessary and sufficient condition for a commutative Hilbertian Frobenius algebra to be semisimple is that its regular representation is faithful, or equivalently that its multiplication $H\hat{\otimes}_2 H \xrightarrow{\mu} H$ has a dense range or that its comultiplication is one-to-one (Theorem 2.17). (In particular every commutative *special* Hilbertian Frobenius algebra is semisimple.) It is a remarkable fact that in the finite-dimensional situation this may be interpreted as the existence of a unit (Corollary 3.1). Consequently we recover (with a proof free of a C^*-argument) the result of [6] stating that finite-dimensional commutative Hilbertian Frobenius *monoids* are semisimple.

Because group-like elements of a commutative Hilbertian Frobenius algebra, are non-zero and pairwise orthogonal (even orthonormal when furthermore the algebra is special) several bijective correspondences (Theorem 3.2) between structures of Frobenius algebras of certain kinds available on a given Hilbert space, and some of its orthogonal (or orthonormal) sets are obtained, which extend the main result of [6]. It is worth mentioning that contrary to the finite-dimensional situation, not all orthogonal sets correspond to a structure of a commutative Hilbertian Frobenius algebra but only those which are bounded above (or below by a strictly positive constant), including the empty set; in fact one cannot expect unbounded (above or below) orthogonal families to be in the range of the above bijections as boundedness of the norm of the group-like elements is a direct consequence of the fact that in a Banach algebra, the multiplication, as a bilinear map, is jointly continuous. Moreover, the easy description of semisimple commutative Hilbertian Frobenius algebras makes it possible to provide a characterization of Ambrose's commutative H^*-algebras, from a setting free of an involution (Proposition 3.4).

Besides the above structure theorem also has some important consequences at the level of the category ${}_c\mathbf{FrobSem}(\mathbb{Hilb})$ of commutative Hilbertian Frobenius algebras and semigroup morphisms. Most notably it is shown that every semigroup morphism between commutative Hilbertian Frobenius algebras arises from a unique set-theoretic base-point preserving map (of some specific kind), from the set of minimal ideals of its codomain to the set of minimal ideals of its domain, both with zero added as base-point. We also prove the following results.

1. ${}_c\mathbf{FrobSem}(\mathbb{Hilb})$ is equivalent to ${}_{\mathsf{semisimple},c}\mathbf{FrobSem}(\mathbb{Hilb}) \times \mathbf{Hilb}$ (Proposition 4.5) where the first factor is the full subcategory of ${}_c\mathbf{FrobSem}(\mathbb{Hilb})$ spanned by the semisimple algebras. The splitting into a semisimple Hilbertian Frobenius algebra and the radical provides the equivalence.

2. ${}_{\mathsf{semisimple},c}\mathbf{FrobSem}(\mathbb{Hilb})$ is dually equivalent to a category of pointed weighted sets (Theorem 5.8). Other but related equivalences of categories are provided.

1 Preliminaries

To begin, some terminology has to be set: vector spaces are over $\mathbb{C}$ and algebras are implicitly assumed associative and *commutative* (unless stated explicitly), but not necessarily unital. Algebra maps (also called morphisms of algebras or semigroup maps or semigroup morphisms) are thus not required to preserve a unit. An *orthogonal set* (or *family*) of a Hilbert space is a subset of pairwise orthogonal *non-zero* vectors, such as e.g. $\varnothing$ but not $\{0\}$. An *orthonormal set* (or *family*) is an orthogonal set consisting of vectors of norm 1. An *orthogonal basis* is an orthogonal set X such that $X^\perp = 0$.

In Table 1 below, are listed the categories of algebras or of (pointed) sets used hereafter together with the page number of their introduction.

Categories of algebras

Name	**Objects**	**Morphisms**
$_c\mathbf{Sem}(\mathbb{Hilb})$ (p. 6)	Hilbertian algebras (H.A.)	Bounded algebra maps
$_c\mathbf{Sem}(\mathbb{FdHilb})$	Finite dimensional (F.D.) H.A.	Algebra maps
$^\dagger_c\mathbf{Sem}(\mathbb{Hilb})$ (p. 7)	Special H.A.	Bounded algebra maps
$_{coc}\mathbf{Cosem}(\mathbb{Hilb})$ (p. 6)	Hilbertian coalgebras	Bounded coalgebra maps
$_{\mathsf{semisimple}}\mathbf{C}$ (p. 8)	Semisimple algebras in $\mathbf{C}$	$\mathbf{C}$-morphisms
$_c\mathbf{FrobSem}(\mathbb{Hilb})$ (p. 9)	Frobenius H.A.	Bounded algebra maps
$_c\mathbf{FrobSem}(\mathbb{FdHilb})$ (p. 9)	F.D. Frobenius H.A.	Algebra maps
$^\dagger_c\mathbf{FrobSem}(\mathbb{Hilb})$ (p. 9)	Special Frobenius H.A.	Bounded algebra maps
$^\dagger_c\mathbf{FrobSem}(\mathbb{FdHilb})$ (p. 9)	F.D. special Frobenius H.A.	Algebra maps
$_c\mathbf{Mon}(\mathbb{FdHilb})$ (p. 9)	Monoids in $\mathbb{FdHilb}$	Unital algebra maps
$_c\mathbf{FrobMon}(\mathbb{FdHilb})$ (p. 9)	Frobenius monoids in $\mathbb{FdHilb}$	Unital algebra maps
$_{coc}\mathbf{Comon}(\mathbb{FdHilb})$ (p. 9)	Comonoids in $\mathbb{FdHilb}$	Counital coalgebra maps
$_{coc}\mathbf{FrobComon}(\mathbb{FdHilb})$ (p. 9)	Frobenius comonoids in $\mathbb{FdHilb}$	Counital coalgebra maps
$_{\mathsf{radical},c}\mathbf{FrobSem}(\mathbb{Hilb})$ (p. 23)	Radical Frobenius H.A.	Bounded algebra maps
$_{\mathsf{partiso},c}\mathbf{FrobSem}(\mathbb{Hilb})$ (p. 25)	Frobenius H.A. with a partial isometric comultiplication	Bounded algebra maps
$_{\mathsf{bnd},c}\mathbf{FrobSem}(\mathbb{Hilb})$ (p. 28)	"Bounded" Frobenius H.A.	Bounded algebra maps
$_{\mathsf{unbnd},c}\mathbf{FrobSem}(\mathbb{Hilb})$ (p. 28)	"Unbounded" Frobenius H.A.	Bounded algebra maps
$_{\mathsf{partiso},c}\mathbf{FrobBisem}(\mathbb{Hilb})$ (p. 33)	Frobenius H.A. with a partial isometric comultiplication	Bounded algebra and coalgebra maps
$^\dagger_c\mathbf{FrobBisem}(\mathbb{Hilb})$ (p. 33)	Special Frobenius H.A.	Bounded algebra and coalgebra maps
$_c\mathbf{Frob}(\mathbb{Hilb})_{\mathsf{ambi}}$ (p. 34)	Frobenius H.A.	Bounded algebra and coalgebra maps
$_c\mathbf{FrobSem}(\mathbb{Hilb})_{\mathsf{proper}}$ (p. 35)	Frobenius H.A.	Proper algebra maps
$_{1,c}\mathbf{Frob}(\mathbb{FdHilb})_{\mathsf{ambi}}$ (p. 37)	Frobenius monoids in $\mathbb{FdHilb}$	Unital algebra and counital coalgebra maps

Set-like categories

Name	**Objects**	**Morphisms**
$\mathbf{WSet}_\bullet$ (p. 25)	Weighted pointed sets	$\mathbf{WSet}_\bullet$-morphisms
$\mathbf{Set}_{\bullet,<+\infty}$ (p. 25)	Pointed sets	Base-point preserving maps with bounded fibers
$\mathsf{bnd}\mathbf{WSet}_\bullet$ (p. 25)	Bounded weighted pointed sets	$\mathbf{WSet}_\bullet$-morphisms
$\mathsf{unbnd}\mathbf{WSet}_\bullet$ (p. 25)	Unbounded weighted pointed sets	$\mathbf{WSet}_\bullet$-morphisms
$\mathbf{FinSet}_\bullet$ (p. 30)	Pointed finite sets	Base-point preserving maps
$\mathbf{WFinSet}_\bullet$ (p. 31)	Weighted pointed finite sets	$\mathbf{WSet}_\bullet$-morphisms
$\mathbf{PInj}_\bullet$ (p. 34)	Pointed sets	Partial injections
$\mathbf{PInj}_{\bullet,\mathsf{w}}$ (p. 34)	Weighted pointed sets	Weight-preserving partial injections
$\mathbf{WSet}$ (p. 35)	Weighted sets	$\mathbf{WSet}$-morphisms
$\mathbf{WFinSet}$ (p. 36)	Weighted finite sets	$\mathbf{WSet}$-morphisms
$\mathbf{FinSet}$ (p. 36)	Finite sets	Maps
$\mathbf{FinSet}_{\mathsf{bij,w}}$ (p. 37)	Weighted finite sets	Weight-preserving bijections

Table 1: Categories.

Let us now summarize some notation and results from [22] as far as they are needed hereafter.

1.1 Hilbert spaces

When E is a Banach space, $\mathcal{B}(E)$ stands for the Banach space of all bounded linear endomorphisms of E with the operator norm $\|-\|_{\mathsf{op}}$. The obvious forgetful functor from Hilbert spaces to Banach spaces, with bounded linear maps as morphisms for both categories, is denoted U but there is no risk to identify – as we shall do hereafter – a Hilbert space H with its underlying Banach space as U is injective on objects.[1] The inner product (linear in its first variable) of a Hilbert space H is denoted $\langle\cdot,\cdot\rangle_H$ or simply $\langle\cdot,\cdot\rangle$. Basic properties about the Hilbertian tensor product (or tensor product of Hilbert spaces) $\hat{\otimes}_2$ are provided in [15]. Given a bounded linear map $\mu\colon H\hat{\otimes}_2 K\to L$, where H,K,L are Hilbert spaces, $\mu_{\mathsf{bil}}\colon H\times K\to L$ denotes its (unique) associated bounded bilinear map. For a bounded multilinear or linear map f, $\|f\|_{\mathsf{op}}$ stands for its usual *operator norm*. The *Hilbert direct sum* (or *orthogonal direct sum*) of Hilbert spaces H,K is denoted $H\oplus_2 K$. Finally given a closed subspace V of H, $p_V\colon H\to H$ denotes the orthogonal projection onto V, that is, $p_V = i_V\circ\pi_V$, where $\pi_V\colon H\to V$ is the canonical projection $H\simeq V\times V^{\perp}\to V$ and $i_V\colon V\hookrightarrow H$ is the canonical inclusion.

1.2 Hilbertian algebras

Let $\mathbb{Hilb} := (\mathbf{Hilb},\hat{\otimes}_2,\mathbb{C})$ be the symmetric monoidal category of complex Hilbert spaces and bounded linear maps together with the tensor product of Hilbert spaces. The associativity constraint $\alpha_{H,K,L}\colon(H\hat{\otimes}_2 K)\hat{\otimes}_2 L\simeq H\hat{\otimes}_2(K\hat{\otimes}_2 L)$ and the symmetry constraint $\sigma_{H,K}\colon H\hat{\otimes}_2 K\simeq K\hat{\otimes}_2 H$ are unitary transformations. Let $\mathbb{FdHilb} := (\mathbf{FdHilb},\otimes_2,\mathbb{C})$ be its monoidal subcategory of finite-dimensional Hilbert spaces and (necessarily bounded) linear maps. $H\otimes_2 K$ thus is the finite-dimensional vector space $H\otimes_{\mathbb{C}} K$ together with the inner product $\langle u\otimes v, u'\otimes v'\rangle = \langle u,u'\rangle_H\langle v,v'\rangle_K$, $u,u'\in H$, $v,v'\in K$.

A semigroup object (H,μ) in $\mathbb{Hilb}$ has an *underlying* Banach algebra (H,μ_{bil}). Note that $\|\mu_{\mathsf{bil}}(x,y)\|\leq M\|x\|\|y\|$ for some constant M, where $\|-\|$ is the norm induced by the inner product on H. So it may happen that strictly speaking, (H,μ_{bil}) is not a Banach algebra (i.e., $\|-\|$ is not submultiplicative). However $\|x\|' := \max\{1,\|\mu_{\mathsf{bil}}\|_{\mathsf{op}}\}\|x\|$ defines a submultiplicative norm, that is, $\|\mu_{\mathsf{bil}}(x,y)\|'\leq\|x\|'\|y\|'$, equivalent to $\|-\|$. In other words $((H,\|-\|'),\mu_{\mathsf{bil}})$ is a usual Banach algebra, which is the underlying Banach algebra of (H,μ).

An *ideal* I of (H,μ) is defined as an ideal of the underlying Banach algebra (H,μ_{bil}). When I is closed, this is equivalent to the requirement that $\mu((I^{\perp}\hat{\otimes}_2 I^{\perp})^{\perp})\subseteq I$. Note that $(I^{\perp}\hat{\otimes}_2 I^{\perp})^{\perp} = (I\hat{\otimes}_2 I^{\perp})\oplus_2(I\hat{\otimes}_2 I)\oplus_2(I^{\perp}\hat{\otimes}_2 I)$ (see [22, Lemma 12, p. 13]).

By $\mathbf{Sem}(\mathbb{C})$, ${}_c\mathbf{Sem}(\mathbb{C})$, and ${}_{coc}\mathbf{Cosem}(\mathbb{C})$ are meant respectively the categories of semigroups, commutative semigroups, and cocommutative cosemigroups in a sym-

[1] Let H,K be Hilbert spaces such that $U(H)=U(K)$, then as complex vector spaces $H=K$. The parallelogram law implies that the norms of $U(H)$ and $U(K)$ comes from a common inner product, and thus $H=K$ as Hilbert spaces.

metric monoidal category $\mathbb{C}$. A *Hilbertian (co)algebra* (or *(co)semigroup*) is an object of ${}_c\mathbf{Sem}(\mathbb{Hilb})$ (resp., ${}_{coc}\mathbf{Cosem}(\mathbb{Hilb})$) while by *special* is meant a Hilbertian algebra (H,μ) with a coisometric multiplication, that is, $\mu\circ\mu^\dagger=\mathrm{id}$. ${}_c^\dagger\mathbf{Sem}(\mathbb{Hilb})$ is the full subcategory of ${}_c\mathbf{Sem}(\mathbb{Hilb})$ spanned by the special Hilbertian algebras.

The Hilbert adjoint (or dagger) functor $\mathbf{Hilb}^{\mathsf{op}}\xrightarrow{(-)^\dagger}\mathbf{Hilb}$ lifts to an isomorphism from the category ${}_{coc}\mathbf{Cosem}(\mathbb{Hilb})^{\mathsf{op}}$ into ${}_c\mathbf{Sem}(\mathbb{Hilb})$. This is still true after the substitution of $\mathbb{Hilb}$ by $\mathbb{FdHilb}$.

By a *subalgebra* V of a Hilbertian algebra (H,μ) is meant a *closed* subspace V of H such that $\mu(V\hat{\otimes}_2V)\subseteq V$, where $V\hat{\otimes}_2V$ is identified with the range of $V\hat{\otimes}_2V\xrightarrow{i_V\hat{\otimes}_2 i_V}H\hat{\otimes}_2H$. $(V,\mu_{|V})$ is a Hilbertian algebra in its own right with multiplication $V\hat{\otimes}_2V\xrightarrow{\mu_{|V}}V$ the restriction and co-restriction of $H\hat{\otimes}_2H\xrightarrow{\mu}H$.

By a *subcoalgebra* V of (H,μ) (or of $(H,\mu^\dagger)$) is meant a *closed* subspace V of H such that $\mu^\dagger(V)\subseteq V\hat{\otimes}_2V$. $(V,(\mu^\dagger)_{|V})$ is a Hilbertian coalgebra in its own right with comultiplication $V\xrightarrow{(\mu^\dagger)_{|V}}V\hat{\otimes}_2V$ the restriction and co-restriction of $H\hat{\otimes}_2H\xrightarrow{\mu^\dagger}H$.

When (H,μ) is a Hilbertian algebra, xy stands for $\mu(x\otimes y)$ and x^2 for $\mu(x\otimes x)$, $x,y\in H$. In what follows (co)semigroup morphisms are also referred to as *(co)algebra maps*.

1.1 Lemma *Let $(H,\mu),(K,\gamma)$ be Hilbertian algebras. Let $f\colon H\to K$ be a bounded linear map.*

1. *For all $u,v\in H$, $f(uv)=f(u)f(v)$ if, and only if, $f\colon(H,\mu)\to(K,\gamma)$ is a semigroup morphism if, and only if, $f^\dagger\colon(K,\gamma^\dagger)\to(H,\mu^\dagger)$ is a cosemigroup morphism.*

2. *$f\colon(H,\mu)\to(K,\gamma)$ (resp. $f\colon(H,\mu^\dagger)\to(K,\gamma^\dagger)$) is a (co)semigroup isomorphism if, and only if, f is both a (co)semigroup morphism and a bijection.*

Proof: Only the converse implication of the second equivalence needs a proof: by the Open Mapping Theorem, since $f\colon H\to K$ is bounded and bijective, $f\colon H\to K$ has a bounded inverse $f^{-1}\colon K\to H$. That f^{-1} is a semigroup morphism follows easily from the first point above. □

1.3 Semisimplicity

The *Jacobson radical* or *radical* $J(A)$, or J when there is no risk of confusion, of a not necessarily commutative nor unital algebra A is the intersection of all its maximal modular left (or right) ideals [20, Theorem 4.3.6, p. 476]. The Jacobson radical $J(H,\mu)$ of a Hilbertian algebra (H,μ) is defined as the Jacobson radical $J(H,\mu_{\mathsf{bil}})$ of its underlying Banach algebra $((H,\|-\|'),\mu_{\mathsf{bil}})$. (H,μ) is *semisimple* (resp. *radical*) when so is the Banach algebra $((H,\|-\|'),\mu_{\mathsf{bil}})$.

$J(H,\mu)$ also has an intrinsic description: Let $G(H,\mu) \coloneqq \{ x \in H \setminus \{0\} \colon \mu^\dagger(x) = x \otimes x) \}$ be the set of all *group-like elements* of (H,μ). Then, $J(H,\mu) = G(H,\mu)^\perp$ and $\overline{\langle G(H,\mu)\rangle} = J(H,\mu)^\perp$, where here and elsewhere $\langle X \rangle$ denotes the linear span, and $\overline{X}$ the closure, of a subset X of H. As a consequence of the Riesz representation theorem, the map $G(H,\mu) \xrightarrow{R} \mathsf{char}(H,\mu_{\mathsf{bil}})$, $x \mapsto \langle \cdot, x\rangle$ is bijective ([22, Lemma 19, p. 17]), where $\mathsf{char}(A)$ is the set of non-trivial characters of a Banach algebra A.

1.2 Lemma *Let (H,μ) be a Hilbertian algebra and let V be a closed subspace of H which is both a subalgebra and a subcoalgebra. Then, $(\mu_{|V})^\dagger = (\mu^\dagger)_{|V}$ and $G(V,\mu_{|V}) = G(H,\mu) \cap V$.*

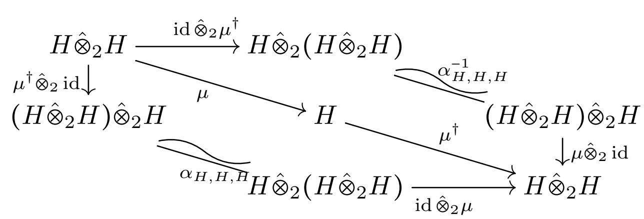
1.3 Lemma *Let $(H,\mu),(K,\gamma)$ be Hilbertian algebras. Let $f\colon H \to K$ be a bounded linear map. If $f\colon (H,\mu^\dagger) \to (K,\gamma^\dagger)$ is a coalgebra map, then $f(G(H,\mu)) \subseteq G(K,\gamma) \cup \{0\}$. When (H,μ) is semisimple, the converse also holds.*

Proof: The first statement is [22, Lemma 25, p. 19]. Let us assume that (H,μ) is semisimple. Then, by assumption and linearity, $(f \hat{\otimes}_2 f) \circ \mu^\dagger = \gamma^\dagger \circ f$ on $\langle G(H,\mu)\rangle$. By continuity the maps are equal on $\overline{\langle G(H,\mu)\rangle} = J(H,\mu)^\perp = H$ (H semisimple). □

1.1 Notation If $\mathbf{C}$ is a subcategory of ${}_c\mathbf{Sem}(\mathbb{Hilb})$ or of ${}_c\mathbf{Sem}(\mathbb{FdHilb})$, then ${}_{\mathsf{semisimple}}\mathbf{C}$ stands for the full subcategory of $\mathbf{C}$ spanned by the Hilbertian algebras in $\mathbf{C}$ which are semisimple.

1.4 Hilbertian Frobenius algebras

Let H be a Hilbert space and let $H \hat{\otimes}_2 H \xrightarrow{\mu} H$ be a bounded linear map. Let us consider the following diagram where $\alpha_{H,H,H}$ is the component at (H,H,H) of the coherence constraint of associativity of $\mathbb{Hilb}$.



(1)

One says that (H,μ) satisfies the *Frobenius condition* – or that (H,μ) is *Frobenius* – when the top and the bottom cells of Diag.(1) commute. In this case, the surrounding diagram commutes too.

By a *Hilbertian Frobenius algebra* (or *semigroup*) is meant a Hilbertian algebra which satisfies the Frobenius condition. (Such objects are referred to as *commutative †-Frobenius semigroups* in [6].) A Hilbertian Frobenius algebra (H,μ) is said to be *special* when furthermore $\mu \circ \mu^\dagger = \mathrm{id}$.

Note that any Hilbert space with the zero multiplication is a Hilbertian Frobenius algebra. Also any semisimple special Hilbertian algebra (H,μ) is a Hilbertian Frobenius algebra: indeed $G(H,\mu)$ is an orthonormal family [22, Lemma 27, p. 21], and so an orthonormal basis of $J(H,\mu)^{\perp}$, thus of H by semisimplicity. For each $x,y \in G(H,\mu)$, $x^2 = \mu(x\otimes x) = \mu(\mu^{\dagger}(x)) = x$ and $xy = \delta_{x,y}x$ (since by [22, Corollary 28, p. 982], $x \neq y \Rightarrow xy \in J(H,\mu)$). So more generally for $u,v \in H$, $\mu(u\otimes v) = \sum_{x\in G(H,\mu)}\langle u,x\rangle\langle v,x\rangle x$. Consequently $\mu^{\dagger}(u) = \sum_{x\in G(H,\mu)}\langle u,x\rangle x\otimes x$ and then one sees by direct inspection that $\mu^{\dagger}$ satisfies the Frobenius compatibility relations.

1.1 Remark Since (H,μ) is assumed commutative, it is not difficult to check that the Frobenius condition actually reduces to the commutativity of only one of the two cells of Diag. (1). One thus recovers the definition of a Frobenius algebra in $\mathbb{Hilb}$ from [2].

Let ${}_c\mathbf{FrobSem}(\mathbb{Hilb})$ and ${}^{\dagger}_c\mathbf{FrobSem}(\mathbb{Hilb})$ be respectively the full subcategories of ${}_c\mathbf{Sem}(\mathbb{Hilb})$ spanned by the Hilbertian Frobenius algebras and by the special Hilbertian Frobenius algebras. One obtains corresponding categories after the replacement of $\mathbb{Hilb}$ by $\mathbb{FdHilb}$. Still in the finite-dimensional situation one may as well consider *Hilbertian Frobenius monoids*, that is, commutative monoid objects (H,μ,η) in $\mathbb{FdHilb}$ such that (H,μ) is a finite-dimensional Hilbertian Frobenius semigroup. Let ${}_c\mathbf{FrobMon}(\mathbb{FdHilb})$ be the full subcategory of the category ${}_c\mathbf{Mon}(\mathbb{FdHilb})$ of monoid objects of $\mathbb{FdHilb}$, they generate. Finally let us call a (cocommutative) comonoid (H,δ,ϵ) in $\mathbb{FdHilb}$ a *finite-dimensional Hilbertian Frobenius comonoid* when $(H,\delta^{\dagger},\epsilon^{\dagger})$ is a Hilbertian Frobenius monoid. The full subcategory ${}_{coc}\mathbf{FrobComon}(\mathbb{FdHilb})$ of the category of comonoid objects ${}_{coc}\mathbf{Comon}(\mathbb{FdHilb})$ in $\mathbb{FdHilb}$, they generate is of course isomorphic to ${}_c\mathbf{FrobMon}(\mathbb{FdHilb})^{\mathsf{op}}$ under the dagger functor.

1.5 Example: Weighted Hilbert spaces

Let X be a non empty set, and let $w: X \to [C,+\infty[$ be a map, where $C > 0$. Let $\ell^2_w(X) := \{ f \in \mathbb{C}^X : \sum_{x\in X} w(x)|f(x)|^2 < +\infty \}$. With inner product $\langle f,g\rangle_w := \sum_{x\in X} w(x)f(x)\overline{g(x)}$ this provides a Hilbert space. The corresponding norm is denoted $\|-\|_w$. For $x\in X$, let us identify $\delta_x: X \to \mathbb{C}$ with x itself. Under this identification, $\{ \frac{x}{w(x)^{\frac{1}{2}}} : x\in X \}$ forms a Hilbertian basis of $\ell^2_w(X)$. Note that $\langle f, \frac{x}{w(x)^{\frac{1}{2}}}\rangle_w = w(x)^{\frac{1}{2}}f(x)$, $x\in X$. The next result is clear.

1.4 Lemma *$\ell^2_w(X) \subseteq \ell_2(X)$, the inclusion is bounded and $\overline{\ell^2_w(X)} = \ell^2(X)$. Furthermore if w is also bounded above, then $\ell^2_w(X) = \ell_2(X)$ as vector spaces and the norms $\|-\|_w$ and $\|-\|$ are equivalent.*

Let $m_X: \ell^2_w(X)\times\ell^2_w(X) \to \ell^2_w(X)$ be given by $m_X(f,g) := fg$ where by juxtaposition is denoted the pointwise product of maps. m_X is a weak Hilbert-Schmidt mapping ([15]) as $\sum_{x,y\in X}|\langle m_X(\frac{x}{w(x)^{\frac{1}{2}}},\frac{x}{w(x)^{\frac{1}{2}}}),f\rangle_w|^2 \le \frac{1}{C}\|f\|_w^2$ for each $f\in\ell_w^2(X)$. Let $\mu_X:\ell_w^2(X)\hat{\otimes}_2\ell_w^2(X)\to\ell_w^2(X)$ be the corresponding bounded linear map, that is, $\mu_X(f\otimes g)=fg$ (see [15, Theorem 2.6.4, p. 132]). In details, $\mu_X(f\otimes g)=\sum_{x\in X}f(x)g(x)x$.

It is clear by its very definition that μ_X is commutative and associative making $(\ell_w^2(X),\mu_X)$ a Hilbertian algebra. Moreover $\|\mu_X\|_{\mathsf{op}}\le\frac{1}{C^{\frac{1}{2}}}$. Such an algebra $(\ell_w^2(X),\mu_X)$ was already considered in [22, pp. 11-12] in the situation where $C=1$. By direct computations one obtains

1.5 Proposition *$G(\ell_w^2(X),\mu_X)=\{\frac{x}{w(x)}:x\in X\}$ and $(\ell_w^2(X),\mu_X)$ is a semisimple Hilbertian Frobenius algebra.*

Letting $w\equiv 1$ one observes that $(\ell^2(X),\mu_X)$ is also a (special) Hilbertian Frobenius algebra, with $G(\ell^2(X),\mu_X)=X$.

The following lemma, the proof of which is not difficult, provides the relation between $\ell_w^2(X)$ and $\ell^2(X)$ as algebras.

1.6 Lemma *$(\ell_w^2(X),m_X)$ is a not necessarily closed ideal of $(\ell^2(X),m_X)$. In fact $\ell_w^2(X)=\ell^2(X)$ if, and only if, $\ell_w^2(X)$ is closed in $\ell^2(X)$ if, and only if, w is bounded above.*

Let us also provide a description of $(\ell_w^2(X),\mu_X)$ under another disguise. Under the unitary transformation $\Phi:f\mapsto\hat{f}$ from $\ell_w^2(X)$ to $\ell^2(X)$, with $\hat{f}(x):=\langle f,\frac{x}{w(x)^{\frac{1}{2}}}\rangle_w=w(x)^{\frac{1}{2}}f(x)$, $x\in X$, one may transport the multiplication μ_X on $\ell^2(X)$. In detail, the inverse $\Phi^\dagger$ of Φ is given by $\Phi^\dagger(f):=w^{-\frac{1}{2}}f$, that is, $(\Phi^\dagger(f))(x)=\frac{1}{w(x)^{\frac{1}{2}}}f(x)$, $x\in X$, and then one may define $\mu_{w,X}:\ell^2(X)\hat{\otimes}_2\ell^2(X)\to\ell^2(X)$ by $\Phi\circ\mu_X\circ(\Phi^\dagger\hat{\otimes}_2\Phi^\dagger)$, so for each $f,g\in\ell^2(X)$, $\mu_{w,X}(f\otimes g)=w^{-\frac{1}{2}}fg$, that is, for each $x\in X$, $(\mu_{w,X}(f\otimes g))(x)=w(x)^{-\frac{1}{2}}f(x)g(x)$. (Note that as $C\le w(x)$, $\frac{1}{w(x)^{\frac{1}{2}}}\le\frac{1}{C^{\frac{1}{2}}}$ for each $x\in X$, and thus pointwise multiplication of functions by $w^{-\frac{1}{2}}$ is an operator on $\ell^2(X)$, and as $\ell^2(X)$ is closed under pointwise products, given $f,g\in\ell^2(X)$, $w^{-\frac{1}{2}}fg\in\ell^2(X)$.) It is clear that $(\ell^2(X),\mu_{w,X})$ is a semisimple Frobenius algebra, unitarily isomorphic to $(\ell_w^2(X),\mu_X)$. Note that $G(\ell^2(X),\mu_{w,X})=\{\frac{x}{w(x)^{\frac{1}{2}}}:x\in X\}$.

1.2 Remark Everything becomes trivial if $X=\varnothing$ (with w the empty map), that is, $\ell_w(X)$ is the zero algebra, which is Frobenius, semisimple and radical.

2 Semisimplicity of commutative Hilbertian Frobenius algebras

The main results of this section are Theorem 2.14 and Theorem 2.17. The former states that both the Jacobson radical of a Hilbertian Frobenius algebra, and its orthogonal complement are subalgebras and subcoalgebras, and that the Jacobson radical actually coincides with the annihilator of the algebra. The latter provides the explicit conditions on the multiplication (or comultiplication) of a Hilbertian Frobenius algebra to be semisimple.

2.1 Commutative Hilbertian Frobenius algebras

Given a not necessarily commutative algebra A, $E(A)$ denotes the set of all its idempotent elements, that is, the members e of A such that $e^2 = e$. It is well-known that when A is a commutative Banach algebra, then $E(A) \cap J(A) = (0)$ (see for instance [20, Proposition 4.3.12.(a), p. 479]). The set of idempotent elements of a Hilbertian algebra (H, μ) is $E(H, \mu) \coloneqq E(H, \mu_{\mathsf{bil}})$.

In this current section, (H, μ) stands for a commutative Hilbertian Frobenius algebra and one denotes $J \coloneqq J(H, \mu)$.

Let us recall the following result from [6] (and recall also that for us an orthogonal family does not contain 0).

2.1 Lemma *$G(H, \mu)$ is an orthogonal family, that is, for each $x, y \in G(H, \mu)$, $x \neq y \Rightarrow \langle x, y \rangle = 0$. In particular $G(H, \mu)$ is an orthogonal basis of $J^{\perp}$.*

Lemma 2.1 has the important consequences listed below.

2.2 Corollary
1. *Let $x, y \in G(H, \mu)$ such that $x \neq y$. Then, $xy \in J$.*
2. *Let $x \in G(H, \mu)$. Then, $p_{J^{\perp}}(x^2) = \|x\|^2 x$.*

Proof:

1. Let $x, y \in G(H, \mu)$ such that $x \neq y$. Let $z \in G(H, \mu)$. Then, $\langle xy, z \rangle = \langle \mu(x \otimes y), z \rangle = \langle x \otimes y, \mu^{\dagger}(z) \rangle = \langle x \otimes y, z \otimes z \rangle = \langle x, z \rangle \langle y, z \rangle = \|x\|^2 \delta_{x,z} \|y\|^2 \delta_{y,z}$ (by Lemma 2.1) $= 0$ as by assumption $x \neq y$. Whence $p_{J^{\perp}}(xy) = 0$ and thus $xy \in J$.

2. Let $x \in G(H, \mu)$. Let $z \in G(H, \mu)$. Then, $\langle x^2, z \rangle = \langle \mu(x \otimes x), z \rangle = \langle x \otimes x, \mu^{\dagger}(z) \rangle = \langle x \otimes x, z \otimes z \rangle = \langle x, z \rangle^2 = \|x\|^4 \delta_{x,z}$ according to Lemma 2.1. As a result, $p_{J^{\perp}}(x^2) = \langle x^2, \frac{x}{\|x\|} \rangle \frac{x}{\|x\|} = \|x\|^2 x$.

□

2.1 Remark According to Corollary 2.2 a semisimple Hilbertian Frobenius algebra (H,μ) is easily described: as $G(H,\mu)$ is an orthogonal basis of $H = J^{\perp}$, one has that each $u \in H$ is the sum of the summable family $\sum_{x\in X}\langle u, \frac{x}{\|x\|}\rangle \frac{x}{\|x\|}$ and given $u,v \in H$, $uv = \sum_{x\in G(H,\mu)} \frac{1}{\|x\|}\langle u,x\rangle\langle v,x\rangle\frac{x}{\|x\|}$ as $\langle uv,x\rangle = \langle u\otimes v, x\otimes x\rangle = \langle u,x\rangle\langle v,x\rangle$, $x\in G(H,\mu)$.

2.3 Lemma *Let $x \in G(H,\mu)$. $x^2 \in J^{\perp}$ if, and only if, $x^2 = \|x\|^2 x$. In this case, $\frac{1}{\|x\|^2}x$ is an idempotent element which belongs to $J^{\perp}$ and $\|x\| \le \|\mu\|_{\mathsf{op}}$.*

Proof: The first equivalence is due to Corollary 2.2. The second statement is immediate. Let e be a non-zero idempotent element of (H,μ). Then, $\|e\| = \|e^2\| = \|\mu(e\otimes e)\| \le \|\mu\|_{\mathsf{op}}\|e\|^2$. If $e \ne 0$, then $\frac{1}{\|\mu\|_{\mathsf{op}}} \le \|e\|$. The last statement thus is obtained by taking $e = \frac{x}{\|x\|^2}$. □

Let $G'(H,\mu) \coloneqq \{\, x \in G(H,\mu) : x^2 \in J^{\perp} \,\} = \{\, x \in G(H,\mu) : x^2 = \|x\|^2 x \,\}$ (by Lemma 2.3).

2.4 Lemma *Let $x,y \in G'(H,\mu)$ such that $x \ne y$. Then, $xy = 0$.*

Proof: According to Corollary 2.2 $xy \in J$. As $x,y \in G'(H,\mu)$, $\frac{x}{\|x\|^2}$ and $\frac{y}{\|y\|^2}$ both are idempotent elements which belong to $J^{\perp}$ by Lemma 2.3. By commutativity of μ, $\frac{xy}{\|x\|^2\|y\|^2}$ is also an idempotent element and it is a member of J by the above. Therefore it reduces to zero (see the beginning of the current section), and thus $xy = 0$ too. □

2.5 Lemma *$G'(H,\mu) = G(H,\mu)$. Consequently, $J^{\perp}$ is a closed subalgebra of (H,μ) and the map $G(H,\mu) \to]0,+\infty[$, $x \mapsto \|x\|$ is bounded above by $\|\mu\|_{\mathsf{op}}$.*

Proof: Let $x \in G(H,\mu)$. Using one of the Frobenius conditions, one obtains $\mu^{\dagger}(x^2) = x^2\otimes x$ and with the other, $\mu^{\dagger}(x^2) = x\otimes x^2$. Let $u,v\in H$. Then, $\langle \mu^{\dagger}(x^2), u\otimes v\rangle = \langle x\otimes x^2, u\otimes v\rangle = \langle x,u\rangle\langle x^2,v\rangle$ and also $\langle\mu^{\dagger}(x^2), u\otimes v\rangle = \langle x^2\otimes x, u\otimes v\rangle = \langle x^2,u\rangle\langle x,v\rangle$. In particular given $u\in J$, one has $\langle \mu^{\dagger}(x^2), u\otimes x\rangle = \langle x^2,x\rangle\langle x,u\rangle = 0$ and $\langle\mu^{\dagger}(x^2), u\otimes x\rangle = \langle x,x\rangle\langle x^2,u\rangle = \|x\|^2\langle x^2,u\rangle$. Therefore $x^2\in J^{\perp}$, that is, $x\in G'(H,\mu)$.

According to Lemmas 2.3 and 2.4, the linear span of $G'(H,\mu) = G(H,\mu)$ is closed under μ. So is its closure, by continuity of μ, which is nothing but $J^{\perp}$. The last assertion is a consequence of the last assertion of Lemma 2.3. □

2.2 Remark It is clear from the definition of a group-like element, that $\mu = 0 \Rightarrow G(H,\mu) = \varnothing \Rightarrow (H,\mu)$ is radical. Actually we will prove below that the converse implications are also true (see Corollary 2.16).

2.3 Remark According to Lemma 2.5 the multiplication of $J^{\perp}$ arising from the restriction of μ is as easily described as in Remark 2.1. $G(H,\mu)$ is an orthogonal basis of $J^{\perp}$, and given $u,v\in H$, $uv = \sum_{x\in G(H,\mu)}\frac{1}{\|x\|}\langle u,x\rangle\langle v,x\rangle\frac{x}{\|x\|} + (p_{J^{\perp}}(u)p_J(v) + p_J(u)p_{J^{\perp}}(v) + p_J(u)p_J(v))$.

Since J is an ideal, $J^{\perp}$ is a subcoalgebra of (H,μ) by [22, Theorem 18, p. 975] and so according to Lemma 2.5, $J^{\perp}$ is both a subalgebra and a subcoalgebra of (H,μ).

2.6 Corollary *Let (H,μ) be a Hilbertian Frobenius algebra. Under the restriction of μ, $J^{\perp}$ is a semisimple Hilbertian Frobenius algebra.*

2.2 Multiplication operators

Let $(E,*)$ be a commutative Banach algebra. The *annihilator* $A(E,*)$ of $(E,*)$ is $\{\, u \in E \colon \forall v \in E,\ u * v = 0 \,\}$. For a Hilbertian algebra (H,μ) let $A(H,\mu) := A(H,\mu_{\mathsf{bil}})$. The following result is almost immediate.

2.7 Lemma $A(H,\mu) \subseteq J(H,\mu)$.

Let $(E,*)$ be a complex commutative Banach algebra. Let $M\colon E \to \mathcal{B}(E)$ be the regular representation of $(E,*)$, that is, $u \mapsto M_u$, where $M_u(v) = u * v$. One notices that $A(E,*) = \ker M$.

Let $(B,*)$ be a not necessarily commutative Banach algebra and let $u \in B$. u is said to be a *quasi-nilpotent* element when its spectral radius is equal to zero, that is, when $\|u^n\|^{\frac{1}{n}} \to 0$ ([20, p. 213]). In general the Jacobson radical of $(B,*)$ is only contained in the set of all quasi-nilpotent elements but as soon as $(B,*)$ is commutative both sets are equal [16, Corollary 2.2.6, p. 55]. If H is a Hilbert space, a linear map $H \xrightarrow{f} H$ is referred to as a *quasi-nilpotent operator* when it is a quasi-nilpotent element of $\mathcal{B}(H)$.

2.8 Lemma *Let $u \in J(H,\mu)$. Then, M_u is a quasi-nilpotent operator on H. In other words, M maps the Jacobson radical of (H,μ) into the set of all quasi-nilpotent operators on H.*

Proof: Let $v \in H$. Then, $\|M_u^n(v)\| = \|u^n v\| \leq \|\mu_{\mathsf{bil}}\|_{\mathsf{op}}\|u^n\|\|v\| \leq \|u^n\|'\|v\|$ so that $\|M_u^n\|_{\mathsf{op}} \leq \|u^n\|'$ for each $n \in \mathbb{N}\setminus\{0\}$. Consequently, $\|M_u^n\|_{\mathsf{op}}^{\frac{1}{n}} \leq (\|u^n\|')^{\frac{1}{n}} \to 0$ as u is a quasi-nilpotent element of the Banach algebra $((H,\|-\|'),\mu_{\mathsf{bil}})$ ([16, Corollary 2.2.6, p. 55]). □

A bounded linear operator $H \xrightarrow{f} H$ over a Hilbert space H is said to be *normal* when $f \circ f^{\dagger} = f^{\dagger} \circ f$.

2.9 Corollary *Let (H,μ) be a Hilbertian algebra. Let $u \in J(H)$. M_u is normal if, and only if, $u \in A(H,\mu)$. So $J(H,\mu) \cap \{\, u \in H \colon M_u \text{ is normal} \,\} = A(H,\mu)$.*

Proof: The converse implication is clear since the zero operator is normal. So let us assume that M_u is normal. By Lemma 2.8, M_u is quasi-nilpotent, whence its spectral radius is equal to zero ([20, p. 213]). But for normal operators the spectral radius coincides with the operator norm ([5, II.1.6.3, p. 58]). Whence $\|M_u\|_{\mathsf{op}} = 0$, that is, $M_u = 0$. The second statement follows from the first one and Lemma 2.7. □

2.3 Frobenius algebras revisited: Hilbertian modules

Let (H,μ) be an object of $\mathbf{Sem}(\mathbb{Hilb})$ and let K be a Hilbert space. Let $g\colon H\hat{\otimes}_2 K \to K$ be a bounded linear map such that the following diagram on the left commutes. (The isomorphism arrow corresponds to the coherence constraint of associativity of $\mathbb{Hilb}$.)

$$
\begin{array}{ccc}
 & H\hat{\otimes}_2(H\hat{\otimes}_2K) \xrightarrow{\mathrm{id}\,\hat{\otimes}_2 g} H\hat{\otimes}_2K & \\
(H\hat{\otimes}_2H)\hat{\otimes}_2K \overset{\sim}{\nearrow} & & \downarrow g \\
\xrightarrow{\mu\hat{\otimes}_2\,\mathrm{id}} & H\hat{\otimes}_2K \xrightarrow{\;g\;} K &
\end{array}
\qquad
\begin{array}{ccc}
 & (K\hat{\otimes}_2H)\hat{\otimes}_2H \xrightarrow{r\hat{\otimes}_2\,\mathrm{id}} K\hat{\otimes}_2H & \\
K\hat{\otimes}_2(H\hat{\otimes}_2H) \overset{\sim}{\nearrow} & & \downarrow r \\
\xrightarrow{\mathrm{id}\,\hat{\otimes}_2\mu} & K\hat{\otimes}_2H \xrightarrow{\;r\;} K &
\end{array}
\tag{2}
$$

The pair (K,g) is referred to as a *(Hilbertian) left* (H,μ)*-module* and g is called the *left action* of (H,μ). The notion of a *(Hilbertian) right* (H,μ)*-module* (K,r), with $r\colon K\hat{\otimes}_2H \to K$, is obtained by symmetry (cf. the above diagram on the right). r is the *right action* of (H,μ).

Let a Hilbert space K be both a left and a right (H,μ)-module (K,g) and (K,r). (H,g,r) is said to be a *Hilbertian* (H,μ)*-bimodule* when furthermore the following diagram commutes.

$$
\begin{array}{ccc}
(H\hat{\otimes}_2K)\hat{\otimes}_2H & \xrightarrow{g\hat{\otimes}_2\,\mathrm{id}} & K\hat{\otimes}_2H \\
 & & \downarrow r \\
\alpha_{H,K,H}\ \big\| \wr & & K \\
 & & \uparrow g \\
H\hat{\otimes}_2(K\hat{\otimes}_2H) & \xrightarrow[\mathrm{id}\,\hat{\otimes}_2 r]{} & H\hat{\otimes}_2K
\end{array}
\tag{3}
$$

2.10 Lemma *Let (H,μ) be an object of* $\mathbf{Sem}(\mathbb{Hilb})$ *and let K be a Hilbert space. Let $g\colon H\hat{\otimes}_2K \to K$ (resp. $r\colon K\hat{\otimes}_2H \to K$) be a bounded linear map. (K,g) (resp. (K,r)) is a left (resp. right) (H,μ)-module if, and only if, for each $x,y \in H$, $z \in K$, $g_{\mathsf{bil}}(x, g_{\mathsf{bil}}(y,z)) = g_{\mathsf{bil}}(xy,z)$ (resp., for each $x \in K$, $y,z \in H$, $r_{\mathsf{bil}}(r_{\mathsf{bil}}(x,y),z) = r_{\mathsf{bil}}(x,yz)$).*

Given left (H,μ)-modules (K_i,g_i), $i = 1,2$, a bounded linear map $K_1 \xrightarrow{f} K_2$ is a *left* (H,μ)*-module map* or is said to be *left* (H,μ)*-linear*, or simply *left linear*, when the following diagram commutes.

$$
\begin{array}{ccc}
H\hat{\otimes}_2K_1 & \xrightarrow{\mathrm{id}\,\hat{\otimes}_2 f} & H\hat{\otimes}_2K_2 \\
g_1\downarrow & & \downarrow g_2 \\
K_1 & \xrightarrow[f]{} & K_2
\end{array}
\tag{4}
$$

By symmetry *right* (H,μ)*-module* (or *right* (H,μ)*-linear*) maps are also obtained.

2.1 Example Let (H,μ) be an object of ${}_c\mathbf{Sem}(\mathbb{Hilb})$.

1. H is itself a left and right (H,μ)-module under $g=\mu=r$, by associativity of μ. Of course, (H,μ,μ) thus is a bimodule over itself by associativity of μ.

2. $H\hat{\otimes}_2 H$ is a left (H,μ)-module with $H\hat{\otimes}_2(H\hat{\otimes}_2 H)\xrightarrow{\alpha^{-1}_{H,H,H}}(H\hat{\otimes}_2 H)\hat{\otimes}_2 H\xrightarrow{\mu\hat{\otimes}_2\mathrm{id}} H\hat{\otimes}_2 H$. It is also a right (H,μ)-module with right action $(H\hat{\otimes}_2 H)\hat{\otimes}_2 H\xrightarrow{\alpha_{H,H,H}} H\hat{\otimes}_2(H\hat{\otimes}_2 H)\xrightarrow{\mathrm{id}\,\hat{\otimes}_2\mu} H\hat{\otimes}_2 H$. Actually $H\hat{\otimes}_2 H$ with the left and the right actions of (H,μ) as above, is a Hilbertian bimodule.

In what follows, one tacitly assumes that H and $H\hat{\otimes}_2 H$ both have the left or right module structures from Example 2.1.

2.4 Remark It is clear that in view of Remark 1.1, a commutative Hilbertian algebra (H,μ) is Frobenius if, and only if, $\mu^\dagger\colon H\to H\hat{\otimes}_2 H$ is either left or right (H,μ)-linear.

Let us introduce the following notations. Let H,K,L be Hilbert spaces. Let $\gamma\colon H\times K\to L$ be a bounded bilinear map. One may define $H\xrightarrow{\gamma_{\mathsf{left}}}\mathcal{B}(K,L)$ and $K\xrightarrow{\gamma_{\mathsf{right}}}\mathcal{B}(H,L)$ by setting $(\gamma_{\mathsf{left}}(x))(y)\coloneqq\gamma(x,y)\eqqcolon(\gamma_{\mathsf{right}}(y))(x)$, $x\in H$, $y\in K$. When $\gamma\colon H\hat{\otimes}_2 K\to L$ is a bounded linear map, or equivalently when $\gamma_{\mathsf{bil}}\colon H\times K\to L$ is a weak Hilbert-Schmidt mapping, then one also defines $\gamma_{\mathsf{left}}\coloneqq(\gamma_{\mathsf{bil}})_{\mathsf{left}}$ and $\gamma_{\mathsf{right}}\coloneqq(\gamma_{\mathsf{bil}})_{\mathsf{right}}$.

2.11 Lemma *Let H,K,L be Hilbert spaces. Let $\gamma\colon H\times K\to L$ be a bounded bilinear map. Then, γ_{left} and γ_{right} are bounded linear maps.*

2.2 Example Let (H,μ) be an object of ${}_c\mathbf{Sem}(\mathbb{Hilb})$.

1. For the structure of left or right module over (H,μ), under $\lambda=\mu=\rho$, one has $\mu_{\mathsf{left}}=M=\mu_{\mathsf{right}}$ (by commutativity).

2. For the structure $g\colon H\hat{\otimes}_2(H\hat{\otimes}_2 H)\to H\hat{\otimes}_2 H$ of left (H,μ)-module on $H\hat{\otimes}_2 H$ from Example 2.1.2, one has $g_{\mathsf{left}}(u)(v\otimes w)=(\mu\hat{\otimes}_2\mathrm{id})((u\otimes v)\otimes w)=\mu(u\otimes v)\otimes w=M_u(v)\otimes w$, $u,v,w\in H$, so that $g_{\mathsf{left}}(u)=M_u\hat{\otimes}_2\mathrm{id}$ on $H\otimes_{\mathbb{C}} H$. As $g_{\mathsf{left}}(u)$ and $M_u\hat{\otimes}_2\mathrm{id}$ are both linear and continuous, and $H\otimes_{\mathbb{C}} H$ is dense in $H\hat{\otimes}_2 H$, these maps are equal on the whole $H\hat{\otimes}_2 H$.

   For the structure $r\colon(H\hat{\otimes}_2 H)\hat{\otimes}_2 H\to H\hat{\otimes}_2 H$ of right (H,μ)-module on $H\hat{\otimes}_2 H$ also from Example 2.1.2, one has $r_{\mathsf{right}}(u)(v\otimes w)=w\otimes M_u(v)$ by commutativity of μ. Therefore, $r_{\mathsf{right}}(u)=\mathrm{id}\,\hat{\otimes}_2 M_u$.

2.12 Lemma *Let $(K,g),(K,g')$ (resp. $(K,r),(K',r')$) be Hilbertian left (resp. right) (H,μ)-modules, and let $K \xrightarrow{f} K'$ be a bounded linear map. It is left (resp. right) (H,μ)-linear if, and only if, for each $u \in H$, $g'_{\mathsf{left}}(u) \circ f = f \circ g_{\mathsf{left}}(u)$ (resp. $r'_{\mathsf{right}}(u) \circ f = f \circ r_{\mathsf{right}}(u)$).*

2.13 Proposition *Let (H,μ) be a Hilbertian Frobenius algebra. Then, for each $u \in H$, M_u is normal. In particular, $J(H,\mu) = A(H,\mu)$.*

Proof: Let $u, v, w \in \mathcal{H}$. Then,

$$\begin{array}{rcl} \langle M_u^\dagger(M_u(v)), w\rangle & = & \langle M_u(v), M_u(w)\rangle \\ & = & \langle uv, uw\rangle. \end{array} \tag{5}$$

Now let us assume that $\mu^\dagger$ is right linear. One has

$$\begin{array}{rcl} \langle M_u(M_u^\dagger(v)), w\rangle & = & \langle M_u^\dagger(v), M_u^\dagger(w)\rangle \\ & = & \langle v, M_u(M_u^\dagger(w))\rangle \\ & = & \langle v, uM_u^\dagger(w)\rangle \\ & = & \langle \mu^\dagger(v), u \otimes M_u^\dagger(w)\rangle \\ & = & \langle (\mathrm{id}\,\hat{\otimes}_2 M_u)(\mu^\dagger(v)), u \otimes w\rangle \\ & = & \langle \mu^\dagger(uv), u \otimes w\rangle \text{ (according to Lemma 2.12)} \\ & = & \langle uv, uw\rangle. \end{array} \tag{6}$$

The case of left linearity would be treated similarly using commutativity of μ. The last statement is a direct consequence of Corollary 2.9. □

We are now in position to state the following structure theorem and a corollary that extends Corollary 2.6. But before let us recall [4, p. 111] that a complex Banach algebra A has a *strong Wedderburn decomposition* if there exists a closed subalgebra B of A such that $A = B \oplus J(A)$, where $\oplus$ stands for the linear space direct sum.

2.14 Theorem *(Structure Theorem for Hilbertian Frobenius Algebras) Let (H,μ) be a commutative Hilbertian Frobenius algebra. Then it has a strong Wedderburn decomposition. More precisely, $H = J \oplus_2 J^\perp$ (orthogonal direct sum of Hilbert spaces), $J^\perp$ is both a closed subalgebra and subcoalgebra of (H,μ), and $J = A(H,\mu)$ is also both a closed subalgebra and subcoalgebra. In particular, J and $J^\perp$ are ideals.*

Proof: So far it is already known that $J^\perp$ is both a subalgebra and a subcoalgebra. By Proposition 2.13, $A(H,\mu) = J$ and by [22, Theorem 18, p. 15], J is also a subcoalgebra. □

2.15 Corollary *Let (H,μ) be a commutative Hilbertian Frobenius algebra. Under the corresponding restrictions of μ, $J^\perp$ is a semisimple Hilbertian Frobenius algebra and J is a radical Hilbertian Frobenius algebra.*

2.5 Remark Let (H,μ) be a Hilbertian Frobenius algebra. By Theorem 2.14, $uv = p_{J^{\perp}}(u)p_{J^{\perp}}(v)$ (since $J = A(H,\mu)$) $= \sum_{x\in G(H,\mu)} \langle u,x\rangle\langle v,x\rangle \frac{x}{\|x\|^2}$ as follows from Remark 2.3. In particular for each $x \in G(H,\mu)$ and $u \in H$, $ux = p_{J^{\perp}}(u)x = \langle u,x\rangle x$ and thus $\mathbb{C}x$ is an ideal. (In particular, $xx = \langle x,x\rangle x = \|x\|^2 x$ as already known.)

2.16 Corollary *Let (H,μ) be a commutative Hilbertian Frobenius algebra. (H,μ) is radical if, and only if, $\mu = 0$.*

2.17 Theorem *Let (H,μ) be a commutative Hilbertian Frobenius algebra. The following assertions are equivalent.*

1. *(H,μ) is semisimple.*
2. *(H,μ) is* faithful, *that is,* $\ker M = (0)$.
3. *μ has a dense range.*
4. *$\mu^{\dagger}$ is one-to-one.*
5. *$\mu \circ \mu^{\dagger}$ is one-to-one.*

In particular, any commutative special Hilbertian Frobenius semigroup is semisimple.

Proof: The last statement is a consequence of the presumed equivalences. That the two first points are equivalent is clear as $J(H,\mu) = A(H,\mu)$ (Proposition 2.13). That the three other assertions are equivalent is due to the general fact that for a bounded linear map $K \xrightarrow{f} L$ between Hilbert spaces, $\ker f^{\dagger} = \ker(f\circ f^{\dagger}) = \operatorname{ran}(f)^{\perp}$ (see e.g., [18, Proposition 5.76, p. 390]). It remains for instance to prove that semisimplicity is equivalent to injectivity of $\mu^{\dagger}$. So let us assume that $\mu^{\dagger}$ is one-to-one. According to Theorem 2.14, $J(H,\mu) = A(H,\mu)$ is a subcoalgebra, that is, $\mu^{\dagger}(J(H,\mu)) \subseteq J(H,\mu)\hat{\otimes}_2 J(H,\mu)$. Whence for each $x \in J(H,\mu)$, $\mu(\mu^{\dagger}(x)) = 0$. But as $\mu\circ\mu^{\dagger}$ is one-to-one, $x = 0$, that is, $J(H,\mu) = (0)$. Finally, let us assume that (H,μ) is semisimple. Thus $G(H,\mu)$ is an orthogonal basis of (H,μ) according to Lemma 2.1. Let $u = \sum_{x\in G(H,\mu)} u_x x$ be an arbitrary element of H with $u_x = \frac{1}{\|x\|^2}\langle u,x\rangle$. Then, $\mu^{\dagger}(u) = \sum_{x\in G(H,\mu)} u_x x\otimes x$, and thus $\mu^{\dagger}(u) = 0 \Leftrightarrow u = 0$, that is, $\mu^{\dagger}$ is one-to-one. □

2.6 Remark The last statement of Theorem 2.17 answers in the affirmative one of the main questions left open by [2], that is, if all special Hilbertian Frobenius algebras are semisimple. Moreover it implies that the set of group-like elements of such an algebra is an orthonormal family (since by [22, Lemma 27, p. 21] it is an orthonormal set).

The result below uses the notation from Section 1.5 and states that the weighted Hilbert spaces are the only semisimple Hilbertian Frobenius algebras, up to unitary isomorphisms.

2.18 Proposition *Let (H,μ) be a Hilbertian Frobenius algebra. Then, $J(H,\mu)^\perp \simeq (\ell^2_{w_{(H,\mu)}}(G(H,\mu)), \mu_{G(H,\mu)})$ (unitarily so), where $w_{(H,\mu)}\colon G(H,\mu) \to [\frac{1}{\|\mu\|^2_{\mathsf{op}}}, +\infty[$, $x \mapsto \frac{1}{\|x\|^2}$. (If $G(H,\mu) = \varnothing$, then $w_{(H,\mu)}$ stands for the empty map, which vacuously, is bounded below by any positive constant.)*

Proof: It suffices to prove the result for (H,μ) semisimple. Let us consider the unitary transformation $\Lambda\colon H \to \ell^2_{w_{(H,\mu)}}(G(H,\mu))$ given by $\Lambda(\frac{x}{\|x\|}) := \frac{1}{w_{(H,\mu)}(x)^{\frac{1}{2}}}\delta_x = \|x\|\delta_x$, $x \in G(H,\mu)$. By simple verification, Λ is an isomorphism of semigroups. □

3 Some direct consequences

3.1 The finite-dimensional case

The following result explains why every finite-dimensional commutative Hilbertian Frobenius monoid is automatically semisimple ([2, 6]).

3.1 Corollary *Let (H,μ) be a finite-dimensional commutative Hilbertian Frobenius algebra. Then, (H,μ) has a unit $\Leftrightarrow$ μ is onto $\Leftrightarrow$ $\mu^\dagger$ is one-to-one $\Leftrightarrow$ (H,μ) is semisimple.*

Proof: The last three equivalences are already given by Theorem 2.17 in view of finite dimensionality. If (H,μ) has a unit, then of course μ is onto. Conversely, assuming μ onto, by [9, Corollary 3.3, p. 47] (H,μ) is unital. □

3.2 A dictionary of bases

Let $H \xrightarrow{f} K$ be a bounded linear map between Hilbert spaces. f is a *partial isometry* if $f \circ f^\dagger \circ f = f$. Actually f is a partial isometry if, and only if, $f^\dagger$ is so ([18, pp. 401–402]).

By a *structure* of a Hilbertian algebra of some specific kind on a given Hilbert space H is meant a bounded linear map $\mu\colon H\hat{\otimes}_2 H \to H$ which makes (H,μ) a Hilbertian algebra of the desired kind. The following result may be considered as an extension of the summary [6, p. 565] to infinite-dimensional spaces.

It is usual to call *bounded above* (resp. *bounded below*) a set X of a Hilbert space H such that there exists $C > 0$ with $\|x\| \le C$ (resp. $C \le \|x\|$) for each $x \in X$. C is referred to as a *bound* of X. Observe that the empty set is bounded above and below, with any bound $C > 0$. Recall also that following our terminology (Section 1), an orthogonal set does not contain 0.

3.2 Theorem *Let H be a Hilbert space. There are one-to-one correspondences between*

1. *Non void bounded above orthogonal sets of H and structures of commutative Hilbertian Frobenius algebras on H with a non-zero multiplication.*

2. *Bounded above orthogonal bases of H and structures of semisimple commutative Hilbertian Frobenius algebras on H.*

3. *Non void orthonormal sets of H and structures of commutative Hilbertian Frobenius algebras on H, whose comultiplication is a non-zero partial isometry.*

4. *Orthonormal bases of H and structures of commutative special Hilbertian Frobenius algebras. The corresponding algebras are all unitarily isomorphic.*

5. *The empty orthogonal set corresponds to the unique structure of radical Frobenius algebra on H.*

Proof: Let X be a non void bounded above orthogonal family of H with bound $C > 0$. Let $u, v \in H$. Then, $\sum_{x\in X} \|x\|^2 |\langle u, \frac{x}{\|x\|}\rangle|^2 |\langle v, \frac{x}{\|x\|}\rangle|^2 \leq C^2\|u\|^2\|v\|^2$. One thus defines $m_X : H \times H \to H$ by $m_X(u, v) \coloneqq \sum_{x\in X}\langle u, x\rangle\langle v, x\rangle \frac{x}{\|x\|^2}$. As $m_X(x, x) = \|x\|^2 x$, $x \in X$, m_X is non-zero. One notices that $m_X(u, v) = 0$ whenever $u \in X^\perp$ or $v \in X^\perp$. m_X is a weak Hilbert-Schmidt mapping since $\sum_{x,y\in X} |\langle m_X(\frac{x}{\|x\|}, \frac{y}{\|y\|}), u\rangle|^2 \leq C^2\|u\|^2$, $u \in H$. Let $\mu_X : H\hat{\otimes}_2 H \to H$ be the unique linear extension of m_X, that is, $\mu_X(u \otimes v) = \sum_{x\in X}\langle u, x\rangle\langle v, x\rangle\frac{x}{\|x\|^2}$. (H, μ_X) is of course a Hilbertian algebra, with a non-zero multiplication. As for $u \in H$, $\mu_X(\frac{x}{\|x\|} \otimes \frac{y}{\|y\|}) = \delta_{x,y}x$, $x, y \in X$, and $\langle \mu_X^\dagger(u), v \otimes w\rangle = \langle u, \mu_X(v \otimes w)\rangle = 0$ for $v \in X^\perp$ or $w \in X^\perp$, it follows that $\mu_X^\dagger(u) = \sum_{x\in X}\langle u, x\rangle \frac{x}{\|x\|} \otimes \frac{x}{\|x\|}$ from which one sees that $G(H, \mu_X) = X$. Moreover $\mu_X^\dagger(\mu_X(u \otimes v)) = \mu_X^\dagger(\sum_{x\in X}\langle u, x\rangle\langle v, x\rangle \frac{x}{\|x\|^2}) = \sum_{x\in X}\langle u, x\rangle\langle v, x\rangle \frac{x}{\|x\|} \otimes \frac{x}{\|x\|}$ while one has $(\mathrm{id}\,\hat{\otimes}_2\mu_X)(\alpha_{H,H,H}(\mu_X^\dagger(u) \otimes v)) = (\mathrm{id}\otimes\mu_X)(\sum_{x\in X}\langle u, x\rangle \frac{x}{\|x\|} \otimes (\frac{x}{\|x\|} \otimes v)) = \sum_{x\in X}\langle u, x\rangle \frac{x}{\|x\|} \otimes \mu_X(\frac{x}{\|x\|} \otimes v) = \sum_{x\in X}\langle u, x\rangle\langle v, x\rangle \frac{x}{\|x\|} \otimes \frac{x}{\|x\|}$ as $\mu(\frac{x}{\|x\|} \otimes v) = \langle v, x\rangle \frac{x}{\|x\|}$. This proves that (H, μ_X) is Frobenius. By the way, if X is an orthogonal basis, then (H, μ_X) is a semisimple commutative Hilbertian Frobenius algebra.

Conversely, if (H, μ) is a (semisimple) commutative Hilbertian Frobenius algebra with a non-zero multiplication, thus (H, μ) is not radical, then $G(H, \mu)$ is a non-void bounded orthogonal family (basis), with bound $\|\mu\|_{\mathsf{op}} > 0$ by Lemmas 2.1 and 2.3. It is easily checked that $\mu_{G(H,\mu)} = \mu$. Therefore the first two statements of the theorem are proved.

The third and fourth statements are proved similarly by considering orthonormal families (bases) rather than orthogonal families (bases), and by the following discussion. If X is an orthonormal family, then for $x \in X$, $\mu_X^\dagger(\mu_X(\mu_X^\dagger(x))) = x \otimes x = \mu_X^\dagger(x)$ so that $\mu_X^\dagger$ is indeed a partial isometry since also for $u \in X^\perp$,

$\mu_X^\dagger(u) = 0 = \mu_X^\dagger(\mu_X(\mu_X^\dagger(u)))$. Of course if $X \neq \varnothing$, then $\mu_X \neq 0$ and thus so is $\mu_X^\dagger$.

Conversely assuming that (H,μ) is a commutative Hilbertian Frobenius algebra with $\mu^\dagger$ (or μ) a non-zero partial isometry, then for each $x \in G(H,\mu)$, $\|x\|^2 x \otimes x = \mu^\dagger(x^2) = \mu^\dagger(\mu(\mu^\dagger(x))) = \mu^\dagger(x) = x \otimes x$, so that $\|x\| = 1$, and $G(H,\mu)$ is indeed an orthonormal family.

Let X, Y be two orthonormal bases of H. Let $X \xrightarrow{\pi} Y$ be a bijection. Then it is easily checked that $(H,\mu_X) \xrightarrow{\Pi} (H,\mu_Y)$ is a semigroup isomorphism where $\Pi(x) := \pi(x)$, $x \in X$. The last statement is obvious. ◻

Due to the lemma below, one may substitute in Theorem 3.2, "bounded above" by "bounded below" and the resulting statements are still valid.

3.3 Lemma *Let H be a Hilbert space. There is an involution Θ on the set of all orthogonal subsets of H such that for each orthogonal set X of H, $X \simeq \Theta(X)$ under $x \mapsto \frac{x}{\|x\|^2}$. Under Θ bounded above orthogonal sets correspond one-to-one to bounded below orthogonal sets. For an orthogonal set X, the corresponding orthonormal sets $\{ \frac{x}{\|x\|} : x \in X \}$ and $\{ \frac{x}{\|x\|} : x \in \Theta(X) \}$ are equal.*

3.1 Remark Let H be a Hilbert space and X be a non-void orthogonal set of H which is not bounded above. Define $m_X : \langle X \rangle \times \langle X \rangle \to \langle X \rangle$ by $m_X(u,v) := \sum_{x \in X} \langle u, x \rangle \langle v, x \rangle \frac{x}{\|x\|^2}$ (sum with finitely many non zero terms) in a way similar to the proof of Theorem 3.2. With $\langle X \rangle$ being a pre-Hilbert space and thus a normed space under the induced inner product, m_X is not jointly continuous. Assuming the contrary, for each $x \in X$, $m_X(e_x, e_x) = e_x$ where $e_x := \frac{x}{\|x\|^2} \in \langle X \rangle$, and thus $\|e_x\| = \|m_X(e_x,e_x)\| \le C\|e_x\|^2 \Rightarrow \frac{1}{C} \le \|e_x\| = \frac{1}{\|x\|}$, that is, $\|x\| \le C$, a contradiction. So one cannot extend m_X to the whole H. Nevertheless with, for $u \in \langle X \rangle$, $u^* := \sum_{x \in X} \langle \frac{x}{\|x\|}, u \rangle \frac{x}{\|x\|}$, $(\langle X \rangle, m_X, (-)^*)$ is a Hilbert algebra [8] of a somewhat special kind since $\langle X \rangle = \{ m_X(u,v) : u, v \in \langle X \rangle \}$.

3.3 H^*-algebras

It is possible to characterize Hilbertian Frobenius algebras using Ambrose's concept of H^*-algebras [3] or conversely to characterize commutative H^*-algebras in an involution-free way. Let $(E, *)$ be a commutative Banach algebra where E is the underlying Banach space of a Hilbert space. By a H^**-adjoint* of u is meant a member v of E such that $M_u^\dagger = M_v$, that is, for every $w, w' \in E$, $\langle u * w, w' \rangle = \langle w, v * w' \rangle$. $(E, *)$ is a H^**-algebra* when every element of E has a H^*-adjoint [3].

3.4 Proposition *Let (H,μ) be a Hilbertian algebra. The underlying Banach algebra of (H,μ) is a H^*-algebra if, and only if, (H,μ) a Hilbertian Frobenius algebra. Alternatively, let (E,m) be a commutative Banach algebra where $E = U(H)$ for* *a Hilbert space H. (E,m) is a H^*-algebra if, and only if, (H,μ) is a Hilbertian Frobenius algebra, where $\mu\colon H\hat{\otimes}_2 H \to H$ is the unique extension of m.*

Proof: The direct assertion of the first statement is [2, Lemma 6, p. 9].

Let us assume that (H,μ) is a Hilbertian Frobenius algebra. Let $u \in H$. Let us define $u^* := \sum_{x\in G(H,\mu)} \frac{1}{\|x\|^2}\langle x,u\rangle x$. Of course $p_J(u^*) = 0$. Now, let $u,v,w \in H$. Then,

$$\begin{array}{rcl}\langle uv,w\rangle & = & \langle p_{J^\perp}(u)p_{J^\perp}(v),w\rangle \text{ (according to Remark 2.5)}\\ & = & \langle p_{J^\perp}(u)p_{J^\perp}(v),p_{J^\perp}(w)\rangle \text{ (as } J^\perp \text{ is a subalgebra by Theorem 2.14)}\\ & = & \sum_{x\in G(H,\mu)} \frac{1}{\|x\|^2}\langle u,x\rangle\langle v,x\rangle\overline{\langle w,x\rangle}\\ & = & \sum_{x\in G(H,\mu)} \frac{1}{\|x\|^2}\langle v,x\rangle\overline{\langle x,u\rangle\langle w,x\rangle}\\ & = & \langle p_{J^\perp}(v),u^*p_{J^\perp}(w)\rangle\\ & = & \langle p_{J^\perp}(v),u^*w\rangle\\ & = & \langle v,u^*w\rangle \text{ (as } u^*w \in J^\perp).\end{array} \tag{7}$$

Therefore (H,μ_{bil}) is a H^*-algebra.

Now let (E,m) be a commutative H^*-algebra with $E = U(H)$ for a Hilbert space H and E has a norm equivalent to the norm $\|-\|$ induced by the inner product of H. For each minimal ideal I of (E,m), let $e(I)$ be the idempotent generator of I. By the work of [3], $(\frac{e(I)}{\|e(I)\|})_I$, where I runs over the set of all minimal ideals of (E,m), forms an orthonormal basis of the orthocomplement of the annihilator of (E,m) or equivalently its Jacobson radical. Since $\frac{1}{\|m\|_{\mathsf{op}}} \le \|e(I)\|$ for each I, it follows that $m\colon H\times H \to H$ is a weak Hilbert-Schmidt mapping, and thus extends uniquely to a bounded linear map $\mu\colon H\hat{\otimes}_2 H \to H$ with $\mu_{\mathsf{bil}} = m$. The second statement of the proposition now follows as (E,m) is the underlying Banach algebra of (H,μ). □

3.5 Corollary *Let (H,μ) be a Hilbertian Frobenius algebra. $(-)^\perp$ provides an order-reversing involution on the set of closed ideals of (H,μ). In particular a closed subspace of H is an ideal if, and only if, it is a subcoalgebra.*

Proof: The first statement follows from the existence of H^*-adjoints (by Proposition 3.4). The second statement is due to [22, Lemmas 14 and 15, p. 14] which jointly assert that I is a closed ideal if, and only if, $I^\perp$ is a closed subcoalgebra. □

3.4 Approximate (co)units

Let (H,μ) be a Hilbertian algebra and let $(e_\lambda)_{\lambda\in\Lambda}$ be a directed net on H, that is, Λ is a directed set and $e_\lambda \in H$, $\lambda \in \Lambda$. $(e_\lambda)_\lambda$ is an *approximate unit* if $ue_\lambda \to u$ in the norm topology, for each $u \in H$. By its very definition the existence of an approximate unit forces the multiplication μ to have a dense range. Any semisimple commutative Hilbertian Frobenius algebra (H,μ) has an approximate unit (this was already noticed in [2] for H separable and μ isometric, under the assumption, redundant by Theorem 2.17, of semisimplicity of (H,μ)), namely $(e_F)_{F\in\mathfrak{P}_{\mathsf{fin}}(G(H,\mu))}$ where $e_F := \sum_{x\in F}\frac{x}{\|x\|^2}$ and $\mathfrak{P}_{\mathsf{fin}}(G(H,\mu))$ is the set of all finite subsets of group-like elements, directed under inclusion (because $p_{J^\perp}(u)$ is precisely the sum of the summable family $(\langle u,\frac{x}{\|x\|}\rangle\frac{x}{\|x\|})_{x\in G(H,\mu)}$ for each $u\in H$.)

Let (H,μ) be a Hilbertian algebra and let $(H\xrightarrow{\epsilon_\lambda}\mathbb{C})_{\lambda\in\Lambda}$ where Λ is a directed set. Call $(\epsilon_\lambda)_\lambda$ an *approximate counit* when for each $u\in H$, $\|(\mathrm{id}\otimes\epsilon_\lambda)(\mu^\dagger(u)) - u\otimes 1\|_{H\hat{\otimes}_2 H}\to 0$. The existence of such an approximate counit forces $\mu^\dagger$ to be one-to-one. Given a Hilbertian Frobenius algebra (H,μ), let $\epsilon_F(u) := \sum_{x\in F}\langle u,\frac{x}{\|x\|}\rangle$ for $F\in\mathfrak{P}_{\mathsf{fin}}(G(H,\mu))$ and $u\in H$, that is, $\epsilon_F = e_F^\dagger$, when e_F is identified with a map $\mathbb{C}\xrightarrow{e_F}H$. Then, $(\mathrm{id}\,\hat{\otimes}_2\epsilon_F)(\mu^\dagger(u)) = \sum_{x\in F}\frac{x}{\|x\|}\otimes\langle u,\frac{x}{\|x\|}\rangle\to p_{J^\perp}(u)\otimes 1$. Whence if (H,μ) is semisimple, then ϵ_F is an approximate counit.

All of this may be combined as a corollary of Theorem 2.17.

3.6 Corollary *A Hilbertian Frobenius semigroup is semisimple if, and only if, it has an approximate unit if, and only if, it has an approximate counit. In particular each special Hilbertian Frobenius semigroup has an approximate unit and an approximate counit.*

4 Reformulation of the Structure Theorem as an equivalence of categories

In this section it is shown that the splitting of an Hilbertian Frobenius algebra into the orthogonal direct sum of a semisimple and a radical Hilbertian Frobenius semigroups may in fact be recasted into an equivalence between ${}_c\mathbf{FrobSem}(\mathbb{Hilb})$ and the product category ${}_{\mathsf{semisimple},c}\mathbf{FrobSem}(\mathbb{Hilb})\times\mathbf{Hilb}$.

Let (H,μ) be a commutative Hilbertian Frobenius algebra. The closure $\overline{\mathrm{ran}(\mu)}$ of the range of μ is also equal to $\overline{H^2}$, with $H^2 := \langle xy\colon x,y\in H\rangle$. Now $(H^2)^\perp = A(H,\mu)$ as it follows easily from the existence of H^*-adjoints (Proposition 3.4). Consequently, $\overline{\mathrm{ran}(\mu)} = J^\perp$ by Proposition 2.13. One now defines $P(H,\mu) := (J^\perp,\mu_{|_{J^\perp}})$ (see Corollary 2.15).

4.1 Proposition $(H,\mu)\mapsto P(H,\mu)$ *extends to a functor P from ${}_c\mathbf{FrobSem}(\mathbb{Hilb})$ to ${}_{\mathsf{semisimple},c}\mathbf{FrobSem}(\mathbb{Hilb})$ which is a right adjoint left inverse of the full embedding functor $E\colon{}_{\mathsf{semisimple},c}\mathbf{FrobSem}(\mathbb{Hilb})\hookrightarrow{}_c\mathbf{FromSem}(\mathbb{Hilb})$.*

Proof: Let $(H,\mu)\xrightarrow{f}(K,\gamma)$ be an algebra map between Hilbertian Frobenius algebras. As $f(H^2)\subseteq K^2$, it follows that $f(\overline{H^2})\subseteq\overline{f(H^2)}\subseteq\overline{K^2}$. Whence $P(H,\mu)\xrightarrow{f}P(K,\gamma)$ is defined as the co-restriction of f. In particular, $P(f) = \pi_{P(K,\gamma)}\circ f\circ i_{P(H,\mu)}$ and thus $P(f)$ is bounded. By Corollary 2.15, $P(H,\mu) = J^\perp$ is a semisimple Hilbertian Frobenius algebra, under the co-restriction of μ. $i_{P(H,\mu)}$ is an algebra morphism since $J(H,\mu)^\perp$ is a subalgebra, and $\pi_{P(K,\gamma)} = i^\dagger_{P(K,\gamma)}$ is an algebra morphism too as $J(K,\gamma)^\perp$ is a subcoalgebra, so that $P(f)$ is indeed a morphism of algebras, and thus one obtains the desired functor. P is of course a left inverse of E.

Now let (H,μ) be a semisimple Hilbertian Frobenius algebra and let (K,γ) be a Hilbertian Frobenius algebra. Let $(H,\mu) \xrightarrow{f} P(K,\gamma)$ be a morphism of semigroups. Define $f^\sharp \coloneqq (H,\mu) \xrightarrow{f} P(K,\gamma) \xrightarrow{i_{P(K,\gamma)}} (K,\gamma)$. Then $P(f^\sharp) = f$ as $i_{P(K,\gamma)} \circ P(f^\sharp) = f^\sharp \circ i_{P(H,\mu)} = f^\sharp = i_{P(K,\gamma)} \circ f$ (as $P(H,\mu) = (H,\mu)$) and $i_{P(K,\gamma)}$ is a monomorphism. Now let $(H,\mu) = P(H,\mu) \xrightarrow{g} (K,\gamma)$ such that $P(g) = f$. Then $f^\sharp = i_{P(K,\gamma)} \circ f = i_{P(K,\gamma)} \circ P(g) = g \circ i_{P(H,\mu)} = g$. □

One may also define a functor $J\colon {}_c\mathbf{FrobSem}(\mathbb{Hilb}) \to \mathbf{Hilb}$ as follows. Let $f\colon (H,\mu) \to (K,\gamma)$ be a semigroup map. Then, $f^\dagger\colon (K,\gamma^\dagger) \to (H,\mu^\dagger)$ is a cosemigroup map, and thus $f^\dagger(G(K,\gamma)) \subseteq G(H,\mu) \cup \{0\}$. By linearity and continuity, $f^\dagger(J(K,\gamma)^\perp) \subseteq J(H,\mu)^\perp$ and thus $f(J(H,\mu)) \subseteq J(K,\gamma)$ by [22, Lemma 1, p. 964]. Then let $J(f)$ be the co-restriction of f thus obtained. Clearly this provides a functor $J\colon {}_c\mathbf{FrobSem}(\mathbb{Hilb}) \to \mathbf{Hilb}$.

In the opposite direction let $T\colon \mathbf{Hilb} \to {}_c\mathbf{FrobSem}(\mathbb{Hilb})$ be the full embedding functor, $T(H \xrightarrow{f} K) \coloneqq (H,0) \xrightarrow{f} (K,0)$.

4.2 Lemma *The full subcategory* ${}_{\mathsf{radical},c}\mathbf{FrobSem}(\mathbb{Hilb})$ *of* ${}_c\mathbf{FrobSem}(\mathbb{Hilb})$ *spanned by the radical commutative Hilbertian Frobenius algebras, is isomorphic to* $\mathbf{Hilb}$.

Proof: The co-restriction $\mathbf{Hilb} \xrightarrow{T} {}_{\mathsf{radical},c}\mathbf{FrobSem}(\mathbb{Hilb})$ of T is the inverse of the obvious forgetful functor ${}_{\mathsf{radical},c}\mathbf{FrobSem}(\mathbb{Hilb}) \xrightarrow{|-|} \mathbf{Hilb}$. □

In the two results below are identified external and internal orthogonal direct sums.

Let (H,μ) and (K,γ) be commutative Hilbertian algebras. By additivity of $\hat{\otimes}_2$, $(H \oplus_2 K)\hat{\otimes}_2(H \oplus_2 K)$ has the following coproduct presentation.

$$\begin{array}{ccccc} H\hat{\otimes}_2 H & \xrightarrow{i_H\hat{\otimes}_2 i_H} & & \xleftarrow{i_H\hat{\otimes}_2 i_K} & H\hat{\otimes}_2 K \\ & & (H \oplus_2 K)\hat{\otimes}_2(H \oplus_2 K) & & \\ K\hat{\otimes}_2 H & \xrightarrow{i_J\hat{\otimes}_2 i_H} & & \xleftarrow{i_K\hat{\otimes}_2 i_K} & K\hat{\otimes}_2 K \end{array} \tag{8}$$

4.3 Proposition *Let* (H,μ) *and* (K,γ) *be Hilbertian algebras. Let us define* $\rho\colon (H\oplus_2 K)\hat{\otimes}_2(H\oplus_2 K) \to H\oplus_2 K$ *by* $\rho\circ(i_H\hat{\otimes}_2 i_H) \coloneqq i_H\circ\mu$, $\rho\circ(i_K\hat{\otimes}_2 i_K) = i_K\circ\gamma$ *and* $\rho\circ((i_H\hat{\otimes}_2 i_K)\oplus_2(i_K\hat{\otimes}_2 i_H)) = 0$. *Then,* $(H\oplus_2 K,\rho)$ *is a Hilbertian algebra,* $(H,\mu) \xrightarrow{i_H} (H \oplus_2 K,\rho) \xleftarrow{i_K} (K,\gamma)$ *are morphisms of algebras, and* $\rho^\dagger \circ i_H = (i_H\hat{\otimes}_2 i_H)\circ\mu^\dagger$, $\rho^\dagger\circ i_K = (i_K\hat{\otimes}_2 i_K)\circ\gamma^\dagger$, *that is,* $(H,\mu^\dagger) \xrightarrow{i_H} (H\oplus_2 K,\rho^\dagger) \xleftarrow{i_K} (K,\gamma^\dagger)$ *are morphisms of coalgebras as well.*

Moreover if both (H,μ) and (K,γ) are Frobenius, then so is $(H \oplus_2 K, \rho)$, and in this case, $J(H \oplus_2 K, \rho) = J(H,\mu) \oplus_2 J(K,\gamma)$ and $J(H \oplus_2 H, \rho)^\perp = J(H,\mu)^\perp \oplus_2 J(K,\gamma)^\perp$. In particular, if (H,μ) is semisimple and (K,γ) is radical, then $J(H \oplus_2 K, \rho) = K$ and $J(H \oplus_2 K, \rho)^\dagger = H$.

Proof: That $(H \oplus_2 K, \rho)$ is indeed a Hilbertian algebra is easily checked. It is then clear, from the very definition of ρ, that $(H,\mu) \xrightarrow{i_H} (H \oplus_2 K, \rho) \xleftarrow{i_K} (K,\gamma)$ are morphisms of algebras.

Let $x \in H$, $u, u' \in H$, $v, v' \in K$. Then,

$$\begin{array}{rcl} \langle \rho^\dagger(x), (u+v) \otimes (u'+v') \rangle & = & \langle \rho^\dagger(x), u \otimes u' + u \otimes v' + v \otimes u' + v \otimes v' \rangle \\ & = & \langle x, \mu(u \otimes u') + \gamma(v \otimes v') \rangle \\ & = & \langle x, \mu(u \otimes u') \rangle \text{ (since } \gamma(v,v') \in K) \\ & = & \langle \mu^\dagger(x), u \otimes u' \rangle \\ & = & \langle i_{H \hat{\otimes}_2 H}(\mu^\dagger(x)), u \otimes u' + u \otimes v' + v \otimes u' + v \otimes v' \rangle \\ & = & \langle i_{H \hat{\otimes}_2 H}(\mu^\dagger(x)), (u+v) \otimes (u'+v') \rangle. \end{array} \tag{9}$$

As a matter of fact, $\rho^\dagger(x) \in ((H \hat{\otimes}_2 K) \oplus_2 (K \hat{\otimes}_2 H) \oplus_2 (K \hat{\otimes}_2 K))^\perp = H \hat{\otimes}_2 H$. Likewise $\rho^\dagger(x) \in K \hat{\otimes}_2 K$ for each $x \in K$. Consequently, $\rho^\dagger \circ i_H = (i_H \hat{\otimes}_2 i_H) \circ \mu^\dagger$, $\rho^\dagger \circ i_K = (i_K \hat{\otimes}_2 i_K) \circ \gamma^\dagger$, and thus $(H, \mu^\dagger) \xrightarrow{i_H} (H \oplus_2 K, \rho^\dagger) \xleftarrow{i_K} (K, \gamma^\dagger)$ are morphisms of coalgebras.

Let us now assume that both (H,μ) and (K,γ) are Frobenius. It follows easily, by a direct computation, that $(H \oplus_2 K, \rho)$ is Frobenius as well.

Let $u + v \in A(H \oplus_2 K, \rho)$, $u \in H$, $v \in K$. Then $0 = \rho_{\mathsf{bil}}(u+v, u'+v') = \mu_{\mathsf{bil}}(u,u') + \gamma_{\mathsf{bil}}(v,v')$. In particular, $u \in A(H,\mu) = J(H,\mu)$ and $v \in A(K,\gamma) = J(K,\gamma)$, and $u + v \in J(H,\mu) \oplus_2 J(K\gamma)$. Conversely, let $u \in J(H,\mu)$, $v \in J(K,\gamma)$. Then, $u + v \in A(H \oplus_2 K, \rho)$ since $\rho_{\mathsf{bil}}(u+v, u'+v') = \mu_{\mathsf{bil}}(u,u') + \gamma_{\mathsf{bil}}(v,v') = 0$ for all $u' \in H$, $v' \in K$. Thus, $J(H \oplus_2 K, \rho) = A(H \oplus_2 K, \rho) = J(H,\mu) \oplus_2 J(K,\gamma)$. As a result, $J(H \oplus_2 K, \rho)^\perp = (J(H,\mu) \oplus_2 J(K,\gamma))^\perp = J(H,\mu)^\perp \oplus_2 J(K,\gamma)^\perp$. The last assertion is immediate. □

4.4 Lemma *Let $f : (H,\mu) \to (K,\gamma)$ be a ${}_c\mathbf{FrobSem}(\mathbb{Hilb})$-morphism, then $f = P(f) \oplus_2 J(f)$.*

4.5 Proposition *${}_c\mathbf{FrobSem}(\mathbb{Hilb})$ is equivalent to ${}_{\mathsf{semisimple},c}\mathbf{FrobSem}(\mathbb{Hilb}) \times \mathbf{Hilb}$.*

Proof: The functor $\langle P, J \rangle$ from ${}_c\mathbf{FrobSem}(\mathbb{Hilb})$ to ${}_{\mathsf{semisimple},c}\mathbf{FrobSem}(\mathbb{Hilb}) \times \mathbf{Hilb}$, $(H,\mu) \mapsto (P(H,\mu), J(H,\mu))$, is the required equivalence of categories. Indeed that $\langle P, J \rangle$ is full is easily observed. Proposition 4.3 implies essential surjectivity. By Lemma 4.4, $\langle P, V \rangle$ is faithful. □

4.6 Corollary *The categories ${}_c\mathbf{FrobSem}(\mathbb{FdHilb})$ and ${}_{\mathsf{semisimple},c}\mathbf{FrobSem}(\mathbb{FdHilb}) \times \mathbf{FdHilb}$ are equivalent.*

Let ${}_{\mathsf{partiso},c}\mathbf{FrobSem}(\mathbb{Hilb})$ be the full subcategory of ${}_{c}\mathbf{FrobSem}(\mathbb{Hilb})$ spanned by those Hilbertian Frobenius algebras (H,μ) where $\mu^{\dagger}$ is a partial isometry. Recall also that ${}^{\dagger}_{c}\mathbf{FrobSem}(\mathbb{Hilb})$ is the category of special commutative Hilbertian Frobenius algebras. Let (H,μ) be an object of ${}_{\mathsf{partiso},c}\mathbf{FrobSem}(\mathbb{Hilb})$. As $\mu^{\dagger}$ restricts to an isometry from $(\ker\mu^{\dagger})^{\perp} = (\operatorname{ran}\mu)^{\perp\perp} = J^{\perp}$ to $H\hat{\otimes}_2 H$ ([18, p. 404]), it is clear that $P(H,\mu)$ is an object of ${}^{\dagger}_{c}\mathbf{FrobSem}(\mathbb{Hilb})$. The following result then follows easily.

4.7 Corollary *The equivalence from Proposition 4.5 restricts to an equivalence between* ${}_{\mathsf{partiso},c}\mathbf{FrobSem}(\mathbb{Hilb})$ *and* ${}^{\dagger}_{c}\mathbf{FrobSem}(\mathbb{Hilb}) \times \mathbf{Hilb}$.

5 Equivalences between semisimple Frobenius algebras and weighted pointed sets

The one-to-one correspondence between structures of semisimple Hilbertian Frobenius algebras on a given Hilbert space and its bounded below (or above) orthogonal bases (Theorem 3.2) may be upgraded into an equivalence of categories.

5.1 Categories of pointed sets with a weight function

Let $\mathbf{WSet}_{\bullet}$ be the following category of *weighted pointed sets*. Its objects are pointed sets (X,x_0,α) together with a *weight function*, i.e., a map $\alpha\colon X\setminus\{x_0\}\to[C_\alpha,+\infty[$ for some given $C_\alpha>0$. A morphism $(X,x_0,\alpha)\xrightarrow{f}(Y,y_0,\beta)$ is a base-point preserving map $(X,x_0)\xrightarrow{f}(Y,y_0)$ such that

- for each $y\neq y_0$, $|f^{-1}(\{y\})|<+\infty$,
- there exists a real number $M_f\geq 0$ such that for all $y\neq y_0$, $\sum_{x\in f^{-1}(\{y\})}\alpha(x)\leq M_f\beta(y)$.

Under the usual composition this indeed forms a category.

In [22] is introduced the category $\mathbf{Set}_{\bullet,<+\infty}$ the objects of which are pointed sets (X,x_0) and morphisms $(X,x_0)\xrightarrow{f}(Y,y_0)$ are those base-point preserving maps such that (1) $|f^{-1}(\{y\})|$ is finite for each $y\neq y_0$ and (2) $\mathsf{B}_f := \sup_{y\neq y_0}|f^{-1}(\{y\})|<+\infty$.

5.1 Lemma $\mathbf{Set}_{\bullet,<+\infty}$ *fully embeds into* $\mathbf{WSet}_{\bullet}$ *under the identity-on-arrows functor* E *such that* $E(X,x_0):=(X,x_0,\mathbf{1})$, *with* $\mathbf{1}(x)=1$, $x\neq x_0$.

5.2 Lemma *Let* $f\in\mathbf{WSet}_{\bullet}((X,x_0,\alpha),(Y,y_0,\beta))$ *with* β *bounded above. Then,* $f\in\mathbf{Set}_{\bullet,<+\infty}((X,x_0),(Y,y_0))$.

Let us define ${}_{\mathsf{bnd}}\mathbf{WSet}_{\bullet}$ (resp. ${}_{\mathsf{unbnd}}\mathbf{WSet}_{\bullet}$) be the full subcategory of $\mathbf{WSet}_{\bullet}$ spanned by those objects (X,x_0,α) with α bounded above (resp. unbounded).

5.3 Lemma ${}_{\mathsf{bnd}}\mathbf{WSet}_{\bullet}$ *is equivalent to* $\mathbf{Set}_{\bullet,<+\infty}$*. Moreover no object of* ${}_{\mathsf{bnd}}\mathbf{WSet}_{\bullet}$ *is isomorphic to an object of* ${}_{\mathsf{unbnd}}\mathbf{WSet}_{\bullet}$*.*

Proof: By Lemma 5.2, for $(X, x_0, \alpha) \xrightarrow{f} (Y, y_0, \beta)$ with α, β bounded above, $(X, x_0) \xrightarrow{f} (Y, y_0)$ is a morphism in $\mathbf{Set}_{\bullet,<+\infty}$. This defines in an obvious way a functor $U: {}_{\mathsf{bnd}}\mathbf{WSet}_{\bullet} \to \mathbf{Set}_{\bullet,<+\infty}$. This functor is readily faithful.

Let $(X, x_0, \alpha), (Y, y_0, \beta)$ with bounded α, β and let $(X, x_0) \xrightarrow{f} (Y, y_0)$ in $\mathbf{Set}_{\bullet,<+\infty}$. Then, for $y \neq y_0$, $\sum_{x \in f^{-1}(\{y\})} \alpha(x) \leq \sup_{x \neq x_0} \alpha(x) \mathsf{B}_f \leq \sup_{x \neq x_0} \alpha(x) \mathsf{B}_f \frac{1}{C_\beta} \beta(y)$. So U is full. Of course, U is surjective on objects because $U(X, x_0, \mathbf{1}) = (X, x_0)$.

Now, let (X, x_0, α) and (Y, y_0, β) with α bounded, and β unbounded. Let $(X, x_0, \alpha) \xrightarrow{\phi} (Y, y_0, \beta)$ be an isomorphism in $\mathbf{WSet}_{\bullet}$. In particular, $X \xrightarrow{\phi} Y$ is a bijection with $\phi(x_0) = y_0$. Let $\theta := \phi^{-1}$ which is a morphism in $\mathbf{WSet}_{\bullet}$ too. Then, for all $x \neq x_0$, $\beta(\phi(x)) = \sum_{y \in \theta^{-1}(\{x\})} \beta(y) \leq M_f \alpha(x)$. In particular, $\sup_{y \neq y_0} \beta(y) = \sup_{x \neq x_0} \beta(\phi(x)) \leq M_f \sup_{x \neq x_0} \alpha(x) < +\infty$ which is a contradiction. □

5.2 The set of minimal ideals functor

Let (H, μ) be a Hilbertian Frobenius algebra.

5.4 Lemma $E(H, \mu) \subseteq \langle G(H, \mu) \rangle \subseteq J(H, \mu)^{\perp}$*. More precisely, if e is an idempotent element, then the* support of e, $S_e := \{ g \in G(H, \mu) : \langle e, g \rangle \neq 0 \}$*, is the (unique) finite subset of $G(H, \mu)$ such that* $e = \sum_{g \in S_e} \frac{g}{\|g\|^2}$*.*

Proof: That $E(H, \mu) \subseteq J(H, \mu)^{\perp}$ follows from [4, Lemma 3.3, p. 112] since $H = J(H, \mu)^{\perp} \oplus_2 J(H, \mu)$ and $J(H, \mu)^{\perp}$ is a subalgebra by the Structure Theorem (Theorem 2.14).

Let $e = \sum_{g \in G(H,\mu)} \langle e, \frac{g}{\|g\|} \rangle \frac{g}{\|g\|}$ be an idempotent element. $e^2 = \sum_{g \in G(H,\mu)} \frac{1}{\|g\|^2} \langle e, g \rangle^2 g$. But $e^2 = e$ so that for each $g \in G(H, \mu)$, $\langle e, g \rangle \in \{ 0, 1 \}$. Let $S_e := \{ g \in G(H, \mu) : \langle e, g \rangle \neq 0 \}$. Then, $e = \sum_{g \in S_e} \frac{1}{\|g\|} \frac{g}{\|g\|}$ and thus $(\frac{1}{\|g\|^2})_{g \in S_e}$ is summable and $\|e\|^2 = \sum_{g \in S_e} \frac{1}{\|g\|^2}$ (since $G(H, \mu)$ is an orthogonal family). But for each group-like element g, $\frac{1}{\|\mu\|^2_{\mathsf{op}}} \leq \frac{1}{\|g\|^2}$ (by Lemma 2.5) and $\sum_{g \in S_e} \frac{1}{\|\mu\|^2_{\mathsf{op}}} = \frac{|S_e|}{\|\mu\|^2_{\mathsf{op}}} \leq \|e\|^2$, so that $|S_e| < +\infty$. Uniqueness of S_e is clear since $(\frac{g}{\|g\|})_g$ is an orthonormal basis. □

As usually let $e \leq f$ be defined by $ef = e$, for $e, f \in E(H, \mu)$, and call *minimal* an idempotent which is minimal in $(E(H, \mu) \setminus \{ 0 \}, \leq)$. Let $\mathrm{Min}(E(H, \mu))$ be the set of all these minimal (non-zero) idempotent elements.

As Lemma 5.4 actually establishes a one-one correspondence $e \mapsto S_e$, between $E(H, \mu)$ and the set $\mathfrak{P}_{\mathsf{fin}}(G(H, \mu))$ of finite subsets of group-like elements, and as under this bijection the product of idempotents corresponds to the intersection of their supports, $\leq$ corresponds to the usual inclusion of sets. Consequently $\mathrm{Min}(E(H, \mu)) = \{ \frac{g}{\|g\|^2} : g \in G(H, \mu) \}$.

5.1 Remark Let (H, μ) be a Hilbertian Frobenius algebra. Lemma 3.3 tells us that (1) $G(H, \mu) \simeq \text{Min}(E(H, \mu))$ under $g \mapsto \frac{g}{\|g\|^2}$ with inverse $e \mapsto \frac{e}{\|e\|^2}$, (2) $\text{Min}(E(H, \mu))$ is a bounded below orthogonal family. More precisely if $G(H, \mu)$ is non void, that is, $\mu \neq 0$ (by Corollary 2.16), $\text{Min}(E(H, \mu))$ is a non void orthogonal family. (3) $\{ \frac{e}{\|e\|} : e \in \text{Min}(E(H, \mu)) \} = \{ \frac{g}{\|g\|} : g \in G(H, \mu) \}$.

5.5 Lemma *Let $I = He$ for some idempotent e of (H, μ). Then, $I \subseteq J(H, \mu)^{\perp}$ and $I^{\perp}$ is a modular ideal with modular unit e.*

Proof: That $I \subseteq J(H, \mu)^{\perp}$ is clear by Lemma 5.4 since $J(H, \mu)^{\perp}$ is a subalgebra (Lemma 2.5) and $J(H, \mu)J(H, \mu)^{\perp} = 0$ (Proposition 2.13). $I^{\perp}$ is indeed an ideal (Corollary 3.5). To prove that e is a modular unit of $I^{\perp}$ it suffices to check that $\langle u - ue, ve \rangle = 0$ for each $u, v \in H$, which is left to the readers. □

Let I be an ideal of (H, μ) contained in $J(H, \mu)^{\perp}$ and such that $I \neq (0)$. Then, $I^2 \neq (0)$. (Indeed if $I^2 = (0)$, then for each $x \in I$, $x^2 = 0$. But then $x \in J(H, \mu) \cap I \subseteq J(H, \mu) \cap J(H, \mu)^{\perp} = (0)$, that is, $x = 0$.)

5.6 Lemma *Let I be a minimal ideal. Then, $I = \mathbb{C}g = \mathbb{C}\frac{g}{\|g\|^2}$ for a unique group-like element g, in particular $I \subseteq J(H, \mu)^{\perp}$. Equivalently, $I = \mathbb{C}e = \mathbb{C}\frac{e}{\|e\|^2}$ for a unique minimal idempotent element e. e may be characterized as the unit of I. Finally, $(-)^{\perp}$ establishes a one-one correspondence between the set of all maximal regular ideals of (H, μ) and the set $\text{Min}(H, \mu)$ of all its minimal ideals.*

Proof: By the above observation and by Brauer's lemma ([19, p. 162]), $I = He$ for a non-zero idempotent e of (H, μ).

Since I is minimal, $I^{\perp}$ is maximal (Corollary 3.5). It is modular by Lemma 5.5. Therefore $I^{\perp} = (\mathbb{C}g)^{\perp}$ for a unique group-like element g of (H, μ). Since $\mathbb{C}g$ is finite dimensional, it is closed and thus $\overline{I} = I^{\perp\perp} = (\mathbb{C}g)^{\perp\perp} = \mathbb{C}g$. Since $I \subseteq \mathbb{C}g$, I also is finite dimensional so $I = \mathbb{C}g$. (This in particular shows that $I \mapsto I^{\perp}$ is a bijection from the set of all minimal ideals of (H, μ) to the set of all its maximal regular ideals since $\mathbb{C}g$ certainly is a minimal ideal for each group-like element g by Remark 2.5.) Uniqueness of the group-like generator is clear since $G(H, \mu)$ is a linearly independent set. What remains of the proof is easy. □

Let I be a minimal ideal of (H, μ). Let $\textcircled{g}(I)$ and $\textcircled{e}(I)$ be respectively its group-like and its idempotent generators, provided by Lemma 5.6 and which are related by the equalities $\textcircled{e}(I) = \frac{\textcircled{g}(I)}{\|\textcircled{g}(I)\|^2}$ and thus also $\textcircled{g}(I) = \frac{\textcircled{e}(I)}{\|\textcircled{e}(I)\|^2}$. This defines maps $\text{Min}(H, \mu) \xrightarrow{\textcircled{g}} G(H, \mu)$ and $\text{Min}(H, \mu) \xrightarrow{\textcircled{e}} \text{Min}(E(H, \mu))$. The proof of the next result is essentially provided by the proof of Lemma 5.6.

5.7 Lemma $\textcircled{g}$ *is a bijection with inverse* $G(H,\mu) \xrightarrow{\mathbb{C}(-)} \mathrm{Min}(H,\mu)$, $g \mapsto \mathbb{C}g$, *and* $\textcircled{e}$ *is a bijection from* $\mathrm{Min}(H,\mu)$ *to* $\mathrm{Min}\, E(H,\mu)$ *with inverse* $\mathrm{Min}(E(H,\mu)) \xrightarrow{\mathbb{C}(-)} \mathrm{Min}(H,\mu)$, $e \mapsto \mathbb{C}e$.

5.1 Notation For convenience one still denotes by $\textcircled{g}$ (resp. $\textcircled{e}$) the bijection from $\mathrm{Min}(H,\mu)\cup\{0\}$ (resp. $\mathrm{Min}(H,\mu)\cup\{0\}$) onto $G(H,\mu)\cup\{0\}$ (resp. $\mathrm{Min}(E(H,\mu))\cup\{0\}$) obtained from the original $\textcircled{g}$ (resp. $\textcircled{e}$) by setting $\textcircled{g}(0) \coloneqq 0$ (resp. $\textcircled{e}(0) \coloneqq 0$).

Let (H,μ) be a Hilbertian Frobenius algebra. One recalls from Lemma 2.3, that for each $g \in G(H,\mu)$, $\|g\| \le \|\mu\|_{\mathsf{op}}$. (This is true even for $G(H,\mu) = \varnothing$, or equivalently for $\mu = 0$.) So in particular for each $e \in \mathrm{Min}(E(H,\mu))$, $\frac{1}{\|e\|} = \|\frac{e}{\|e\|^2}\| \le \|\mu\|_{\mathsf{op}}$. This is equivalent to $\frac{1}{\|\mu\|_{\mathsf{op}}} \le \|e\|$, $e \in \mathrm{Min}(E(H,\mu))$ but only for a non void $\mathrm{Min}(E(H,\mu))$ or equivalently for $\mu \neq 0$. Nevertheless even when $\mu = 0$, $\mathrm{Min}(E(H,\mu))$ is bounded below since it is void. To avoid statements by cases where $\mu \neq 0$ or $\mu = 0$, one introduces the bound of (H,μ).

5.1 Definition Let (H,μ) be a Hilbertian Frobenius algebra. Define the *bound* $b(H,\mu) > 0$ of (H,μ) by $b(H,\mu) \coloneqq \begin{cases} 1 & \text{if } \mu = 0 \\ \|\mu\|_{\mathsf{op}} & \text{if } \mu \neq 0 \end{cases}$. For each $g \in G(H,\mu)$, $\|g\| \le b(H,\mu)$. So in particular for each $e \in \mathrm{Min}(E(H,\mu))$, $\frac{1}{\|e\|} = \|\frac{e}{\|e\|^2}\| \le b(H,\mu)$. This is equivalent, even for $\mu = 0$, to $\frac{1}{b(H,\mu)} \le \|e\|$, $e \in \mathrm{Min}(E(H,\mu))$. One defines $w_{(H,\mu)}\colon \mathrm{Min}(H,\mu) \to [\frac{1}{b(H,\mu)^2}, +\infty[$ by $w_{(H,\mu)}(I) \coloneqq \frac{1}{\|\textcircled{g}(I)\|^2} = \|\textcircled{e}(I)\|^2$. Let $\mathrm{Min}_{\bullet}(H,\mu) \coloneqq (\mathrm{Min}(H,\mu) \cup \{0\}, 0, w_{(H,\mu)})$. One defines ${}_{\mathsf{bnd},c}\mathbf{FrobSem}(\mathbb{Hilb})$ and ${}_{\mathsf{unbnd},c}\mathbf{FrobSem}(\mathbb{Hilb})$ as the full subcategories of ${}_{c}\mathbf{FrobSem}(\mathbb{Hilb})$ spanned by the algebras (H,μ) with $w_{(H,\mu)}$ bounded above, and respectively unbounded above.

Let $(H,\mu) \xrightarrow{f} (K,\gamma)$ be a semigroup map between Hilbertian Frobenius alegbras. Therefore $(K,\gamma^{\dagger}) \xrightarrow{f^{\dagger}} (H,\mu^{\dagger})$ is a coalgebra map and thus $f^{\dagger}(G(K,\gamma)) \subseteq G(H,\mu) \cup \{0\}$.

Let $\ell\colon \mathrm{Min}(K,\gamma) \to \mathrm{Min}(H,\mu) \cup \{0\}$ be given by $\ell(J) = 0$ if, and only if, $f^{\dagger}(\textcircled{g}(J)) = 0$, and $\ell(J) = I$ if, and only if, $f^{\dagger}(\textcircled{g}(J)) = \textcircled{g}(I)$. Therefore, for each $J \in \mathrm{Min}(K,\gamma) \cup \{0\}$, $\ell(J) = \mathbb{C}f^{\dagger}(\textcircled{g}(J))$. (Observe that for $J = 0$ or for $f^{\dagger}(\textcircled{g}(J)) = 0$, $\ell(J) = \mathbb{C}0 = 0$.)

Let $I \in \mathrm{Min}(H,\mu)$. Then,

$$\langle f(\textcircled{e}(I)), \frac{\textcircled{e}(J)}{\|\textcircled{e}(J)\|}\rangle \quad = \quad \langle \textcircled{e}(I), \|\textcircled{e}(J)\| f^{\dagger}(\frac{\textcircled{e}(J)}{\|\textcircled{e}(J)\|^2})\rangle. \tag{10}$$

Consequently, for $J \in \mathrm{Min}(K,\gamma)$, $\langle f(\textcircled{e}(I)), \frac{\textcircled{e}(J)}{\|\textcircled{e}(J)\|}\rangle = 0$ if, and only if, $f^{\dagger}(g(J)) = 0$ or $f^{\dagger}(g(J)) = g(I')$ with $I' \neq I$, and $\langle f(\textcircled{e}(I)), \frac{\textcircled{e}(J)}{\|\textcircled{e}(J)\|}\rangle = \langle \textcircled{e}(I), \|\textcircled{e}(J)\| g(I)\rangle =$

$\frac{\|ⓔ(J)\|}{\|ⓔ(I)\|^2}\langle ⓔ(I), ⓔ(I)\rangle = \|ⓔ(J)\|$ if, and only if, $f^\dagger(ⓖ(J)) = ⓖ(I)$ if, and only if, $\ell(J) = I$.

Since f is a semigroup map, it sends an idempotent element of (H,μ) to one of (K,γ). Whence the image by f of an idempotent belongs to $J(K,\gamma)^\perp$. Therefore, for each $I \in \mathrm{Min}(H,\mu)$, $f(ⓔ(I)) = \sum_{J\in\mathrm{Min}(K,\gamma)}\langle f(ⓔ(I)), \frac{ⓔ(J)}{\|ⓔ(J)\|}\rangle \frac{ⓔ(J)}{\|ⓔ(J)\|} = \sum_{J\in\ell^{-1}(\{I\})}\|ⓔ(J)\|\frac{ⓔ(J)}{\|ⓔ(J)\|}$.

Now for $I \in \mathrm{Min}(H,\mu)$, $\sum_{J\in\ell^{-1}(\{I\})} b(K,\gamma)^2\|ⓔ(J)\|^2 = b(K,\gamma)^2\|f(ⓔ(I))\|^2 \le b(K,\gamma)^2\|f\|^2_{\mathsf{op}}\|ⓔ(I)\|^2 < +\infty$. But $1 \le b(K,\gamma)^2\|ⓔ(J)\|^2$ for each $J \in \mathrm{Min}(K,\gamma)$, so necessarily $|\ell^{-1}(\{I\})|$ is finite.

From the equality $f(ⓔ(I)) = \sum_{J\in\ell^{-1}(\{I\})}\|ⓔ(J)\|\frac{ⓔ(J)}{\|ⓔ(J)\|} = \sum_{J\in\ell^{-1}(\{I\})}ⓔ(J)$ it follows that $\sum_{J\in\ell^{-1}(\{I\})}\|ⓔ(J)\|^2 = \|f(ⓔ(I))\|^2 \le \|f\|^2\|ⓔ(I)\|^2$ for each $I \in \mathrm{Min}(H,\mu)$.

Consequently, $\ell \in \mathbf{WSet}_\bullet(\mathrm{Min}_\bullet(K,\gamma), \mathrm{Min}_\bullet(H,\mu))$, where ℓ is extended to the whole of $\mathrm{Min}(K,\gamma)\cup\{0\}$ by setting $\ell(0) \coloneqq 0$.

Contravariance of $f \mapsto \ell$ is clear. So are obtained the *set of minimal ideals* functor $\mathrm{Min}_\bullet: {}_c\mathbf{FrobSem}(\mathbb{Hilb}) \to \mathbf{WSet}_\bullet^{\mathsf{op}}$ and $\mathrm{Min}_\bullet: {}_{\mathsf{semisimple},c}\mathbf{FrobSem}(\mathbb{Hilb}) \to \mathbf{WSet}_\bullet^{\mathsf{op}}$ by restriction.

5.3 The $\ell^2_\bullet$ functor

Let (X, x_0, α) be an object of $\mathbf{WSet}_\bullet$. Define $\alpha^+: X \to [C_\alpha, +\infty[$, $0 < C_\alpha$, by $\alpha^+(x) = \alpha(x)$, $x \neq x_0$, $\alpha^+(x_0) \coloneqq C_\alpha$. One may consider the semisimple Hilbertian Frobenius algebra $((\ell^2_{\alpha^+}(X), \langle\cdot,\cdot\rangle_{\alpha^+}), \mu_X)$ (Proposition 1.5).

Let $\ell^2_\bullet(X, x_0, \alpha) \coloneqq \{u \in \ell^2_{\alpha^+}(X): u(x_0) = 0\} = \delta_{x_0}^\perp$. It is a closed subalgebra, since it is even a closed (maximal) ideal as the kernel of $\langle\cdot,\delta_{x_0}\rangle$, and $\delta_{x_0} \in G(\ell^2_{\alpha^+}(X), \mu_X)$.

The Hilbertian algebra $((\ell^2_\bullet(X,x_0,\alpha), \langle\cdot,\cdot\rangle_{\alpha^+}), (\mu_X)_{|\delta_{x_0}^\perp})$ is clearly unitarily isomorphic to $((\ell^2_\alpha(X\setminus\{x_0\}), \langle\cdot,\cdot\rangle_\alpha), \mu_{X\setminus\{x_0\}})$. As a matter of fact, $\ell^2_\bullet(X,x_0,\alpha)$ is an object of ${}_{\mathsf{semisimple},c}\mathbf{FrobSem}(\mathbb{Hilb})$, with $G(\ell^2_\bullet(X,x_0,\alpha)) = \{\frac{\delta_x}{\alpha(x)}: x \in X\setminus\{x_0\}\}$ by Proposition 1.5.

In particular, when α is bounded above (resp. unbounded), then $\ell^2_\bullet(X,x_0,\alpha)$ is an object of ${}_{\mathsf{semisimple,bnd},c}\mathbf{FrobSem}(\mathbb{Hilb})$ (resp. ${}_{\mathsf{semisimple,unbnd},c}\mathbf{FrobSem}(\mathbb{Hilb})$).

Let $f \in \mathbf{WSet}_\bullet((X,x_0,\alpha),(Y,y_0,\beta))$. Let $u \in \ell^2_\bullet(Y,y_0,\beta)$. Then, $u\circ f \in \ell^2_\bullet(X,x_0,\alpha)$. Indeed, $u(f(x_0)) = u(y_0) = 0$. Let $A \subseteq X$ be a finite set. Then,

$$\begin{array}{rcl} \sum_{x\in A\setminus\{x_0\}}\alpha(x)|u(f(x))|^2 & = & \sum_{y\in f(A)\setminus\{y_0\}}\left(\sum_{x\in f^{-1}(\{y\})}\alpha(x)\right)|u(y)|^2 \\ & \le & \sum_{y\neq y_0} M_f\beta(y)|u(y)|^2 \\ & = & M_f\|u\|^2_\beta. \end{array} \tag{11}$$

In particular $\|u\circ f\|_\alpha \le M_f^{\frac{1}{2}}\|u\|_\beta$.

Since $\ell^2_\bullet(f): \ell^2_\bullet(Y,y_0,\beta) \to \ell^2_\bullet(X,x_0,\alpha)$, $u \mapsto u\circ f$, is clearly a semigroup morphism, it follows easily that one has a functor $\ell^2_\bullet: \mathbf{WSet}_\bullet^{\mathsf{op}} \to {}_c\mathbf{FrobSem}(\mathbb{Hilb})$ and thus also the following co-restriction $\ell^2_\bullet\colon \mathbf{WSet}^{\mathsf{op}}_\bullet \to {}_{\mathsf{semisimple},c}\mathbf{FrobSem}(\mathbb{Hilb})$.

5.4 The main equivalences

5.8 Theorem *One has an adjunction* $\mathrm{Min}_\bullet \dashv \ell^2_\bullet\colon \mathbf{WSet}^{\mathsf{op}}_\bullet \to {}_c\mathbf{FrobSem}(\mathbb{Hilb})$ *that restricts to an adjoint equivalence* $\mathrm{Min}_\bullet \dashv \ell^2_\bullet\colon \mathbf{WSet}^{\mathsf{op}}_\bullet \simeq {}_{\mathsf{semisimple},c}\mathbf{FrobSem}(\mathbb{Hilb})$. *In particular,* ${}_{\mathsf{semisimple},c}\mathbf{FrobSem}(\mathbb{FdHilb}) \simeq \mathbf{FinSet}^{\mathsf{op}}_\bullet$, *where* $\mathbf{FinSet}_\bullet$ *is the category of finite pointed sets and base-point preserving maps.*

Proof: Let (X, x_0, α) be a weighted pointed set, with $X \setminus \{x_0\} \xrightarrow{\alpha} [C_\alpha, +\infty[$, where $C_\alpha > 0$. Then, $\mathrm{Min}(\ell^2_\bullet(X, x_0, \alpha)) \simeq \mathrm{Min}(\ell^2_\alpha(X \setminus \{x_0\}, \mu_{X\setminus\{x_0\}})) = \{\mathbb{C}\delta_x : x \neq x_0\}$.

Moreover $w_{\ell^2_\bullet(X,x_0,\alpha)} = w_{(\ell^2_\alpha(X\setminus\{x_0\}),\mu_{X\setminus\{x_0\}})}$ so $w_{\ell^2_\bullet(X,x_0,\alpha)}(\mathbb{C}\delta_x) = \|ⓔ(\mathbb{C}\delta_x)\|^2_\alpha = \|\delta_x\|^2_\alpha = \alpha(x)$.

Let $\epsilon_{(X,x_0,\alpha)}\colon (X, x_0) \to (\mathrm{Min}(\ell^2_\bullet(X, x_0, \alpha)) \cup \{0\}, 0)$ be given by $\epsilon_{(X,x_0,\alpha)}(x) \coloneqq \mathbb{C}\delta_x$, $x \neq x_0$, and $\epsilon_{(X,x_0,\alpha)}(x_0) \coloneqq 0$. $\epsilon_{(X,x_0,\alpha)}$ is clearly a pointed bijection. By the above, $\epsilon_{(X,x_0,\alpha)}$ is clearly a $\mathbf{WSet}_\bullet$-isomorphism.

Let (H, μ) be a Hilbertian Frobenius algebra. An orthonormal basis for the semigroup $\ell^2_\bullet(\mathrm{Min}_\bullet(H, \mu))$ is given by $(\frac{\delta_I}{\sqrt{w_{(H,\mu)}(I)}})_{I\in\mathrm{Min}(H,\mu)} = (\frac{\delta_I}{\|ⓔ(I)\|})_{I\in\mathrm{Min}(H,\mu)}$. Whence as a Hilbert space, $\ell^2_\bullet(\mathrm{Min}_\bullet(H, \mu))$ is unitary isomorphic to $J(H, \mu)^\perp$ because an orthonormal basis of the latter is given by $\{\frac{e}{\|e\|} : e \in \mathrm{Min}(E(H, \mu))\}$ (Remark 5.1). Let $\Phi_{(H,\mu)}\colon \ell^2_\bullet(\mathrm{Min}_\bullet(H, \mu)) \simeq J(H, \mu)^\perp$ be the corresponding unitary transformation. For each minimal ideal I of (H, μ), $\Phi_{(H,\mu)}(\delta_I) = ⓔ(I)$, and one has $\Phi_{(H,\mu)}(IJ) = \Phi_{(H,\mu)}(\delta_{I,J}I) = \delta_{I,J}ⓔ(I) = ⓔ(I)ⓔ(J) = \Phi_{(H,\mu)}(\delta_I)\Phi_{(H,\mu)}(\delta_J)$ for a minimal ideal J, from which it follows that $\Phi_{(H,\mu)}$ is actually an isomorphism of semigroups.

Now let $(X, x_0, \alpha) \xrightarrow{f} \mathrm{Min}_\bullet(H, \mu)$ be a $\mathbf{WSet}_\bullet$-morphism, where (H, μ) is an Hilbertian Frobenius semigroup. Let $(H, \mu) \xrightarrow{f^\sharp} \ell^2_\bullet(X, x_0, \alpha) \coloneqq (H, \mu) \xrightarrow{\pi_{J(H,\mu)^\perp}} J(H, \mu)^\perp \xrightarrow{\Phi^{-1}_{(H,\mu)}} \ell^2_\bullet(\mathrm{Min}_\bullet(H, \mu)) \xrightarrow{\ell^2_\bullet(f)} \ell^2_\bullet(X, x_0, \alpha)$. By construction $f^\sharp$ is a morphism of semigroups.

For $g \in G(H, \mu)$, $f^\sharp(\frac{g}{\|g\|}) = \ell^2_\bullet(f)(\Phi^{-1}_{(H,\mu)}(\frac{g}{\|g\|})) = \ell^2_\bullet(f)(\frac{\delta_{\mathbb{C}g}}{\|e(\mathbb{C}g)\|}) = \ell^2_\bullet(f)(\|g\|\delta_{\mathbb{C}g}) = \|g\|\delta_{\mathbb{C}g} \circ f$. Let $u \in \ell^2_\bullet(X, x_0, \alpha)$ and $g \in G(H, \mu)$. Then, $\langle (f^\sharp)^\dagger(u), \frac{g}{\|g\|}\rangle = \langle u, \|g\|\delta_{\mathbb{C}g} \circ f\rangle_\alpha = \sum_{x\neq x_0} \alpha(x)u(x)\|g\|\delta_{\mathbb{C}g}(f(x))$. Consequently,

$$(f^\sharp)^\dagger(u) = \sum_{g\in G(H,\mu)} \|g\| \left(\sum_{x\in f^{-1}(\{\mathbb{C}g\})} \alpha(x)u(x) \right) \frac{g}{\|g\|} = \sum_{g\in G(H,\mu)} \left(\sum_{x\in f^{-1}(\{\mathbb{C}g\})} \alpha(x)u(x) \right) g. \tag{12}$$

In particular, for $x \in X \setminus \{x_0\}$, $(f^\sharp)^\dagger(\frac{\delta_x}{\alpha(x)}) = \frac{\alpha(x)}{\alpha(x)}ⓖ(f(x)) = ⓖ(f(x))$. (Recall that $ⓖ(0) = 0$ by Notation 5.1.) This is equivalent to $\mathrm{Min}_\bullet(f^\sharp)(\mathbb{C}\delta_x) = f(x)$, $x \in X \setminus \{x_0\}$. In other words, $\mathrm{Min}_\bullet(f^\sharp) \circ \epsilon_{(X,x_0,\alpha)} = f$.

Now let $h:(H,\mu)\to\ell^2_\bullet(X,x_0,\alpha)$ be a semigroup map such that $\mathrm{Min}_\bullet(h^\dagger)\circ\epsilon_{(X,x_0,\alpha)}=f$. Then, for each $x\neq x_0$, $x\in X$, $h^\dagger(\frac{\delta_x}{\alpha(x)})=\circledS(f(x))=(f^\sharp)^\dagger(\frac{\delta_x}{\alpha(x)})$. Since $\{\frac{\delta_x}{\alpha(x)}: x\in X\smallsetminus\{x_0\}\}$ is dense into $\ell^2_\bullet(X,x_0,\alpha_x)$ it follows that $h^\dagger=(f^\sharp)^\dagger$, that is, $h=f^\sharp$.

The proof for the adjunction will be concluded as soon as naturality of the family $(\epsilon_{(X,x_0,\alpha)})_{(X,x_0,\alpha)}$ will be proved. This is equivalent to the requirement that for each $(X,x_0,\alpha)\xrightarrow{f}(Y,y_0,\beta)$, $\mathrm{Min}_\bullet(\ell^2_\bullet(f))(\mathbb{C}\delta_x)=\mathbb{C}\delta_{f(x)}$, $x\in X\smallsetminus\{x_0\}$ with $f(x)\neq y_0$, and also that $\mathrm{Min}_\bullet(\ell^2_\bullet(f))(\mathbb{C}\delta_x)=0$, $x\in X\smallsetminus\{x_0\}$ with $f(x)=y_0$.

But $\mathrm{Min}_\bullet(\ell^2_\bullet(f))(\mathbb{C}\delta_x)=\mathbb{C}((\ell^2_\bullet(f))^\dagger(\circledS(\mathbb{C}\delta_x)))$. Naturality thus is equivalent to $(\ell^2_\bullet(f))^\dagger(\frac{\delta_x}{\alpha(x)})=\frac{\delta_{f(x)}}{\beta(f(x))}$, $x\in X\smallsetminus\{x_0\}$, $f(x)\neq y_0$, since $\frac{\delta_x}{\alpha(x)}=\circledS(\mathbb{C}\delta_x)$ and in this case $\circledS(\mathbb{C}\delta_{f(x)})=\frac{\delta_{f(x)}}{\beta(f(x))}$, and $(\ell^2_\bullet(f))^\dagger(\frac{\delta_x}{\alpha(x)})=0$, $x\in X\smallsetminus\{x_0\}$, $f(x)=y_0$. So one has to compute $(\ell^2_\bullet(f))^\dagger(\frac{\delta_x}{\alpha(x)})$. Let $u\in\ell^2_\bullet(X,x_0,\alpha)$ and let $y\in Y\smallsetminus\{y_0\}$. Then,

$$\begin{array}{rcl}\langle(\ell^2_\bullet(f))^\dagger(u),\frac{\delta_y}{\beta(y)^{\frac12}}\rangle_\beta &=& \langle u,\ell^2_\bullet(f)(\frac{\delta_y}{\beta(y)^{\frac12}})\rangle_\alpha\\ &=& \sum_{x\in X\smallsetminus\{x_0\}}\alpha(x)u(x)\frac{\delta_y(f(x))}{\beta(y)^{\frac12}}.\end{array}\tag{13}$$

Therefore

$$\begin{array}{rcl}(\ell^2_\bullet(f))^\dagger(u) &=& \sum_{y\in Y\smallsetminus\{y_0\}}\frac{1}{\beta(y)^{\frac12}}\left(\sum_{x\in f^{-1}(\{y\})}\alpha(x)u(x)\right)\frac{\delta_y}{\beta(y)^{\frac12}}\\ &=& \sum_{y\in Y\smallsetminus\{y_0\}}\left(\sum_{x\in f^{-1}(\{y\})}\alpha(x)u(x)\right)\frac{\delta_y}{\beta(y)}.\end{array}\tag{14}$$

In particular, for each $x\in X\smallsetminus\{x_0\}$ with $f(x)\neq y_0$

$$\begin{array}{rcl}(\ell^2_\bullet(f))^\dagger(\frac{\delta_x}{\alpha(x)}) &=& \sum_{y\neq y_0}\frac{\delta_y}{\beta(y)}\left(\sum_{x'\in f^{-1}(\{y\})}\alpha(x')\frac{\delta_x(x')}{\alpha(x)}\right)\\ &=& \frac{\delta_{f(x)}}{\beta(f(x))}.\end{array}\tag{15}$$

For each $x\in X\smallsetminus\{x_0\}$ with $f(x)=y_0$, the same computation as above leads as expected to $(\ell^2_\bullet(f))^\dagger(\frac{\delta_x}{\alpha(x)})=0$.

It remains to prove the statement about the equivalence of categories. The component at (H,μ) of the unit of the above adjunction is by definition, $\mathrm{id}^\sharp_{\mathrm{Min}_\bullet(H,\mu)}=\Phi^{-1}_{(H,\mu)}\circ\pi_{J(H,\mu)^\perp}$. Since the counit ϵ is an isomorphism, the above equivalence restricts to an equivalence between $\mathbf{WSet}_\bullet^{\mathsf{op}}$ and the full subcategory of ${}_c\mathbf{FrobSem}(\mathbb{H}\mathbb{i}\mathbb{l}\mathbb{b})$ spanned by those algebras (H,μ) such that $\pi_{J(H,\mu)^\perp}$ is an isomorphism, that is, the semisimple Hilbertian Frobenius algebras.

Concerning the last statement one first notices that the adjunction $\mathrm{Min}_\bullet\dashv\ell^2_\bullet$ co-resricts to the adjunction $\mathrm{Min}_\bullet\dashv\ell^2_\bullet: {}_c\mathbf{FrobSem}(\mathbb{F}\mathbb{d}\mathbb{H}\mathbb{i}\mathbb{l}\mathbb{b})\to\mathbf{WFinSet}_\bullet$, where $\mathbf{WFinSet}_\bullet$ stands for the full subcategory of $\mathbf{WSet}_\bullet$ spanned by the pointed weighted sets (X,x_0,α) where X is finite. By finiteness the embedding functor $E:\mathbf{Set}_{\bullet,<+\infty}\to\mathbf{WSet}_\bullet$ from Lemma 5.1, provides an equivalence $\mathbf{FinSet}_\bullet\simeq\mathbf{WFinSet}_\bullet$. By restriction again one obtains the expected equivalence. □

It is a consequence of Theorem 5.8 and of Lemma 5.3 that not all the semisimple Hilbertian Frobenius algebras $(H,\mu),(H,\gamma)$ on the same Hilbert space H are isomorphic. (In view of Theorem 3.2 it suffices to consider bounded above orthogonal bases of H, one of which also bounded below and the other not.)

Using some previous results (in particular Proposition 4.5) one obtains the following easily, by obvious restrictions of the equivalences from Theorem 5.8. (The proof of Item 2 requires the use of [22, Theorem 41, p. 28].)

5.9 Corollary *One has the following equivalences of categories.*

1. ${}_{\mathsf{unbnd}}\mathbf{WSet}_\bullet^{\mathsf{op}} \simeq {}_{\mathsf{semisimple,unbnd},c}\mathbf{FrobSem}(\mathbb{Hilb})$.
2. ${}^{\dagger}_{c}\mathbf{FrobSem}(\mathbb{Hilb}) \simeq \mathbf{Set}^{\mathsf{op}}_{\bullet,<+\infty} \simeq {}_{\mathsf{bnd}}\mathbf{WSet}_\bullet^{\mathsf{op}} \simeq {}_{\mathsf{semisimple,bnd},c}\mathbf{FrobSem}(\mathbb{Hilb})$.
3. $\mathbf{WSet}_\bullet^{\mathsf{op}} \times \mathbf{Hilb} \simeq {}_{c}\mathbf{FrobSem}(\mathbb{Hilb})$.
4. ${}_{\mathsf{unbnd}}\mathbf{WSet}_\bullet^{\mathsf{op}} \times \mathbf{Hilb} \simeq {}_{\mathsf{semisimple,unbnd},c}\mathbf{FrobSem}(\mathbb{Hilb}) \times \mathbf{Hilb} \simeq {}_{\mathsf{unbnd},c}\mathbf{FrobSem}(\mathbb{Hilb})$.
5. ${}_{\mathsf{partiso},c}\mathbf{FrobSem}(\mathbb{Hilb}) \simeq {}^{\dagger}_{c}\mathbf{FrobSem}(\mathbb{Hilb}) \times \mathbf{Hilb} \simeq \mathbf{Set}^{\mathsf{op}}_{\bullet,<+\infty} \times \mathbf{Hilb} \simeq {}_{\mathsf{bnd}}\mathbf{WSet}_\bullet^{\mathsf{op}} \times \mathbf{Hilb} \simeq {}_{\mathsf{semisimple,bnd},c}\mathbf{FrobSem}(\mathbb{Hilb}) \times \mathbf{Hilb} \simeq {}_{\mathsf{bnd},c}\mathbf{FrobSem}(\mathbb{Hilb})$.
6. $\mathbf{FinSet}_\bullet^{\mathsf{op}} \simeq {}^{\dagger}_{c}\mathbf{FrobSem}(\mathbb{FdHilb})$ *and* $\mathbf{FinSet}_\bullet^{\mathsf{op}} \times \mathbf{FdHilb} \simeq {}_{c}\mathbf{FrobSem}(\mathbb{FdHilb}) \simeq {}_{\mathsf{partiso},c}\mathbf{FrobSem}(\mathbb{FdHilb})$.

6 Some other (but related) equivalences

In this section are established several equivalences as consequences of Theorem 5.8 by considering different kind of morphisms.

6.1 More on Frobenius algebras with a partial isometric comultiplication

6.1 Proposition *Let (H,μ) be a Hilbertian Frobenius algebra. Let us consider the following (in general non commutative) diagram, with $\sigma_{2,3}\colon (H\hat{\otimes}_2 H)\hat{\otimes}_2(H\hat{\otimes}_2 H) \to (H\hat{\otimes}_2 H)\hat{\otimes}_2(H\hat{\otimes}_2 H)$ the unitary isomorphism given by $(u_1\otimes u_2)\otimes(u_3\otimes u_4) \mapsto (u_1\otimes u_3)\otimes(u_2\otimes u_4)$.*

$$\begin{array}{ccc} H & \xrightarrow{\mu^\dagger} & H\hat{\otimes}_2 H \\ & & \uparrow{\scriptstyle \mu\hat{\otimes}_2\mu} \\ \mu\uparrow & & (H\hat{\otimes}_2 H)\hat{\otimes}_2(H\hat{\otimes}_2 H) \\ & & \wr{\scriptstyle \sigma_{2,3}} \\ H\hat{\otimes}_2 H & \xrightarrow[\mu^\dagger\hat{\otimes}_2\mu^\dagger]{} & (H\hat{\otimes}_2 H)\hat{\otimes}_2(H\hat{\otimes}_2 H) \end{array} \tag{16}$$

$((H,\mu),\mu^\dagger)$ is a bisemigroup in $\mathbb{Hilb}$, that is, Diag. (16) above commutes, if, and only if, for each $g \in G(H,\mu)$, $\|g\| = 1$ if, and only if, $\mu^\dagger$ is a partial isometry.

Proof: It is easily checked by a direct computation that $((H,\mu),\mu^\dagger)$ is a bisemigroup in $\mathbb{Hilb}$ if, and only if, for each $g \in G(H,\mu)$, $\|g\| = 1$. Now let $u \in H$, then

$$\begin{array}{rcl} \mu^\dagger(\mu(\mu^\dagger(u))) & = & \mu^\dagger(\mu(\sum_{g\in G(H,\mu)}\langle u, \frac{g}{\|g\|}\rangle g\otimes g)) \\ & = & \mu^\dagger(\sum_{g\in G(H,\mu)}\langle u, \frac{g}{\|g\|}\rangle \|g\|^2 g) \\ & = & \sum_{g\in G(H,\mu)}\langle u, g\rangle \|g\| g\otimes g. \end{array} \tag{17}$$

Therefore $\mu^\dagger$ is a partial isometry if, and only if, $\|g\| = 1$ for each $g \in G(H,\mu)$. (The converse implication is due to the fact that for each $g \in G(H,\mu)$, $\|g\|^2 g\otimes g = \mu^\dagger(\mu(\mu^\dagger(g))) = \mu^\dagger(g) = g\otimes g$.) □

6.1 Remark For any Hilbertian Frobenius algebra (H,μ), $((H,\mu),0)$ is a *trivial* bisemigroup in $\mathbb{Hilb}$, that is, the diagram, obtained from Diag. (16) by replacing $\mu^\dagger$ by 0, commutes. This does not contradict Proposition 6.1 since the conjunction of (H,μ) Frobenius and $\mu^\dagger = 0$ implies $H = (0)$.

Let ${}_{\mathsf{partiso},c}\mathbf{FrobBisem}(\mathbb{Hilb})$ be the subcategory of ${}_{\mathsf{partiso},c}\mathbf{FrobSem}(\mathbb{Hilb})$ with the same objects but with morphisms preserving both the algebra and the coalgebra structures, that is, with morphism of bisemigroups. Let ${}^\dagger_c\mathbf{FrobBisem}(\mathbb{Hilb})$ be its full subcategory spanned by the Frobenius algebras with an isometric comultiplication.

Being a morphism of bisemigroups is rather restrictive as show the following result and remark below.

6.2 Proposition *Let $f \in {}_{\mathsf{partiso},c}\mathbf{FrobBisem}(\mathbb{Hilb})((H,\mu),(K,\gamma))$ where $\mu^\dagger$ is an isometry (that is, (H,μ) is semisimple). Then, f is a partial isometry.*

Proof: Since both f and $f^\dagger$ are coalgebra maps, $f(G(H,\mu)) \subseteq G(K,\gamma)\cup\{0\}$ and $f^\dagger(G(K,\gamma)) \subseteq G(H,\mu)\cup\{0\}$. Therefore for each $g\in G(H,\mu)$ and $h\in G(K,\gamma)$, $f(g) = h \Leftrightarrow \langle f(g),h\rangle = 1 \Leftrightarrow \langle g, f^\dagger(h)\rangle = 1 \Leftrightarrow g = f^\dagger(h)$. Consequently, for each $h\in G(K,\gamma)$, $f(f^\dagger(h)) = h$ when $f^\dagger(h)\neq 0$ and $f(f^\dagger(h)) = 0$ when $f^\dagger(h) = 0$ and in any case $f^\dagger(f(f^\dagger(h))) = f^\dagger(h)$. Let $u\in H$. Then, $f^\dagger(u) = \sum_{h\in G(K,\gamma)}\langle u,h\rangle f^\dagger(h) = \sum_{h\in G(K,\gamma)}\langle u,h\rangle f^\dagger(f(f^\dagger(h))) = f^\dagger(f(f^\dagger(u)))$. Then, $f^\dagger$ is a partial isometry and so is also f. □

6.2 Remark In general for $f\colon (Y,y_0,\beta)\to(X,x_0,\alpha)$, $\ell^2_\bullet(f)$ is not a coalgebra morphism. Indeed if it was the case, then for each $x\in X\smallsetminus\{x_0\}$, $\frac{\delta_x\circ f}{\alpha(x)} = \ell^2(f)(\frac{\delta_x}{\alpha(x)}) = 0$ or $\ell^2(f)(\frac{\delta_x}{\alpha(x)}) \in G(\ell^2_\bullet(Y,y_0,\beta))$, that is, $\frac{\delta_x\circ f}{\alpha(x)} = \frac{\delta_y}{\beta(y)}$ for some $y\in Y\smallsetminus\{y_0\}$. Equivalently, for each $x\in X\smallsetminus\{x_0\}$, $f^{-1}(\{x\}) = \varnothing$ or there exists $y\in Y\smallsetminus\{y_0\}$ such that $f(y) = x$ and $\alpha(f(y)) = \beta(y)$, and for each $y'\neq y$, $y'\in Y\smallsetminus\{y_0\}$, $f(y')\neq x$. In particular, $|f^{-1}(\{x\})|\leq 1$ for each $x\in X\smallsetminus\{x_0\}$.

Let $\mathbf{PInj}_\bullet$ be the category of *partial injections*, that is, the objects are pointed sets and morphism $(X, x_0) \xrightarrow{f} (Y, y_0)$ are base-point preserving maps such that for all $y \in Y \setminus \{ y_0 \}$, $|f^{-1}(\{ y \})| \leq 1$. ($\mathbf{PInj}_\bullet$ is isomorphic to the category $\mathbf{PInj}$ from [12].) $\mathbf{PInj}_\bullet$ embeds into $\mathbf{WSet}_\bullet$ under $E(X, x_0) := (X, x_0, \mathbf{1})$ (where $\mathbf{1}(x) = 1$, $x \neq x_0$) and $E(f) := f$ as for each $y \neq y_0$, (1) $|f^{-1}(\{ y \})| \leq 1$, and (2) $\sum_{x \in f^{-1}(\{ y \})} \mathbf{1}(x) \leq 1 = \mathbf{1}(y)$. So one may consider the functor $\mathbf{PInj}_\bullet^{\mathsf{op}} \xrightarrow{\ell^2_\bullet \circ E} {}_c\mathbf{FrobSem}(\mathbb{Hilb})$. Of course it factors through ${}^\dagger_c\mathbf{FrobSem}(\mathbb{Hilb}) \hookrightarrow {}_{\mathsf{partiso},c}\mathbf{FrobSem}(\mathbb{Hilb}) \hookrightarrow {}_c\mathbf{FrobSem}(\mathbb{Hilb})$. But actually, for each partial injection $(X, x_0) \xrightarrow{f} (Y, y_0)$, $\ell^2_\bullet$ is even a coalgebra map as $\mu_X^\dagger(\ell^2_\bullet(f)(u)) = \sum_{x \neq x_0} u(f(x))\delta_x \otimes \delta_x$ and $(\ell^2(f) \hat{\otimes}_2 \ell^2(f))(\delta_Y^\dagger(u)) = (\ell^2(f) \hat{\otimes}_2 \ell^2(f))(\sum_{y \neq y_0} u(y)\delta_y \otimes \delta_y) = \sum_{y \neq y_0} u(y)(\delta_y \circ f) \otimes (\delta_y \circ f) = \sum_{x \neq x_0} u(f(x))\delta_x \otimes \delta_x$ for each $u \in \ell^2_\bullet(Y, y_0, \mathbf{1})$. So one has a functor $\ell^2_\bullet\colon \mathbf{PInj}_\bullet^{\mathsf{op}} \to {}_{\mathsf{partiso},c}\mathbf{FrobBisem}(\mathbb{Hilb})$ together with its co-restriction $\ell^2_\bullet\colon \mathbf{PInj}_\bullet^{\mathsf{op}} \to {}^\dagger_c\mathbf{FrobBisem}(\mathbb{Hilb})$.

In the opposite direction one has a functor $G_\bullet\colon {}_{\mathsf{partiso},c}\mathbf{FrobBisem}(\mathbb{Hilb}) \to \mathbf{PInj}_\bullet^{\mathsf{op}}$ given as follows: $G_\bullet(H, \mu) := (G(H, \mu) \cup \{ 0 \}, 0)$ and given a morphism of bisemigroups $f\colon ((H, \mu), \mu^\dagger) \to ((K, \gamma), \gamma^\dagger)$, $G_\bullet(f)\colon G_\bullet(K, \gamma) \to G_\bullet(H, \mu)$ is the restriction of $f^\dagger$. $G_\bullet(f)\colon G_\bullet(K, \gamma) \to G_\bullet(H, \mu)$ is indeed a partial injection because for each $g, h \in (f^\dagger)^{-1}(G(H, \mu)) \cap G(K, \gamma)$, $g \neq h$, $0 = f^\dagger(gh) = f^\dagger(g) f^\dagger(h)$ (since $f^\dagger$ is also a semigroup map), so that $f^\dagger(g) \neq f^\dagger(h)$ as $f^\dagger(g), f^\dagger(h) \in G(H, \mu)$.

The proof of the following result is left to the readers.

6.3 Proposition *One has an adjunction* $G_\bullet \dashv \ell^2_\bullet\colon {}_{\mathsf{partiso},c}\mathbf{FrobBisem}(\mathbb{Hilb}) \to \mathbf{PInj}_\bullet^{\mathsf{op}}$ *which restricts to an equivalence of categories* ${}^\dagger_c\mathbf{FrobBisem}(\mathbb{Hilb}) \simeq \mathbf{PInj}_\bullet^{\mathsf{op}}$.

6.2 Ambidextrous morphisms: Algebra-and-coalgebra maps

Even in the non partial isometric case, that is, even if $((H, \mu), \mu^\dagger)$ is not a bisemigroup, it is tempting to see what happens when morphisms of Frobenius alegbras are chosen as those bounded linear maps which are both algebra and coalgebra morphisms. Let ${}_c\mathbf{Frob}(\mathbb{Hilb})_{\mathsf{ambi}}$ be the corresponding non full subcategory of ${}_c\mathbf{FrobSem}(\mathbb{Hilb})$. (One drops the suffix "**Sem**" to emphasize the fact that both the semigroup and the cosemigroup structures are of equal importance.) In view of Remark 6.2 one introduces the category $\mathbf{Pinj}_{\bullet,\mathsf{w}}$ with

1. objects the weighted pointed sets as in $\mathbf{WSet}_\bullet$,
2. arrows $(X, x_0, \alpha) \xrightarrow{f} (Y, y_0, \beta)$ the partial injections $(X, x_0) \xrightarrow{f} (Y, y_0)$ such that for each $x \in f^{-1}(Y \setminus \{ y_0 \})$, $\alpha(x) = \beta(f(x))$.

It is clear that $\mathbf{PInj}_{\bullet,\mathsf{w}}$ embeds (while not fully) into $\mathbf{WSet}_\bullet$.

6.4 Proposition *One has an adjunction* $\mathrm{Min}_\bullet \dashv \ell^2_\bullet\colon {}_c\mathbf{Frob}(\mathbb{Hilb})_{\mathsf{ambi}} \to \mathbf{PInj}_{\bullet,\mathsf{w}}^{\mathsf{op}}$ *which restricts to an equivalence* ${}_{\mathsf{semisimple},c}\mathbf{Frob}(\mathbb{Hilb})_{\mathsf{ambi}} \simeq \mathbf{PInj}_{\bullet,\mathsf{w}}^{\mathsf{op}}$.

Proof: The functor $\ell^2_\bullet$ (resp. $\mathrm{Min}_\bullet$) occurring in the statement of the proposition is the only one which makes commute the following diagram on the left (resp. right).

$$\begin{array}{ccc} \mathbf{WSet}_\bullet^{\mathsf{op}} & \xrightarrow{\ell^2_\bullet} & {}_c\mathbf{FrobSem}(\mathbb{Hilb}) \\ \uparrow & & \uparrow \\ \mathbf{PInj}^{\mathsf{op}}_{\bullet,\mathsf{w}} & \xrightarrow[\ell^2_\bullet]{} & {}_c\mathbf{Frob}(\mathbb{Hilb})_{\mathsf{ambi}} \end{array} \qquad \begin{array}{ccc} {}_c\mathbf{FrobSem}(\mathbb{Hilb}) & \xrightarrow{\mathrm{Min}_\bullet} & \mathbf{WSet}_\bullet^{\mathsf{op}} \\ \uparrow & & \uparrow \\ {}_c\mathbf{Frob}(\mathbb{Hilb})_{\mathsf{ambi}} & \xrightarrow[\mathrm{Min}_\bullet]{} & \mathbf{PInj}^{\mathsf{op}}_{\bullet,\mathsf{w}} \end{array} \tag{18}$$

The proof follows from that of Theorem 5.8 (details are left to the readers). □

6.3 Proper morphisms

Call a $\mathbf{WSet}_\bullet$-morphism $(X, x_0, \alpha) \xrightarrow{f} (Y, y_0, \beta)$ *proper* when $f^{-1}(\{ y_0 \}) = \{ x_0 \}$ or equivalently $f(X \setminus \{ x_0 \}) \subseteq Y \setminus \{ y_0 \}$. It is clear that every $\mathbf{WSet}_\bullet$-isomorphism is proper.

Let $\mathbf{WSet}$ be the category of *weighted sets* with

1. objects the pairs $(X, \alpha: X \to [C, +\infty[)$, $C > 0$.
2. Arrows $(X, \alpha) \xrightarrow{f} (Y, \beta)$ the maps $X \xrightarrow{f} Y$ such that
   (a) $|f^{-1}(\{ y \})| < +\infty$ for each $y \in Y$,
   (b) there exists $M_f \geq 0$ such that for each $y \in Y$, $\sum_{x \in f^{-1}(\{ y \})} \alpha(x) \leq M_f \beta(y)$.

For a set X, let $X^+ := X + 1$, where $1 := \{ 0 \}$ and $+$ denotes the disjoint union. Let $X \xrightarrow{f} Y$ be a map. Define $X^+ \xrightarrow{f^+} Y^+$ by $f^+ := f + \mathrm{id}_1$, that is, roughly speaking, $f^+(x) = f(x)$, $x \in X$, $f^+(0) := 0$. This provides a functor $\mathbf{WSet} \xrightarrow{(-)^+} \mathbf{WSet}_\bullet$ which acts on objects as $(X, \alpha)^+ := (X^+, 0, \alpha)$, and which is injective on objects and faithful. Under this functor $\mathbf{WSet}$ is clearly equivalent to the (non full) subcategory of $\mathbf{WSet}_\bullet$ whose objects are those of $\mathbf{WSet}_\bullet$ but with proper morphisms between them.

Let $(H, \mu) \xrightarrow{f} (K, \gamma)$ be a semigroup morphism between Hilbertian Frobenius algebras. It is said to be *proper* when $\mathrm{ran}(f)$ is not included in any maximal modular ideals of (K, γ) or alternatively for each $y \in G(K, \gamma)$, there exists $u \in H$ such that $\langle f(u), y \rangle \neq 0$. Properness for f implies that $f^\dagger(y) \neq 0$ for each $y \in G(K, \gamma)$, and since $f^\dagger(G(K, \gamma)) \subseteq G(H, \mu) \cup \{ 0 \}$, it follows that actually $f^\dagger(G(K, \gamma)) \subseteq G(H, \mu)$. Conversely if $f^\dagger(G(K, \gamma)) \subseteq G(H, \mu)$, then for each $y \in G(K, \gamma)$, $\langle f(f^\dagger(y)), y \rangle \neq 0$ and thus f is proper. One observes that every semigroup isomorphism is proper.

Let ${}_c\mathbf{FrobSem}(\mathbb{Hilb})_{\mathsf{proper}}$ be the category whose objects are Hilbertian Frobenius algebras and morphisms are the proper semigroup morphisms. As usually let ${}_{\mathsf{semisimple},c}\mathbf{FrobSem}(\mathbb{Hilb})_{\mathsf{proper}}$ be its full subcategory spanned by the semisimple objects.

Let (X,α) be a **WSet**-object. Define $\ell^2(X,\alpha) := (\ell^2_\alpha(X),\mu_X)$ as given in Section 1.5. Let $(X,\alpha) \xrightarrow{f} (Y,\beta)$ be a **WSet**-morphism. Define $\ell^2(f)\colon(\ell^2_\beta(Y),\mu_Y) \to (\ell^2_\alpha(X),\mu_X)$ by $\ell^2(f)(u) := u \circ f$.

6.5 Lemma *$\ell^2(f)$ is a proper morphism.*

One obtains a functor $\ell^2\colon \mathbf{WSet}^{\mathsf{op}} \to {}_c\mathbf{FrobSem}(\mathbb{Hilb})_{\mathsf{proper}}$ and a co-restriction still denoted ℓ^2 from $\mathbf{WSet}^{\mathsf{op}}$ to ${}_{\mathsf{semisimple},c}\mathbf{FrobSem}(\mathbb{Hilb})_{\mathsf{proper}}$.

Now let $f\colon(H,\mu) \to (K,\gamma)$ be a proper morphism between Hilbertian Frobenius semigroups. As $f^\dagger(G(K,\gamma)) \subseteq G(H,\mu)$, one defines a map $\ell\colon \mathrm{Min}(K,\gamma) \to \mathrm{Min}(H,\mu)$ by the relation $\ell(J) := \mathbb{C}f^\dagger(ⓖ(J))$, $J \in \mathrm{Min}(K,\gamma)$ as in Section 5.2. Consequently, $\ell_0 \in \mathbf{WSet}_\bullet(\mathrm{Min}_\bullet(K,\gamma),\mathrm{Min}_\bullet(H,\mu))$, where ℓ_0 is the extension of ℓ obtained by setting $\ell_0(0) := 0$. As $\ell_0(J) = \ell(J) \neq 0$, $J \in \mathrm{Min}(K,\gamma)$, it follows that actually $\ell \in \mathbf{WSet}((\mathrm{Min}(K,\gamma),w_{(K,\gamma)}),(\mathrm{Min}(H,\mu),w_{(H,\mu)}))$, and from that one has a functor $\mathrm{Min}\colon {}_c\mathbf{FrobSem}(\mathbb{Hilb})_{\mathsf{proper}} \to \mathbf{WSet}^{\mathsf{op}}$. The adjunction $\mathrm{Min}_\bullet \dashv \ell^2_\bullet\colon {}_c\mathbf{FrobSem}(\mathbb{Hilb}) \to \mathbf{WSet}_\bullet^{\mathsf{op}}$ from Theorem 5.8, clearly restricts to an adjunction $\mathrm{Min} \dashv \ell^2\colon {}_c\mathbf{FrobSem}(\mathbb{Hilb})_{\mathsf{proper}} \to \mathbf{WSet}^{\mathsf{op}}$.

6.6 Proposition *One has an adjunction* $\mathrm{Min} \dashv \ell^2\colon {}_c\mathbf{FrobSem}(\mathbb{Hilb})_{\mathsf{proper}} \to \mathbf{WSet}^{\mathsf{op}}$ *that restricts to an equivalence* ${}_{\mathsf{semisimple},c}\mathbf{FrobSem}(\mathbb{Hilb})_{\mathsf{proper}} \simeq \mathbf{WSet}^{\mathsf{op}}$. *In particular,* ${}_{\mathsf{semisimple},c}\mathbf{FrobSem}(\mathbb{FdHilb})_{\mathsf{proper}} \simeq \mathbf{WFinSet}^{\mathsf{op}} \simeq \mathbf{FinSet}^{\mathsf{op}}$, *where* **FinSet** *is the category of finite sets with all maps between them and* **WFinSet** *is the full subcategory of* **WSet** *spanned by the weighted* finite *sets.*

The second statement of the following corollary corresponds to [6, Corollary 7.2, p. 566] as the categories ${}_c\mathbf{FrobComon}(\mathbb{FdHilb})$ and ${}_c\mathbf{FrobMon}(\mathbb{FdHilb})^{\mathsf{op}}$ are isomorphic under the dagger functor (cf. Section 1.4).

6.7 Corollary *There are equivalences* ${}_c\mathbf{FrobSem}(\mathbb{Hilb})_{\mathsf{proper}} \simeq \mathbf{WSet}^{\mathsf{op}} \times \mathbf{Hilb}$ *and* ${}_{\mathsf{semisimple},c}\mathbf{FrobSem}(\mathbb{FdHilb})_{\mathsf{proper}} \simeq {}_c\mathbf{FrobMon}(\mathbb{FdHilb}) \simeq \mathbf{FinSet}^{\mathsf{op}}$.

Proof: As the first statement is clear, one only needs to prove the second, and it is clear that one only needs to prove that the categories ${}_c\mathbf{FrobMon}(\mathbb{FdHilb})$ and ${}_{\mathsf{semisimple},c}\mathbf{FrobSem}(\mathbb{FdHilb})_{\mathsf{proper}}$ are equivalent. According to Corollary 3.1, the obvious forgetful functor ${}_c\mathbf{FrobMon}(\mathbb{FdHilb}) \xrightarrow{|-|} {}_c\mathbf{FrobSem}(\mathbb{FdHilb})$ factors through the embedding ${}_{\mathsf{semisimple},c}\mathbf{FrobSem}(\mathbb{FdHilb}) \hookrightarrow {}_c\mathbf{FrobSem}(\mathbb{FdHilb})$. One thus only needs to check that $|f|$ is actually proper for each monoid morphism f and that the co-restricted functor ${}_c\mathbf{FrobMon}(\mathbb{FdHilb}) \xrightarrow{|-|} {}_{\mathsf{semisimple},c}\mathbf{FrobSem}(\mathbb{FdHilb})_{\mathsf{proper}}$ is full.

Given a finite-dimensional Hilbertian Frobenius monoid (H,μ,η), by a direct inspection $\eta(1) = \sum_{g\in G(H,\mu)} \frac{g}{\|g\|^2}$. The corresponding counit $\eta^\dagger\colon H \to \mathbb{C}$ thus is given by $\eta^\dagger(u) = \sum_{g\in G(H,\mu)}\langle u, \frac{g}{\|g\|^2}\rangle$. In particular for each $g \in G(H,\mu)$, $\eta^\dagger(g) = 1$.

Let $(H,\mu,\eta) \xrightarrow{f} (H',\mu',\eta')$ be a monoid morphism between finite-dimensional Frobenius monoids. Then, $f^\dagger(G(H',\mu')) \subseteq G(H,\mu) \cup \{0\}$ and since $f^\dagger$ is compatible with the counits, for each $h \in G(H',\mu')$, $\eta^\dagger(f^\dagger(h)) = (\eta')^\dagger(h) = 1$. As a consequence f is proper.

Let $(H,\mu,\eta), (H',\mu',\eta')$ be finite-dimensional Frobenius monoids and let $(H,\mu) \xrightarrow{f} (H',\mu')$ be a proper semigroup morphism. Then for each $h \in G(H',\mu')$, there is one $g_h \in G(H,\mu)$ such that $f^\dagger(h) = g_h$ as $f^\dagger(G(H',\mu')) \subseteq G(H,\mu)$. Therefore, for each $h \in G(H',\mu')$, $\eta^\dagger(f^\dagger(h)) = \eta^\dagger(g_h) = 1 = (\eta')^\dagger(h)$. Whence $(\eta')^\dagger$ and $\eta^\dagger \circ f^\dagger$ are equal on $G(H',\mu')$ which spans H', so they are equal on the whole H', and as a consequence $f^\dagger : (H',\mu',\eta') \to (H,\mu,\eta)$ is a comonoid morphism, so that $(H,\mu,\eta) \xrightarrow{f} (H',\mu',\eta')$ is a morphism of monoids. This proves that ${}_c\mathbf{FrobMon}(\mathbb{FdHilb}) \xrightarrow{|-|} {}_{\mathsf{semisimple},c}\mathbf{FrobSem}(\mathbb{FdHilb})_{\mathsf{proper}}$ is full. $\square$

Let us define ${}_{1,c}\mathbf{Frob}(\mathbb{FdHilb})_{\mathsf{ambi}}$ to be the category of finite-dimensional Frobenius monoids whose morphisms preserve both the monoid and the comonoid structures. Let $\mathbf{FinSet}_{\mathsf{bij,w}}$ be the category of finite weighted sets and bijections between them preserving the weight functions, that is, $(X,\alpha) \xrightarrow{f} (Y,\beta)$ is given as a bijection $X \xrightarrow{f} Y$ such that $\alpha(x) = \beta(f(x))$ for each $x \in X$. $\mathbf{FinSet}_{\mathsf{bij,w}}$ clearly embeds into $\mathbf{WSet}$. The following result essentially is [6, Corollary 7.1, p. 365] but can be deduced as well from Corollary 6.7.

6.8 Corollary ${}_{1,c}\mathbf{Frob}(\mathbb{FdHilb})_{\mathsf{ambi}} \simeq \mathbf{FinSet}_{\mathsf{bij,w}}$ *so* ${}_{1,c}\mathbf{Frob}(\mathbb{FdHilb})_{\mathsf{ambi}}$ *is equivalent to the category of finite-dimensional semisimple Frobenius algebras with unitary isomorphisms of semigroups. Moreover every isomorphism of semigroups between finite-dimensional semisimple Frobenius algebras is a unitary transformation.*

7 Epilogue: And non-commutativity in all that?

Let X be a non-void set and let $x_0 \in X$. Let $m_{x_0} : \ell^2(X) \times \ell^2(X) \to \ell^2(X)$ be given by $m_{x_0}(u,v) := v(x_0)u$. It is of course bounded since $\|m_{x_0}(u)\|^2 = |v(x_0)|^2\|u\|^2 \le \|v\|^2\|u\|^2$. Then m_{x_0} is a weak Hilbert-Schmidt mapping as $\sum_{x,y} |\langle m_{x_0}(\delta_x,\delta_y), u\rangle|^2 = \sum_{x\in X} |\langle \delta_x, u\rangle|^2 = \|u\|^2$. Let $\mu_{x_0} : \ell^2(X)\hat{\otimes}_2\ell^2(X) \to \ell^2(X)$ be its unique bounded linear extension. $(\ell^2(X), \mu_{x_0})$ is a Hilbertian semigroup, non-commutative as soon as $X \setminus \{x_0\} \neq \varnothing$. Since $\langle \mu^\dagger_{x_0}(u), \delta_x \otimes \delta_y\rangle = \delta_{y,x_0} u(x)$ for each $x,y \in X$ it follows that $\mu^\dagger_{x_0}(u) = u \otimes \delta_{x_0}$. Consequently $\mu_{x_0}(\mu^\dagger_{x_0}(u)) = u$, $u \in \ell^2(X)$, that is, $\mu^\dagger_{x_0}$ is an isometry.

Moreover for each $u,v \in \ell^2(X)$, $\mu^\dagger_{x_0}(\mu_{x_0}(u\otimes v)) = v(x_0)\mu^\dagger_{x_0}(u) = v(x_0)u\otimes\delta_{x_0}$, $(\mathrm{id}\,\hat{\otimes}_2\mu_{x_0})(\alpha((\mu^\dagger_{x_0}\hat{\otimes}_2\,\mathrm{id})(u\otimes v))) = (\mathrm{id}\,\hat{\otimes}_2\mu_{x_0})(u\otimes(\delta_{x_0}\otimes v)) = v(x_0)u\otimes\delta_{x_0}$ and $(\mu_{x_0}\hat{\otimes}_2\,\mathrm{id})(\alpha^{-1}((\mathrm{id}\,\hat{\otimes}_2\mu^\dagger_{x_0})(u\otimes v))) = (\mu_{x_0}\hat{\otimes}_2\,\mathrm{id})((u\otimes v)\otimes\delta_{x_0}) = v(x_0)u\otimes\delta_{x_0}$. Therefore $(\ell^2(X), \mu_{x_0})$ is Frobenius. To summarize, $(\ell^2(X), \mu_{x_0})$ is a not necessarily commutative special Frobenius Hilbertian semigroup.

It is easily seen that $G(\ell^2(X), \mu_{x_0}) = \{ \delta_{x_0} \}$. It is also clear that $A(\ell^2(X), \mu_{x_0}) = \{ 0 \}$ as δ_{x_0} is a right unit, and that $\{ \delta_{x_0} \}^{\perp}$ consists entirely of nilpotent elements. Also $E(\ell^2(X), \mu_{x_0}) = \{ u \in \ell^2(X) : u(x_0) = 1 \} \cup \{ 0 \}$.

As $\ell^2(X) \to \mathbb{C}$, $u \mapsto u(x_0)$, is a morphism of algebras it follows that its kernel, namely $\{ \delta_{x_0} \}^{\perp} = \{ u : u(x_0) = 0 \}$ is a two-sided maximal modular ideal, with modular unit δ_{x_0}. Let I be a modular right ideal of $(\ell^2(X), m_{x_0})$, that is, I is right ideal with a left-unit e, that is, $u - eu \in I$ for each $u \in \ell^2(X)$. As for each $u \in \{ \delta_{x_0} \}^{\perp}$, $u = u - u(x_0)e = u - eu \in I$, it follows that $\{ \delta_{x_0} \}^{\perp} \subseteq I$. By a codimensionality argument it follows that either $I = \{ \delta_{x_0} \}^{\perp}$ or $I = \ell^2(X)$. Consequently $J(\ell^2(X), \mu_{x_0}) = \{ \delta_{x_0} \}^{\perp}$ and $(\ell^2(X), \mu_{x_0})$ is not semisimple as soon as $X \setminus \{ x_0 \} \neq \varnothing$. It is not a H^*-algebra either for if the annihilator would be equal to the Jacobson radical [21, Theorem 11.6.12, p. 1210].

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