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The solution to the embedding problem of a (differential) Lie algebra into its Wronskian envelope

Laurent Poinsot[a,b]

[a]LIPN - UMR CNRS 7030, University Paris 13, Sorbonne Paris Cité, Villetaneuse, France; [b]CReA, French Air Force Academy, Base aérienne 701, Salon-de-Provence, France

ABSTRACT
Any commutative algebra equipped with a derivation may be turned into a Lie algebra under the Wronskian bracket. This provides an entirely new sort of a universal envelope for a Lie algebra, the *Wronskian envelope*. The main result of this paper is the characterization of those Lie algebras which embed into their Wronskian envelope as Lie algebras of vector fields on a line. As a consequence we show that, in contrast to the classical situation, free Lie algebras almost never embed into their Wronskian envelope.

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1. Introduction and motivations

As is well known each associative algebra becomes a Lie algebra under its commutator bracket. This relation is functorial and admits a left adjoint, namely the universal enveloping algebra construction. As a consequence of the Poincaré-Birkhoff-Witt theorem [5, Théorème 1, p. 30], it is also known that as soon as a Lie algebra is free as a module over its base ring (in particular, any free Lie algebra), then it embeds as a Lie sub-algebra into its universal enveloping algebra. In other words, such Lie algebras have a faithful representation in an associative algebra. There are still more elaborate embedding conditions. E.g., this remains true over a Dedekind domain without assuming freeness of the Lie algebra [6], and actually it suffices to consider a Lie algebra without additive torsion [8]. Nevertheless it is not true for all Lie algebras [6].

If one only considers differential commutative algebras, there is another way to proceed, significantly different, which consists in replacing the commutator bracket (here, the zero bracket) by another Lie bracket, namely, the Wronskian: $W_d(a, b) = a * d(b) - d(a) * b$, where d is a derivation of the algebra $(A, *)$ under consideration. Once again this provides an algebraic functor (between varieties of universal algebras), which as such also admits a left adjoint (see [25, Section 6.2, pp. 68–71]). This leads to the notion of a "Wronskian envelope" of a (differential) Lie algebra. Therefore, given any (differential) Lie algebra $\mathfrak{g}$, there is a unique (up to a canonical isomorphism) differential commutative algebra $\mathcal{W}(\mathfrak{g})$ and a canonical Lie map $can_{\mathfrak{g}}$ from $\mathfrak{g}$ to $\mathcal{W}(\mathfrak{g})$ (seen as a Lie algebra under the Wronskian bracket) which is universal among all such pairs $(((A, *), d), \phi \colon \mathfrak{g} \to (A, W_d))$.

This being said one may ask under which conditions the canonical Lie map from a Lie algebra to its Wronskian envelope is one-to-one. For instance, over a field $\mathbb{K}$ of characteristic zero, $\mathfrak{sl}_2(\mathbb{K})$ embeds into its Wronskian envelope (Example 4).

This question about the embedding conditions was raised by Lawvere, in his PhD thesis [16], in the following terms "is the adjunction [Author's comment: between Lie algebras and differential

CONTACT Laurent Poinsot laurent.poinsot@lipn.univ-paris13.fr LIPN - UMR CNRS 7030, University Paris 13, Sorbonne Paris Cité, Villetaneuse 93140, France; CReA, French Air Force Academy, Base aérienne 701, Salon-de-Provence 13661, France.

commutative algebras under the Wronskian envelope construction] always an embedding, giving an entirely different sort of «universal enveloping algebra»for a Lie algebra?" and was later reformulated as follows "given a Lie algebra L, whether a sufficiently complicated vector field on a sufficiently high-dimensional variety can be found so that L can be faithfully represented by commutators of those very special vector fields which are in the module (over the ring of functions) generated by that particular one" in [17].

In the work presented here one shows that the answer to the first question is negative, since it not true that all Lie algebras embed into their Wronskian envelope. Notwithstanding this provides "an entirely different sort of «universal enveloping algebra»". Of course, the answer to the second question also is negative in general, even if Lie algebras of vector fields (on a line) play an important rôle.

As the main result of this paper (Theorem 17) is provided a solution to this "embedding problem".

The example of $\mathfrak{sl}_2(\mathbb{K})$ is rather relevant because it served as a guideline to solve the embedding problem. For our purpose, the most important property of $\mathfrak{sl}_2(\mathbb{K})$ is the fact that it may be faithfully realized as a sub-algebra of the Lie algebra of polynomial vector fields.

So a reasonable guess for a natural candidate to a positive answer for the embedding problem is a Lie sub-algebra of "vector fields on a line". Without providing all the details here, let us illustrate this notion.

Let $(A, *)$ be any commutative algebra and let d be a derivation of $(A, *)$. Let $A \cdot d$ be the set of all derivations of $(A, *)$ of the form $a \cdot d$ for some $a \in A$ (where $(a \cdot d)(x) = a * d(x)$, $x \in A$). With the commutator of derivations, it becomes a Lie algebra of *vector fields on the line* (A, d). One notices that $[a \cdot d, b \cdot d] = W_d(a, b) \cdot d$. Although $A \cdot d$ is not, in general, (isomorphic to) a Lie sub-algebra of (A, W_d) (because some members of A may annihilate d), it is isomorphic to $(A/I, W_{\tilde{d}})$ for some differential ideal I, and, in particular, this implies that the canonical map $can_{A\cdot d}$ is one-to-one. This provides a sufficient condition.

Actually, this even provides a necessary condition. Whence a Lie algebra embeds into its Wronskian envelope if, and only if, it is isomorphic to a sub-algebra of a Lie algebra of vector fields on some line (Theorem 17). E.g., the one-sided Witt algebra $(\mathbb{C}[x], W_{\frac{d}{dx}})$, the two-sided Witt algebra $(\mathbb{C}[x, x^{-1}], W_{\frac{d}{dx}})$, the Virasoro algebra (without central extension) or the Lie algebra of smooth vector fields on the circle S^1 ([20, Definitions 4.1, 4.2, 4.3.(a), 4.4, pp. 384–385]) embed into their respective Wronskian envelope.

After establishing the above characterization, in the remaining parts of the note one analyzes some of its consequences which, in particular, show that the Wronskian enveloping construction, while not completely unrelated to the usual enveloping algebra construction (Corollary 19), provides a rather, if not entirely, different theory. For instance, in the classical situation, a free Lie algebra is free as a module, whence embeds into its universal enveloping algebra, in consequence of what a free Lie algebra is isomorphic to its canonical image (consisting of "Lie polynomials") into its universal envelope. In contrast to this situation, it is never true that a free differential Lie algebra **DiffLie**$[X]$, for some non-void set X, embeds into its Wronskian envelope (see Theorem 37 and Corollary 41). In particular, the Lie algebra of "differential Lie polynomials", i.e., the smallest differential Lie sub-algebra of $\mathcal{W}(\mathbf{DiffLie}[X])$ generated by X, is generally not isomorphic to **DiffLie**$[X]$. This turns the differential Lie polynomials into interesting objects on their own, and not just a specific realization of a free Lie object (Section 3).

Some relations with other universal envelopes (related to Lie-Rinehart and Jacobi algebras) are finally studied in Section 4.

Because ultimately one compares algebraic structures, one uses rather intensively, except in Section 2, some definitions and results from universal algebra, in particular, the fundamental fact that any algebraic functor between varieties of universal algebras admits a left adjoint. In order to keep this text reasonably self-contained, very few basic notions from universal algebra are recalled in the Subsection 3.1, Section 3. The reader is kindly invited to refer to [9] for other known results. Category theory is also used mainly for presentation purpose. The reader is assumed comfortable with standard definitions and basic facts related to functors, in particular left adjoints. Not surprisingly, [19] is our reference on these subjects.

2. Statement of the main result

2.1. Preliminaries

Throughout this text, R denotes a non-trivial[1] commutative ring with a unit and every (Lie or associative) algebra is taken over R. Every module is unital, and every morphism between algebras with a unit is assumed to preserve the units.

Let $\mathsf{A} = (A, *)$ be a not necessarily associative nor unital R-algebra. Let $\mathbf{End}_R(A)$ be the R-algebra of all R-linear endomorphisms of the R-module A. Let $\mathfrak{Der}_R(\mathsf{A}) \subseteq \mathbf{End}_R(A)$ be the Lie R-algebra of all *derivations* on A, whence any $d \in \mathfrak{Der}_R(\mathsf{A})$ is a R-linear map $d\colon A \to A$ which satisfies Leibniz's rule: $d(a * b) = d(a) * b + a * d(b)$, $a, b \in A$. This definition applies equally for associative and Lie algebras. (A, d) is called a *differential (non-associative) algebra*. In what follows, one denotes by $|\mathsf{A}|$ the underlying module of an algebra A, whence if $\mathsf{A} = (A, *)$, then $|\mathsf{A}| = A$ (this notation will also serve to represent the underlying set of an algebra, see Section 3.2). One notices that the *zero derivation* 0 (such that $0(a) = a$, $a \in \mathsf{A}$) is available for all algebras (associative or not).

In what follows in order to simplify we feel free to make the harmless and standard abuse of notation that consists in identifying a differential algebra with its underlying algebra or an algebra with its underlying module or even more generally it will be allowed to identify some algebraic structure with the underlying object supporting it or with this carrier object with only a part of the algebraic structure as e.g., $(A, *)$ may represent an algebra with a unit, so without explicit mention of its unit.

Given a differential (not necessarily associative) algebra (A, d), a differential (two-sided) ideal is a usual ideal (i.e., a set I such that $\mathsf{A}I \subseteq I \supseteq I\mathsf{A}$) closed under the derivation ($d(I) \subseteq I$). The quotient of (A, d) by a differential ideal I becomes a differential (not necessarily associative) algebra denoted by $(\mathsf{A}/I, \tilde{d})$ (see, e.g., [12, 15]), where $\tilde{d}$ is the quotient derivation. If $X \subseteq \mathsf{A}$, then $\langle X \rangle_{diff}$ denotes the differential ideal of (A, d) generated by X.

By a *differential algebra* is meant a pair (A, d) where A is an associative R-algebra with a unit. If A is commutative, then (A, d) is referred to as a *differential commutative algebra*. A *morphism* of differential algebras (or a *differential algebra map*) is a usual algebra map (which is assumed to respect the identity elements) commuting with the derivations. One thus gets categories $\mathbf{DiffCom} \hookrightarrow \mathbf{DiffAss}$, and the same without the derivations $\mathbf{Com} \hookrightarrow \mathbf{Ass}$ (with obvious full inclusion functors). Of course "**Ass**" (respectively, "**Com**") stands for "associative (respectively, "associative and commutative") algebras (with a unit)". There are obvious forgetful functors $dCom\colon \mathbf{DiffCom} \to \mathbf{Com}$ and $dAss\colon \mathbf{DiffAss} \to \mathbf{Ass}$ which make commute the following diagram.

$$\begin{tikzcd} \mathbf{DiffCom} \arrow[r, hook] \arrow[d, "dCom"'] & \mathbf{DiffAss} \arrow[d, "dAss"] \\ \mathbf{Com} \arrow[r, hook] & \mathbf{Ass} \end{tikzcd} \tag{1}$$

Let **Lie** be the category of all Lie R-algebras (with usual Lie maps), and let **DiffLie** be that of all differential Lie R-algebras (once again the morphisms, also referred to as *differential Lie algebra maps*, are assumed to commute with the derivations). There is an evident forgetful functor $dLie\colon \mathbf{DiffLie} \to \mathbf{Lie}$ which forgets the derivation.

2.2. The Wronskian envelope and some of its elementary properties

Any differential commutative algebra $((A, *), d)$ may be seen as a Lie algebra (A, W_d) under the Wronskian bracket W_d as defined in the Introduction. Actually, $((A, W_d), d)$ is even a differential Lie algebra. This gives rise to forgetful functors $\mathbf{DiffCom} \xrightarrow{W} \mathbf{DiffLie} \xrightarrow{dLie} \mathbf{Lie}$. By abuse of language

[1] The fact that $R \neq (0)$ prevents us from dealing with trivial varieties (within the meaning of universal algebra) of R-algebras, see Section 3.1.

one also denotes the previous composite functor by W. These functors are algebraic functors between (concrete categories concretely isomorphic to) varieties of universal algebras and thus admit a left adjoint (see [25, Sections 3 and 4] for the details and also cf. [24]), the *Wronskian envelope* constructions for the W's functors, and the *differential envelope* construction for $dLie$.

In details this means the following. Let $\mathfrak{g}$ be any Lie algebra. There exists a differential Lie algebra $(\mathcal{Diff}(\mathfrak{g}), d)$, called the *differential envelope* of $\mathfrak{g}$, and a Lie map $can_{\mathfrak{g}} \colon \mathfrak{g} \to \mathcal{Diff}(\mathfrak{g})$, called the *canonical Lie map*, such that for each differential Lie algebra $(\mathfrak{h}, e)$ and each Lie map $\phi \colon \mathfrak{g} \to \mathfrak{h}$ there is a unique differential algebra map $\psi \colon (\mathcal{Diff}(\mathfrak{g}), d) \to (\mathfrak{h}, e)$ such that $\Psi \circ can_{\mathfrak{g}} = \phi$ (more formally one should write $dLie(\Psi) \circ can_{\mathfrak{g}} = \phi$, but there is not much risk forgetting a reference to $dLie$ since it is a faithful functor).

Similarly, given any differential Lie algebra $(\mathfrak{g}, d)$, there is a differential commutative algebra $(\mathcal{W}(\mathfrak{g}, d), D)$, called the *Wronskian envelope* of $(\mathfrak{g}, d)$, and a Lie map $can_{(\mathfrak{g},d)} \colon (\mathfrak{g}, d) \to (|\mathcal{W}(\mathfrak{g}, d)|, W_D) = W(\mathcal{W}(\mathfrak{g}, d), D)$, such that for every differential commutative algebra (A, e) and every differential Lie map $\phi \colon (\mathfrak{g}, d) \to (|\mathsf{A}|, W_e)$, there is a unique differential algebra map $\psi \colon (\mathcal{W}(\mathfrak{g}, d), D) \to (\mathsf{A}, e)$ such that $\psi \circ can_{(\mathfrak{g},d)} = \phi$ (or, strictly speaking, $W(\psi) \circ can_{(\mathfrak{g},d)} = \phi$).

By the usual composition of left adjoints, $(\mathcal{W}(\mathcal{Diff}(\mathfrak{g}), d), D)$ with the canonical Lie map $dLie(can_{(\mathcal{Diff}(\mathfrak{g}),d)}) \circ can_{\mathfrak{g}} \colon \mathfrak{g} \to (|\mathcal{W}(\mathcal{Diff}(\mathfrak{g}), d)|, W_D)$ provides the Wronskian envelope of a usual Lie algebra. When there is no risk of confusion one still denotes by $can_{\mathfrak{g}}$ the previous canonical Lie map. Moreover one defines $(\mathcal{W}(\mathfrak{g}), D) := (\mathcal{W}(\mathcal{Diff}(\mathfrak{g}), d), D)$ for a Lie algebra $\mathfrak{g}$.

Of course, as usually, each of these universal constructions is unique up to a canonical isomorphism.

Remark 1. It is not reflected in the notation $(\mathcal{W}(\mathfrak{g}, d), D)$ but the derivation D itself depends on the differential Lie algebra $(\mathfrak{g}, d)$.

Remark 2. Given a differential commutative algebra $((A, *), d)$, the multiplication $a \circ b := a * d(b)$ endows A with a structure of a pre-Lie algebra. This in turn provides a functor $PL \colon \mathbf{DiffCom} \to \mathbf{PreLie}$. We notice that the commutator bracket of $PL((A, *), d) = (A, \circ)$ is the Wronskian bracket W_d. So there is a factorization $W = \mathbf{DiffCom} \xrightarrow{PL} \mathbf{PreLie} \xrightarrow{C_{PreLie}} \mathbf{Lie}$, where by C_{PreLie} is meant the functor which consists in turning any pre-Lie algebra into a Lie algebra under the commutator bracket. Each of these functors has a left adjoint (all of them left unchanged the carrier sets of the algebras, and the categories under consideration are varieties in the sense of universal algebra). Therefore in a standard way the Wronskian enveloping functor $\mathcal{W}$ may be seen as the composition of two left adjoints $\mathcal{W}_{\mathbf{PreLie}} \circ \mathcal{U}_{\mathbf{PreLie}}$, where $\mathcal{W}_{\mathbf{PreLie}} \colon \mathbf{PreLie} \to \mathbf{DiffCom}$ (respectively, $\mathcal{U}_{\mathbf{PreLie}} \colon \mathbf{Lie} \to \mathbf{PreLie}$) is a left adjoint of PL (respectively, C_{PreLie}). In this text we do not study the functor $\mathcal{W}_{\mathbf{PreLie}}$ nor the functor $\mathcal{U}_{\mathbf{PreLie}}$, focusing only on $\mathcal{W}$. There is another notion [22] known as the "universal enveloping algebra of a pre-Lie algebra", defined by Oudom and Guin, which is essentially the composition $\mathcal{U} \circ C_{PreLie} \colon \mathbf{PreLie} \to \mathbf{Lie} \to \mathbf{Ass}$, i.e., the usual universal enveloping algebra $\mathcal{U}(A, [\cdot, \cdot])$ of the pre-Lie algebra $(A, \circ)$ seen as a Lie algebra $(A, [\cdot, \cdot])$ under its commutator bracket; in particular, this construction rarely provides a commutative algebra. It follows immediately that $\mathcal{W}$ is not the same as this universal enveloping algebra of a pre-Lie algebra.

Corollary 3 (of [25, Proposition 4.9, p. 71])**.** *Let $(\mathfrak{g}, d)$ be a differential Lie algebra which is free as a module with basis X. Then, $(\mathcal{W}(\mathfrak{g}, d), D)$ is $(R[X]/\langle xd(y) - d(x)y - \sum_{z \in X} \gamma^z_{x,y} z\rangle, \tilde{D}_d)$, with $R[X]$ the algebra of polynomials with variables in X, $D_d(x) = d(x)$ for each $x \in X$, $[x, y] = \sum_{z \in X} \gamma^z_{x,y} z$, $x, y \in X$, using the constants of structure of the Lie algebra $\mathfrak{g}$ and $\langle xd(y) - d(x)y - \sum_{z \in X} \gamma^z_{x,y} z\rangle$ the algebraic ideal generated by $xd(y) - d(x)y - \sum_{z \in X} \gamma^z_{x,y} z$, $x, y \in X$, in $R[X]$.*

Proof. This follows immediately from [25, Proposition 4.9, p. 71] where it is stated that the Wronskian envelope $\mathcal{W}(\mathfrak{g}, d)$ of a differential Lie algebra $(\mathfrak{g}, d)$ is the quotient of the symmetric algebra $\mathcal{S}(|\mathfrak{g}|)$ of the underlying module of $\mathfrak{g}$ by the algebraic ideal generated by $x\otimes d(y)-d(x)\otimes y-[x,y]$, $x,y\in\mathfrak{g}$, together with the quotient derivation $\tilde{D}_d$, where D_d is the unique derivation on $\mathcal{S}(|\mathfrak{g}|)$ which extends d. □

Example 4. Let R be a ring in which 2 is invertible. Let us consider the Lie algebra $\mathfrak{g}$ freely generated by e,f,h as a module, with bracket $[e,h]=e$, $[e,f]=2h$ and $[h,f]=f$. (When R is a field of characteristic zero, one recovers $\mathfrak{sl}_2(R)$.) It is a differential Lie algebra with derivation $d(e)=0$, $d(h)=e$ and $d(f)=2h$. Then, $\mathcal{W}(\mathfrak{g},d)=R[e,h,f]/\langle e^2-e,h(e-1),2h^2-f(1+e)\rangle$ with the quotient derivation $\tilde{D}_d$, where $D_d\in\mathfrak{Der}_R(R[e,h,f])$ is given by $D_d(e)=0$, $D_d(h)=e$, $D_d(f)=2h$. Moreover, the assignment $f\mapsto x^2$, $h\mapsto x$ and $e\mapsto 1$ defines a differential algebra map from $(R[e,h,f],D_d)$ onto $(R[x],\frac{d}{dx})$, which factors through $R[e,h,f]\to R[e,h,f]/\langle e^2-e,h(e-1),2h^2-f(1+e)\rangle$ and provides the differential algebra map $(\mathcal{W}(\mathfrak{g},d),\tilde{D}_d)\to(R[x],\frac{d}{dx})$ which is also induced by the one-to-one map $(\mathfrak{g},d)\hookrightarrow(R[x],W_{\frac{d}{dx}})$, $e\mapsto 1$, $h\mapsto x$ and $f\mapsto x^2$, using the universal property of the Wronskian envelope.

Let us enumerate a few elementary properties of the Wronskian envelope. First of all, it is an augmented differential algebra (the augmentation ideal is a differential ideal). The trivial Lie algebra (0) is a zero object for **Lie**, i.e., both an initial and a terminal object [19]. Let $t_{\mathfrak{g}}\colon\mathfrak{g}\to(0)$ and $i_{\mathfrak{g}}\colon(0)\hookrightarrow\mathfrak{g}$ be the terminal and initial Lie maps of $\mathfrak{g}$. With the zero derivation, (0) is also a zero object for **DiffLie** (one even has $t_{(\mathfrak{g},d)}=dLie(t_{\mathfrak{g}})$ and $i_{(\mathfrak{g},d)}=dLie(i_{\mathfrak{g}})$). So given a Lie algebra (respectively, differential Lie algebra) $\mathfrak{g}$ (respectively, $(\mathfrak{g},d)$), there is a unique differential algebra map $\epsilon\colon(\mathcal{W}(\mathfrak{g}),D)\to(\mathcal{W}(0),D)$ (respectively, $\epsilon\colon(\mathcal{W}(\mathfrak{g},d),D)\to(\mathcal{W}(0),D)$) such that $W(\epsilon)\circ can_{\mathfrak{g}}=can_{(0)}\circ t_{\mathfrak{g}}$ (respectively, $W(\epsilon)\circ can_{(\mathfrak{g},d)}=can_{(0)}\circ t_{(\mathfrak{g},d)}$). Now, because $\mathcal{W}$ is a left adjoint functor, it preserves initial objects and thus $(\mathcal{W}(0),D)\simeq(R,0)$ (canonically). Therefore, $\epsilon\colon(\mathcal{W}(\mathfrak{g}),D)\to(R,0)$ (respectively, $\epsilon\colon(\mathcal{W}(\mathfrak{g},d),D)\to(R,0)$) provides the augmentation map. Of course, $\ker\epsilon$ is a differential ideal.

Remark 5. The initial map $i_{\mathfrak{g}}\colon(0)\hookrightarrow\mathfrak{g}$ also canonically provides a differential algebra map $\eta\colon(R,0)\to(\mathcal{W}(\mathfrak{g}),D)$ which coincides with the unit map $1_R\mapsto 1_{\mathcal{W}(\mathfrak{g})}$.

Let $\mathfrak{g}$ be a (differential) Lie algebra and $\mathfrak{h}$ be a (differential) Lie sub-algebra of $\mathfrak{g}$ (both of the same type, differential or not). Let $incl\colon\mathfrak{h}\hookrightarrow\mathfrak{g}$ be the canonical inclusion. Then, there is a unique differential algebra map $I\colon(\mathcal{W}(\mathfrak{h}),D)\to(\mathcal{W}(\mathfrak{g}),D)$ such that the following diagram commutes.

$$\begin{tikzcd} W(\mathcal{W}(\mathfrak{h}),D) \arrow[r, "W(I)"] & W(\mathcal{W}(\mathfrak{g}),D) \\ \mathfrak{h} \arrow[u, "can_{\mathfrak{h}}"] \arrow[r, hook, "incl"'] & \mathfrak{g} \arrow[u, "can_{\mathfrak{g}}"'] \end{tikzcd} \tag{2}$$

Of course there is no reason why I would be one-to-one in general. The following result may be proved similarly to [5, Proposition 3, p. 25].

Proposition 6. *Let $\mathfrak{g}$ be a (differential) Lie algebra, and let I be a (differential) Lie ideal of $\mathfrak{g}$ (both of the same kind, i.e., both differential or both not). The canonical morphism of differential algebras $\mathcal{W}(\pi_I)\colon(\mathcal{W}(\mathfrak{g}),D)\to(\mathcal{W}(\mathfrak{g}/I),D)$ induced by the canonical epimorphism $\pi_I\colon\mathfrak{g}\to\mathfrak{g}/I$ is onto and its kernel is the differential ideal $J:=\langle can_{\mathfrak{g}}(I)\rangle_{diff}$ of $(\mathcal{W}(\mathfrak{g}),D)$ generated by $can_{\mathfrak{g}}(I)$. In other words, $(\mathcal{W}(\mathfrak{g}/I),D)\simeq(\mathcal{W}(\mathfrak{g})/J,\tilde{D})$ (canonically).*

Remark 7. As an application of Proposition 6 one gets $\mathcal{W}(\mathfrak{g}/\ker can_{\mathfrak{g}})\simeq W(\mathfrak{g})/\langle can_{\mathfrak{g}}(\ker can_{\mathfrak{g}})\rangle_{diff}\simeq\mathcal{W}(\mathfrak{g})$ (as differential algebras). Hence the (differential) Lie algebras $\mathfrak{g}$ and $\mathfrak{g}/\ker can_{\mathfrak{g}}$ share the same Wronskian envelope.

2.3. The main result

Because any Lie algebra $\mathfrak{g}$ becomes a differential Lie algebra $(\mathfrak{g}, 0)$ with the zero derivation, it follows that $can_{\mathfrak{g}} \colon \mathfrak{g} \to (\mathcal{Diff}(\mathfrak{g}), d)$ is one-to-one. (Indeed, there is a unique differential Lie algebra map $\psi \colon (\mathcal{Diff}(\mathfrak{g}), d) \to (\mathfrak{g}, 0)$ such that $dLie(\psi) \circ can_{\mathfrak{g}} = id_{\mathfrak{g}}$.) So every Lie algebra embeds into its differential envelope.

The situation is a little bit more subtle for the question of whether or not a (differential) Lie algebra embeds into its Wronskian envelope, or, in other terms, when the corresponding canonical map is one-to-one. The following easy result is useful in many situations (and is true in a more general setting, see Lemma 22).

Lemma 8. *Let $(\mathfrak{g}, d)$ (respectively, $\mathfrak{g}$) be a differential Lie algebra (respectively, Lie algebra). The following assertions are equivalent.*

1. *The canonical differential Lie map $can_{(\mathfrak{g},d)} \colon (\mathfrak{g}, d) \to W(\mathcal{W}(\mathfrak{g}, d), D)$ (respectively, the canonical Lie map $can_{\mathfrak{g}} \colon \mathfrak{g} \to W(\mathcal{W}(\mathfrak{g}), D)$) is one-to-one.*
2. *There exists a differential commutative algebra (A, e) and a differential Lie map $(\mathfrak{g}, d) \to ((|\mathsf{A}|, W_e), e)$ (respectively, a Lie map $\mathfrak{g} \to (|\mathsf{A}|, W_e)$) which is one-to-one.*

Proof. One only treats the case of differential Lie algebras. That 1. implies 2. is immediate. Let us assume the existence of a one-to-one differential Lie map $\phi \colon (\mathfrak{g}, d) \hookrightarrow (|\mathsf{A}|, W_e)$ for some differential commutative algebra (A, e). Then, there is a unique differential algebra map $\psi \colon (\mathcal{W}(\mathfrak{g}, d), D) \to (\mathsf{A}, e)$ such that $W(\psi) \circ can_{(\mathfrak{g},d)} = \phi$ which in turn shows that $can_{(\mathfrak{g},d)}$ is one-to-one. □

Remark 9. In particular, it follows from Lemma 8 that a (differential or not) Lie algebra $\mathfrak{g}$ embeds *canonically* into its Wronskian envelope, i.e., $can_{\mathfrak{g}}$ is one-to-one, if, and only if, there is a not necessarily canonical embedding from $\mathfrak{g}$ into its Wronskian envelope. Therefore there is no need to distinguish between "canonical" or "non canonical" embeddings.

Corollary 10. *Any abelian Lie algebra $\mathfrak{g}$ embeds into its Wronskian envelope. This remains true if $\mathfrak{g}$ is seen as a differential Lie algebra with the zero derivation.*

Proof. Let $\mathfrak{g}$ be any abelian Lie algebra. Let 0 be the zero endomorphism of $\mathfrak{g}$. According to [5, Proposition 7, p. 36], it extends uniquely as a derivation D_0 on the symmetric algebra $\mathcal{S}(\mathfrak{g})$. Of course, because 0 is also a derivation on $\mathcal{S}(\mathfrak{g})$, $0 = D_0$. Now, the canonical embedding $\mathfrak{g} \hookrightarrow \mathcal{S}(\mathfrak{g})$ provides a one-to-one Lie map from $\mathfrak{g}$ into $(|\mathcal{S}(\mathfrak{g})|, 0) = W(\mathcal{S}(\mathfrak{g}), 0)$. Hence, by Lemma 8, $\mathfrak{g}$ embeds into its Wronskian envelope. The second assertion is already provided by the proof. □

Let us now introduce our most fundamental ingredient. Let (A, d) be a differential commutative algebra. Let $d \in \mathfrak{Der}_R(\mathsf{A})$ and $a \in \mathsf{A}$. Then, $a \cdot d \colon x \mapsto ad(x)$ is again a derivation so that $\mathfrak{Der}_R(\mathsf{A})$ becomes a left A-module. Let $\mathsf{A} \cdot d = \{\, a \cdot d \colon a \in \mathsf{A} \,\}$ be the cyclic (left) sub-A-module generated by $d \in \mathfrak{Der}_R(\mathsf{A})$. It is a R-module, and, because $[a \cdot d, b \cdot d] = W_d(a, b) \cdot d$ it is a Lie R-sub-algebra of $\mathfrak{Der}_R(\mathsf{A})$. This is a particular kind of Passman's Lie algebras of Witt type [23].

Definition 11. Let us call $\mathsf{A} \cdot d$ the Lie R-algebra of *vector fields* on the line (A, d).

Let $ann(d) = \{\, a \in \mathsf{A} \colon \forall x \in \mathsf{A},\ ad(x) = 0 \,\}$ be the annihilator of d, i.e., the kernel of the A-linear map $\gamma \colon a \mapsto a \cdot d$, which is an ideal of the R-algebra A. It is even a differential ideal because if $a \in ann(d)$, then $0 = d(ad(x)) = d(a)d(x) + ad^2(x) = d(a)d(x)$ for every x, so that $d(a) \in ann(d)$. Therefore, $\mathsf{A} \cdot d$ becomes a differential Lie R-algebra with derivation $\partial_d(a \cdot d) = d(a) \cdot d$. (This is well defined since $d(ann(d)) \subseteq ann(d)$.) When there is no ambiguity one writes $\partial := \partial_d$.

Observe that in general $\mathsf{A}\cdot d$, while being a A-module and a Lie R-algebra, is not a Lie A-algebra (it is rather related to Lie-Rinehart algebras; see Section 4) because its Lie bracket is not A-linear. Next lemma is fundamental.

Lemma 12. $(\mathsf{A}\cdot d,\partial)\simeq W(\mathsf{A}/ann(d),\tilde{d})=((|\mathsf{A}/ann(d)|,W_{\tilde{d}}),\tilde{d})$ *(as differential Lie R-algebras) under* $a\cdot d\mapsto a+ann(d)$.

Proof. Let us consider $\mathsf{A}\xrightarrow{\gamma}\mathsf{A}\cdot d$, $\gamma(a)=a\cdot d$. It is a A-module map (and a R-module map) whose kernel is $ann(d)$. Since $\mathsf{A}\cdot d=im(\gamma)$, it follows that, as A-modules, and so also as R-modules, $\mathsf{A}/ann(d)\simeq\mathsf{A}\cdot d$ (where $\mathsf{A}/ann(d)$ is a left A-module under the canonical epimorphism $\mathsf{A}\to\mathsf{A}/ann(d)$). Let $\tilde{\gamma}\colon|\mathsf{A}/ann(d)|\to\mathsf{A}\cdot d$, $\tilde{\gamma}(a+ann(d))=\gamma(a)$, be the induced R-linear isomorphism. Since $[\gamma(a),\gamma(b)]=\gamma(W_d(a,b))$, $a,b\in\mathsf{A}$, it follows easily that $\tilde{\gamma}$ lifts to an isomorphism of Lie R-algebras from $(|\mathsf{A}/ann(d)|,W_{\tilde{d}})$ to $\mathsf{A}\cdot d$. Because furthermore $ann(d)$ is a differential ideal, and $\gamma(d(a))=\partial(\gamma(a))$, $a\in\mathsf{A}$, $\tilde{\gamma}$ turns out to be an isomorphism of differential Lie R-algebras from $((|\mathsf{A}/ann(d)|,W_{\tilde{d}}),\tilde{d})$ to $(\mathsf{A}\cdot d,\partial)$. The fact that $((|\mathsf{A}/ann(d)|,W_{\tilde{d}}),\tilde{d})=W(\mathsf{A}/ann(d),\tilde{d})$ is clear. This completes the proof. □

One says that the derivation d on A is *faithful* if $ann(d)=(0)$. In such case, by Lemma 12, $(\mathsf{A}\cdot d,\partial)\simeq W(\mathsf{A},d)$ (as differential Lie algebras). It is clear that the quotient derivation $\tilde{d}$ on $\mathsf{A}/ann(d)$, from Lemma 12, is faithful for every differential commutative algebra (A,d).

Definition 13. A (differential) Lie algebra of *vector fields on a line* is a (differential) Lie algebra which is isomorphic to a (differential) Lie sub-algebra of some $\mathsf{A}\cdot d$ (respectively, $(\mathsf{A}\cdot d,\partial)$).

Let **Vect** (respectively, **DiffVect**) be the full sub-category of **Lie** (respectively, **DiffLie**) spanned by the (differential) Lie algebras of vector fields on a line. The following diagram of functor commutes (where the two vertical arrows are the embedding functors, so that the bottom horizontal arrow is obtained by restriction and corestriction of the top horizontal arrow).

$$\begin{tikzcd}
\mathbf{DiffLie} \arrow[r, "dLie"] & \mathbf{Lie} \\
\mathbf{DiffVect} \arrow[u, hook] \arrow[r, "dLie"'] & \mathbf{Vect} \arrow[u, hook]
\end{tikzcd} \tag{3}$$

The following result is immediate.

Lemma 14.

1. **DiffVect** *and* **Vect** *are closed under sub-algebras, i.e., given any object* $\mathfrak{g}$ *of* **DiffVect** *(respectively,* **Vect***), then every differential Lie sub-algebra (respectively, Lie sub-algebra)* $\mathfrak{h}$ *of* $\mathfrak{g}$ *is also an object of* **DiffVect** *(respectively,* **Vect***).*
2. *Moreover, given any object* $\mathfrak{g}$ *of* **DiffVect***, then every (non-differential) Lie sub-algebra* $\mathfrak{h}$ *of* $dLie(\mathfrak{g})$ *is an object of* **Vect**.
3. **DiffVect** *and* **Vect** *are closed under direct products (even the empty one).*

According to Lemmas 8 and 12, $(\mathsf{A}\cdot d,\partial)$ embeds into its Wronskian envelope for every differential commutative algebra (A,d). This is also true from Lemma 14 for any (differential or not) Lie sub-algebra of $\mathsf{A}\cdot d$. Again by Lemma 8 this remains valid for any (differential) Lie algebra of vector field on a line. Therefore the following lemma is proved.

Lemma 15. *Every (differential or not) Lie algebra of vector fields on a line embeds into its Wronskian envelope.*

Let us now prove an assertion somewhat reciprocal to Lemma 15.

Lemma 16. *If* (A, d) *is a differential commutative algebra, then* $W(\mathsf{A}, d) = ((|\mathsf{A}|, W_d), d)$ *is a differential Lie algebra of vector fields on a line.*

Proof. Let $\mathsf{A}[x]$ be the R-algebra of polynomials on A. Let $D_d \in \mathfrak{Der}_R(\mathsf{A}[x])$ given by $D_d(ax) = d(a)x + a$ for each $a \in \mathsf{A}$ and extended to the whole $\mathsf{A}[x]$ using Leibniz's rule and R-linearity. Of course, the canonical embedding $\mathsf{A} \hookrightarrow \mathsf{A}[x]$, $a \mapsto a1$, lifts to a one-to-one map $(\mathsf{A}, d) \hookrightarrow (\mathsf{A}[x], D_d)$ (because $D_d(a1) = d(a)1 + aD_d(1) = d(a)1$). Furthermore $ann(D_d) = (0)$, i.e., D_d is a faithful derivation. Indeed, if P belongs to $ann(D_d)$, then, in particular, $P = PD_d(x) = 0$. Therefore, $W(\mathsf{A}[x], D_d) = ((|A[x]|, W_{D_d}), D_d) \simeq (\mathsf{A}[x] \cdot D_d, \partial_{D_d})$. Finally, the embedding $a \mapsto a1$ still defines an embedding $W(\mathsf{A}, d) = ((|\mathsf{A}|, W_d), d) \hookrightarrow W(\mathsf{A}[x], D_d)$ and thus provides an embedding $W(\mathsf{A}, d) \hookrightarrow (\mathsf{A}[x] \cdot D_d, \partial_{D_d})$, $a \mapsto a \cdot D_d$. □

We are now in position to establish our main result, providing the necessary and sufficient embedding conditions of a (differential) Lie algebra into its Wronskian envelope, and thus a satisfying solution to the embedding problem.

Theorem 17. *A (differential) Lie algebra embeds into its Wronskian envelope if, and only if, it is a (differential) Lie algebra of vector fields on a line.*

Proof. Lemma 15 provides the sufficient conditions. By Lemma 16, $W(\mathcal{W}(\mathfrak{g}), D)$ is a differential Lie algebra of vector fields on some line, for every (differential or not) Lie algebra $\mathfrak{g}$. When $can_{\mathfrak{g}} : \mathfrak{g} \to W(\mathcal{W}(\mathfrak{g}), D)$ is one-to-one, this implies that $\mathfrak{g}$ is also a (differential or not) Lie algebra of vector fields on some line (by Lemma 14 since $\mathfrak{g} \simeq im(can_{\mathfrak{g}})$). The necessity is proved. □

Given a (differential) Lie algebra $\mathfrak{g}$ (respectively, $(\mathfrak{g}, d)$) which embeds into its Wronskian envelope, according to Theorem 17 and from the proof of Lemma 16, $\mathfrak{g}$ (resp., $(\mathfrak{g}, d)$) may be seen as a (differential) Lie sub-algebra of vector fields on the line $(\mathcal{W}(\mathfrak{g})[x], D_D)$ (resp., $(\mathcal{W}(\mathfrak{g}, d)[x], D_D)$).

Remark 18. Corollary 10 in conjunction with Theorem 17 shows that any abelian Lie algebra is a Lie algebra of vector fields on some line.

As already stated in the Introduction, using Theorem 17 one gets many – and some interesting – examples of Lie algebras which embed into their Wronskian envelope. Would it be possible that every Lie algebra be a Lie algebra of vector fields on a line? No. Indeed, such Lie algebras satisfy a non-trivial identity (i.e., which is not satisfied by all Lie algebras) called the "standard Lie identity of degree 5" in [26], the identity T_4 in [14] (see Appendix A where this identity is displayed in some expanded form). It is defined as follows:

$$T_4(x_1, x_2, x_2, x_4, y) := \sum_{\sigma \in \mathfrak{S}_4} \epsilon(\sigma)[x_{\sigma(1)}, [x_{\sigma(2)}, [x_{\sigma(3)}, [x_{\sigma(4)}, y]]]] = 0 \tag{4}$$

for every y, x_1, x_2, x_3, x_4 in a Lie algebra, where $\epsilon(\sigma)$ denotes the sign representation of the permutation σ. As shown in [2, pp. 19–20], in the free Lie algebra $\mathbf{Lie}[x, y]$ on two generators, over a field, the Lie polynomial given by $T_4(x, y, [x, y], [x, [x, y]], x)$ is not reduced to zero. Therefore such a Lie algebra does not embed into its Wronskian envelope. Actually, the Lie polynomial $T_4(x, y, [x, y], [x, [x, y]], x)$ is a linear combination of monomials of length 8, with integer coefficients, in the free associative algebra $R\langle x, y\rangle$ over R (see below Eq. (5)). Of course, some coefficients may be equal to zero in R but it remains a non-void linear combination. Thus it is equal to zero if, and only if, the coefficient of each monomial is equal to zero, which is impossible at the exception of $R = (0)$. Whence for each non trivial ring R, no free Lie algebras on two generators embed into its Wronskian envelope, and clearly it remains the same for every free Lie algebra on at least two generators.

$$\begin{aligned} T_4(x, y, [x, y], [x, [x, y]], x) = &-x^4yxy^2 + 2x^4y^2xy - x^4y^3x + 6x^3yx^2y^2 - 13x^3yxyxy + 5x^3yxy^2x \\ &+ 2x^3y^2x^2y + x^3y^2xyx - x^3y^3x^2 - 13x^2yx^3y^2 + 28x^2yx^2yxy \\ &- 7x^2yx^2y^2x - x^2yxyx^2y - 15x^2yxyxyx + 7x^2yxy^2x^2 - 6x^2y^2x^3y \\ &+ 13x^2y^2x^2yx - 7x^2y^2xyx^2 + x^2y^3x^3 + 12xyx^4y^2 - 23xyx^3yxy \\ &+ xyx^3y^2x - 12xyx^2yx^2y + 37xyx^2yxyx - 13xyx^2y^2x^2 + 21xyxyx^3y \\ &- 37xyxyx^2yx + 15xyxyxyx^2 - xyxy^2x^3 - 2xy^2x^4y - xy^2x^3yx \\ &+ 7xy^2x^2yx^2 - 5xy^2xyx^3 + xy^3x^4 - 4yx^5y^2 + 6yx^4yxy + 2yx^4y^2x \\ &+ 10yx^3yx^2y - 21yx^3yxyx + 6yx^3y^2x^2 - 10yx^2yx^3y + 12yx^2yx^2yx \\ &+ yx^2yxyx^2 - 2yx^2y^2x^3 - 6yxyx^4y + 23yxyx^3yx - 28yxyx^2yx^2 \\ &+ 13yxyxyx^3 - 2yxy^2x^4 + 4y^2x^5y - 12y^2x^4yx + 13y^2x^3yx^2 \\ &- 6y^2x^2yx^3 + y^2xyx^4. \end{aligned} \tag{5}$$

Theorem 17 also provides a sufficient condition for the embedding of a Lie algebra into its universal enveloping algebra.

Corollary 19. *If a Lie algebra $\mathfrak{g}$ embeds into its Wronskian envelope, then it embeds also into its universal enveloping algebra (as a Lie sub-algebra under the commutator bracket).*

Proof. From Theorem 17 one may assume the existence of a differential commutative algebra (A, d) such that $\mathfrak{g} \hookrightarrow \mathsf{A} \cdot d$. The result follows from the following sequence of inclusions of Lie algebras because $\mathfrak{g}$ admits a faithful representation into an associative algebra: $\mathfrak{g} \hookrightarrow \mathsf{A} \cdot d \subseteq \mathfrak{Der}_R(\mathsf{A}) \subseteq \mathbf{End}_R(|\mathsf{A}|)$. □

3. Free (differential) Lie algebras of vector fields

Every (differential) Lie algebra freely generates a (differential) Lie algebra of vector fields, namely its canonical image into its Wronskian envelope. By contrast with the classical situation, free (differential) Lie algebras are almost never isomorphic with their canonical images, which equivalently means that they rarely embed into their Wronskian envelope (Theorem 37 and Corollary 41). To see this we first consider more general embedding problems, and related functorial relations, at the level of universal algebra.

3.1. *A glance at universal algebra*

Our main reference about universal algebra is [9]. A *signature* is a $\mathbb{N}$-graded set $(\Sigma(n))_{n\in\mathbb{N}}$. A member f of $\Sigma(n)$ is called a *function symbol of arity n* when $n > 0$, or a *constant symbol* if $n = 0$. A Σ*-algebra* is a pair (A, F) where A is a set and $F = (F_n)_{n\in\mathbb{N}}$ is a family of maps $F_n \colon \Sigma(n) \to A^{A^n}$, $n \in \mathbb{N}$, sometimes referred to as the Σ*-algebra structure* of A (when $c \in \Sigma(0)$, $F_0(c)$ is usually interpreted as a member of A and not as a map from $A^0 \to A$). A *homomophism* $\phi\colon (A, F) \to (B, G)$ between Σ-algebras is a map $\phi\colon A \to B$ that commutes to the "basic operations" of A and B, i.e., for each $n \in \mathbb{N}$ and each $f \in \Sigma(n)$, $\phi(F_n(f)(a_1, \cdots, a_n)) = G_n(f)(\phi(a_1), \cdots, \phi(a_n))$, $a_1, \cdots, a_n$. Σ-algebras and their homomorphisms form a category. This category has a faithful forgetful functor to the category **Set** of sets, and it is well known that it admits a left adjoint that makes possible the construction of the free Σ-algebra $\Sigma[X]$ generated by the set X.

An *equation* is a pair (s,t) of two elements of $\Sigma[\mathbb{N}]$ (the members of $\mathbb{N}$ are customary denoted by x_1, x_2, x_3 and so on, and called the *variables*). A class of Σ-algebras that satisfy a given set of equations (i.e., in which the equations hold) is called a *variety*. For instance, the empty set of equations provides the variety of all Σ-algebras.

Such a variety $\mathbf{V}$ spans a full sub-category of Σ-algebras (and, in what follows we make no distinction between the variety and its associated category) the objects of which are Σ-algebras which satisfy the defining equations of the variety $\mathbf{V}$, hereafter referred to as *Σ-algebras in* $\mathbf{V}$. As such it also has a forgetful functor to **Set**, so is a concrete category, and when the variety is non-trivial (i.e., when it admits algebras with at least two elements), it has a left adjoint, which provides the free Σ-algebra $\mathbf{V}[X]$ relatively to the variety $\mathbf{V}$ generated by a set X.

3.2. A general embedding problem

From now on one only considers non-trivial varieties without further ado.

Given a variety $\mathbf{V}$ of Σ-algebras and a variety $\mathbf{W}$ of Ω-algebras, a functor $U\colon \mathbf{V} \to \mathbf{W}$ is called *algebraic* when the following diagram of functors commutes (where $|\cdot|$ denotes the forgetful functors).

$$\begin{tikzcd} \mathbf{V} \arrow[rr, "U"] \arrow[dr, "|\cdot|"'] & & \mathbf{W} \arrow[dl, "|\cdot|"] \\ & \mathbf{Set} & \end{tikzcd} \tag{6}$$

A fundamental result states that such an algebraic functor admits a left adjoint (see [3, Corollary 8.17, p. 28]).

Definition 20. In the situation depicted by Diagram (6), for each Ω-algebra (A,F) in $\mathbf{W}$, one has a Σ-algebra $\mathbf{V}[A,F]$ (called the *free Σ-algebra* in $\mathbf{V}$ generated by (A,F) or the *universal $\mathbf{V}$-envelope* of (A,F)) and a homomorphism $can_{(A,F)}\colon (A,F) \to U(\mathbf{V}[A,F])$ of Ω-algebras, called the *canonical map*, such that for every Σ-algebra (B,G) in $\mathbf{V}$ and every homomorphism $\phi\colon (A,F) \to U(B,G)$ of Ω-algebras, there is a unique homomorphism $\psi\colon \mathbf{V}[A,F] \to (B,G)$ of Σ-algebras such that $U(\psi) \circ can_{(A,F)} = \phi$. Of course, this provides the "free algebra" functor $\mathbf{V}[\cdot]\colon \mathbf{W} \to \mathbf{V}$.

Remark 21. In a situation where both notations $\mathbf{V}[X]$ for a set X and $\mathbf{V}[A,F]$ for a Ω-algebra (A,F) occur, in order to avoid confusion, one denotes the former Σ-algebra by $\mathbf{V}[A,F]_{\mathbf{W}}$.

There is an associated *embedding problem*: what are the conditions under which $can_{(A,F)}$ is one-to-one? As already announced in Section 2.3 (Lemma 8) one has the following useful result, whose proof follows the same principle as that of Lemma 8.

Lemma 22. *Let (A,F) be a Ω-algebra in $\mathbf{W}$. The following assertions are equivalent.*
1. *The canonical map $can_{(A,F)}\colon (A,F) \to U(\mathbf{V}[A,F])$ is one-to-one.*
2. *There exists a Σ-algebra (B,G) in $\mathbf{V}$ and a homomorphism of Ω-algebras from (A,F) to $U(B,G)$ which is one-to-one.*

3.3. The category of algebras which embed into their envelope

In this section one assumes that we are in the situation illustrated by Diagram (6).

Definition 23. Let $\mathbf{W}_U$ be the full sub-category of $\mathbf{W}$ spanned by those Ω-algebras (A,F) in $\mathbf{W}$ whose canonical map $can_{(A,F)}\colon (A,F) \to U(\mathbf{V}[A,F])$ is one-to-one.

Remark 24. It is clear from Lemma 22 that for all (A, F) in $\mathbf{V}$, $U(A, F)$ is an object of $\mathbf{W}_U$.

The notion of sub-algebras is rather evident. It is clear that $\mathbf{W}_U$ is closed under sub-algebras (i.e., every sub-algebra of an object of $\mathbf{W}_U$ is also an object of $\mathbf{W}_U$). Furthermore, it is also closed under direct products (cartesian products of the underlying sets of algebras with the component-wise algebra structure). Finally, the obvious inclusion functor $E_U : \mathbf{W}_U \hookrightarrow \mathbf{W}$ is (of course) full, but also faithful and injective on objects.

Example 25.
1. With $\mathbf{V} = \mathbf{DiffCom}$, $\mathbf{W} = \mathbf{Lie}$ (respectively, $\mathbf{DiffLie}$), $U = W$ one has $\mathbf{W}_U = \mathbf{Vect}$ (respectively, $\mathbf{DiffVect}$) by Theorem 17.
2. With $\mathbf{V} = \mathbf{Ass}$, $\mathbf{W} = \mathbf{Lie}$, $U = C$ which transforms the algebra into a Lie algebra under the commutator bracket, then $\mathbf{W}_U$ is the category of all Lie algebras which embed into their universal enveloping algebra. In particular, when the base ring R is a field, $\mathbf{W}_U = \mathbf{Lie}$.
3. With $\mathbf{V} = \mathbf{Com}$, $\mathbf{W} = \mathbf{Ass}$, U is the evident forgetful functor, then $\mathbf{W}_U = \mathbf{Com}$ (since an algebra embeds into its abelianization if, and only if, it is commutative).
4. With $\mathbf{V}$ the category $\mathbf{Ab}$ of abelian groups, and $\mathbf{W}$ that of commutative monoids, and with U the obvious forgetful functor, $\mathbf{W}_U$ is the category of all cancellative commutative monoids [7].

Before going on, one recalls the following facts: let (A, F) and (B, G) be two Σ-algebras in a variety $\mathbf{V}$, and let $\phi : (A, F) \to (B, G)$ be a homomorphism.
1. The *image* $im(\phi) = \phi(A)$ is a sub-algebra of (B, G) and, as such, is an object of $\mathbf{V}$ (since a variety is closed under sub-algebras, cartesian products and homomorphic image).
2. The *kernel* $\ker\phi = \{ (a, b) \in A^2 : \phi(a) = \phi(b) \}$ is a congruence (i.e., an equivalence relation on A compatible with the Σ-algebra structure F [8]), the quotient set $A/\ker\phi$ admits a natural structure of Σ-algebra denoted by $(A/\ker\phi, \tilde{F})$, which is an object of $\mathbf{V}$, and, finally, the canonical epimorphism $\pi : A \to A/\ker\phi$ lifts to a homomorphism of Σ-algebras $\pi : (A, F) \to (A/\ker\phi, \tilde{F})$.

Let us go back to the situation illustrated by Diagram (6). One observes that for every Ω-algebra (A, F) in $\mathbf{W}$, $im(can_{(A,F)})$ is a Ω-sub-algebra of $U(\mathbf{V}[A, F])$ which belongs to $\mathbf{W}$, and thus is an object of $\mathbf{W}_U$. It is of course the least Ω-sub-algebra of $U(\mathbf{V}[A, F])$ generated by $im(can_{(A,F)})$ (i.e., the intersection of all Ω-sub-algebras of $U(\mathbf{V}[A, F])$ that contain $im(can_{(A,F)})$).

Definition 26. Hereafter $im(can_{(A,F)})$ as above is referred to as the *canonical image* of (A, F) into its universal $\mathbf{V}$-envelope.

Moreover, as Ω-algebras in $\mathbf{W}$,

$$im(can_{(A,F)}) \simeq (A/\ker can_{(A,F)}, \tilde{F}).$$

This follows from the first isomorphism theorem for (universal) algebras as indicated in the diagram below (where (A, F) is a Ω-algebra in $\mathbf{W}$, $\widetilde{can}_{(A,F)}$ denotes the induced isomorphism, and *incl* is the canonical inclusion).

$$\begin{tikzcd}
(A, F) \arrow[r, "can_{(A,F)}"] \arrow[d, "\pi"'] & U(\mathbf{V}[A, F]) \\
(A/\ker can_{(A,F)}, \tilde{F}) \arrow[r, "\sim", "\widetilde{can}_{(A,F)}"'] & im(can_{(A,F)}) \arrow[u, hook, "incl"']
\end{tikzcd} \tag{7}$$

Remark 27. It follows from Diagram (7) that π is one-to-one (whence an isomorphism) if, and only if, $can_{(A,F)}$ is one-to-one if, and only if, (A, F) is an object of $\mathbf{W}_U$.

Actually, $(A,F) \mapsto im(can_{(A,F)})$ is the object component of a functor $\mathbf{Sp}_U\colon \mathbf{W} \to \mathbf{W}_U$ which is defined as follows. Let $\phi\colon (A,F) \to (B,G)$ be a homomorphism of Ω-algebras in $\mathbf{W}$. Let $\hat{\phi}\colon \mathbf{V}[A,F] \to \mathbf{V}[B,G]$ be the unique homomorphism of Σ-algebras in $\mathbf{V}$ such that the following diagram commutes.

$$\begin{tikzcd} U(\mathbf{V}[A,F]) \arrow[r, "U(\hat{\phi})"] & U(\mathbf{V}[B,G]) \\ (A,F) \arrow[u, "can_{(A,F)}"] \arrow[r, "\phi"'] & (B,G) \arrow[u, "can_{(B,G)}"'] \end{tikzcd} \tag{8}$$

Let $y = can_{(A,F)}(x)$ for some $x \in A$. So,

$$can_{(B,G)}(\phi(x)) = U(\hat{\phi})(can_{(A,F)}(x)) = U(\hat{\phi})(y).$$

Therefore there is a unique map $\phi_0\colon im(can_{(A,F)}) \to im(can_{(B,G)})$ such that the following diagram commutes.

$$\begin{tikzcd} U(\mathbf{V}[A,F]) \arrow[r, "U(\hat{\phi})"] & U(\mathbf{V}[B,G]) \\ im(can_{(A,F)}) \arrow[u, hook, "incl"] \arrow[r, "\phi_0"'] & im(can_{(B,G)}) \arrow[u, hook, "incl"'] \end{tikzcd} \tag{9}$$

That ϕ_0 is a homomorphism of Ω-algebras is obvious. One thus defines $\mathbf{Sp}_U(\phi) := \phi_0$. The remaining details to check functoriality of $\mathbf{Sp}_U\colon \mathbf{W} \to \mathbf{W}_U$ are left to the reader.

Remark 28. "**Sp**" stands for "special" because of [9] where the Lie algebras that embed into their universal enveloping algebra are referred to as "special Lie algebras".

The result below shows that $\mathbf{W}_U$ is a reflective sub-category of $\mathbf{W}$ [19].

Theorem 29. *The functor $\mathbf{Sp}_U$ provides a left adjoint of the canonical embedding $E_U\colon \mathbf{W}_U \hookrightarrow \mathbf{W}$.*

Proof. One uses the notations from Diagram (7). Let (A,F) be a Ω-algebra in $\mathbf{W}_U$. Then, $(\widetilde{can}_{(A,F)} \circ \pi)^{-1}\colon \mathbf{Sp}_U(E_U(A,F)) \simeq (A,F)$ (Remark 27). So one has a natural isomorphism[2] $\epsilon\colon \mathbf{Sp}_U \circ E_U \Rightarrow id_{\mathbf{W}_U}\colon \mathbf{W}_U \to \mathbf{W}_U$. Now, from Diagram (7), one gets $\eta := \pi \circ \widetilde{can}\colon id_{\mathbf{W}} \Rightarrow E_U \circ \mathbf{Sp}_U\colon \mathbf{W} \to \mathbf{W}$. By definition, the composite $E_U(\epsilon_{(A,F)}) \circ \eta_{(A,F)} = id_{(A,F)}$ for every algebra (A,F) in $\mathbf{W}_U$. Now, let (A,F) be a Ω-algebra in $\mathbf{W}$. The following commutative Diagram (10) implies that $\mathbf{Sp}(\eta_{(A,F)}) = \eta_{im(can_{(A,F)})}$ so that $\epsilon_{im(can_{(A,F)})} \circ \mathbf{Sp}(\eta_{(A,F)}) = id_{(A,F)}$. Therefore the two natural transformations satisfy the usual triangle equalities that characterize an adjunction [19], and shows that $\mathbf{Sp}_U$ is a left adjoint to E_U.

$$\begin{tikzcd} im(can_{(A,F)}) \arrow[r, "\mathbf{Sp}_U(\eta_{(A,F)})"] \arrow[d, hook, "incl"] & im(can_{im(can_{(A,F)})}) \arrow[d, hook, "incl"'] \\ U(\mathbf{V}[A,F]) \arrow[r, "\hat{\eta}_{(A,F)}"] & U(\mathbf{V}[im(can_{(A,F)})]) \\ (A,F) \arrow[u, "can_{(A,F)}"] \arrow[r, "\eta_{(A,F)}"'] \arrow[uu, bend left=60, "\eta_{(A,F)}"] & im(can_{(A,F)}) \arrow[u, hook, "can_{im(can_{(A,F)})}"'] \arrow[uu, bend right=60, "\eta_{im(can_{(A,F)})}"'] \end{tikzcd} \tag{10}$$

□

[2] $\alpha\colon F \Rightarrow G\colon \mathbf{C} \to \mathbf{D}$ denotes a natural transformation from F to G which are functors from $\mathbf{C}$ to $\mathbf{D}$.

Remark 30. It follows from Theorem 29 that given a Ω-algebra (A, F) in $\mathbf{W}$, $im(can_{(A,F)}) = \mathbf{Sp}_U(A, F)$ satisfies the following universal property. Let (B, G) be an algebra in $\mathbf{W}_U$ and let $\phi\colon (A, F) \to E_U(B, G)$ be a homomorphism of Ω-algebras. Then, there is a unique homomorphism of Ω-algebras $\psi : im(can_{(A,F)}) \to (B, G)$ such that $E_U(\psi) \circ \pi \circ \widetilde{can}_{(A,F)} = \phi$, under the notation from Diagram (7). This makes it possible to call $im(can_{(A,F)})$ the *free Ω-algebra in $\mathbf{W}$ which embeds into its $\mathbf{V}$-envelope* generated by (A, F).

Let us summarize the situation by the following commutative diagram of functors, where U is the functor from Diagram (6), and U_0 is obtained by co-restricting U to $\mathbf{W}_U$.

$$\begin{tikzcd} \mathbf{V} \arrow[rr, "U"] \arrow[dr, "U_0"'] & & \mathbf{W} \\ & \mathbf{W}_U \arrow[ur, hook, "E_U"'] & \end{tikzcd} \tag{11}$$

One already knows that U and E_U have a left adjoint. In general that does not imply that also U_0 has a left adjoint (in particular because $\mathbf{W}_U$ is not a variety since it is generally not closed under homomorphic images, and thus U_0 is not algebraic), but it does in this particular situation.

Lemma 31. *Let $\mathbf{C}, \mathbf{D}, \mathbf{E}$ be three categories such that the following diagram of functors commutes, where E is fully faithful and U admits a left adjoint $F\colon \mathbf{D} \to \mathbf{C}$.*

$$\begin{tikzcd} \mathbf{C} \arrow[rr, "U"] \arrow[dr, "V"'] & & \mathbf{D} \\ & \mathbf{E} \arrow[ur, "E"'] & \end{tikzcd} \tag{12}$$

Then, $F \circ E\colon \mathbf{E} \to \mathbf{C}$ is a left adjoint of V. Under the previous assumptions, if, moreover, E also has a left adjoint, say $G\colon \mathbf{D} \to \mathbf{E}$, then $F \circ E \circ G\colon \mathbf{D} \to \mathbf{C}$ is also a left adjoint of U, and thus F and $F \circ E \circ G$ are naturally isomorphic.

Proof. Let e be an object of $\mathbf{E}$ and c an object of $\mathbf{C}$. The following holds.

$$\begin{aligned} \mathbf{E}(e, V(c)) &\simeq \mathbf{D}(E(e), E(V(c))) \\ &\quad \text{(because } E \text{ is fully faithful)} \\ &= \mathbf{D}(E(e), U(c)) \\ &\quad \text{(since } E \circ V = U) \\ &\simeq \mathbf{C}(F(E(e)), c) \\ &\quad \text{(since } F \text{ is a left adjoint of } U) \end{aligned} \tag{13}$$

The second assertion follows from the fact that, by composition of adjoints, $F \circ E \circ G$ is a left adjoint of $E \circ V = U$. □

Lemma 31 applies in the situation depicted in Diagram (11), and thus

1. U_0 has a left adjoint given by $\mathbf{V}[\cdot] \circ E_U$,
2. $\mathbf{V}[\cdot] \circ E_U \circ \mathbf{Sp}_U \simeq \mathbf{V}[\cdot]$ (naturally) is a left adjoint of U. This means that given (A, F) a Ω-algebra in $\mathbf{W}$, then $\mathbf{V}[A, F] \simeq \mathbf{V}[E_U(\mathbf{Sp}_U(A, F))] = \mathbf{V}[E_U(im(can_{(A,F)}))]$. (Remark 7 was merely a special instance of this result.)

3.4. The free algebra on a set that embeds into its envelope

Before applying the results of the previous section to the case of (differential) Lie algebras, as some final comments one adds that in the following commutative diagram (combining Diagrams (6) and (11)), each functor has a left adjoint. Here $|\cdot|\colon \mathbf{W}_U \to \mathbf{Set}$ is the evident restriction of the forgetful functor $|\cdot|\colon \mathbf{W} \to \mathbf{Set}$.

$$\begin{tikzcd}
\mathbf{V} \arrow[rr, "U"] \arrow[dr, "U_0"] \arrow[ddr, bend right, "|\cdot|"'] & & \mathbf{W} \arrow[ddl, bend left, "|\cdot|"] \\
& \mathbf{W}_U \arrow[ur, hook, "E_U"'] \arrow[d, "|\cdot|"] & \\
& \mathbf{Set} &
\end{tikzcd} \tag{14}$$

What is new in the above diagram is the fact that $|\cdot|\colon \mathbf{W}_U \to \mathbf{Set}$ also has a left adjoint, although $\mathbf{W}_U$ is in general not a variety. This does not follow from an application of Lemma 31, but from an evident composition of left adjoints: given a set X the free Ω-algebra in $\mathbf{W}_U$ is given by $\mathbf{Sp}_U(\mathbf{W}[X]) = im(can_{\mathbf{W}[X]})$, and thus consists of the canonical image of the free algebra $\mathbf{W}[X]$ into its universal $\mathbf{V}$-envelope. There is another way to describe $\mathbf{W}_U[X]$.

Lemma 32. *Given a set X, the least Ω-algebra $\langle X\rangle_\Omega$ in $U(\mathbf{V}[X])$ generated by X, where $\mathbf{V}[X]$ is the free Σ-algebra in $\mathbf{V}$ over X, is the free Ω-algebra $\mathbf{W}_U[X]$ in $\mathbf{W}_U$ on X.*

Proof. Indeed, first of all, $\langle X\rangle_\Omega$ is an algebra of $\mathbf{W}_U$ (it belongs to $\mathbf{W}$ since $U(\mathbf{V}[X])$ does and $\mathbf{W}$ is closed under sub-algebras, and it embeds into its universal $\mathbf{V}$-envelope because of Lemma 22). Second, let $\phi\colon X \to |(A,F)| = A$ be a map, where (A,F) is a Ω-algebra in $\mathbf{W}_U$, whence $can_{(A,F)}$ is one-to-one. There is a unique homomorphism of Σ-algebras $\psi\colon \mathbf{V}[X] \to \mathbf{V}[A,F]$ such that $|\psi| \circ can_X = |can_{(A,F)}| \circ \phi$, where $can_X\colon X \to |\mathbf{V}[X]|$ is the canonical embedding (the canonical map from a set to the free algebra over this set in a non-trivial variety is always one-to-one [9]). The restriction of $U(\psi)$ on $\langle X\rangle_\Omega$ is a homomorphism of Ω-algebras which extends ϕ. Because it is determined by its values on X, it is unique. Therefore $\langle X\rangle_\Omega$ has to be the free Σ-algebra in $\mathbf{W}_U$ generated by X. □

Remember the use of notation $\mathbf{V}[A,F]_\mathbf{W}$ for $\mathbf{V}[A,F]$ from Remark 21 to distinguish the former from $\mathbf{V}[X]$ when both notations occur.

Corollary 33. *Let X be a set. Then,*

$$\mathbf{V}[X] \simeq \mathbf{V}[\mathbf{W}[X]]_\mathbf{W} = \mathbf{V}[E_U(\mathbf{W}_U[X])]_\mathbf{W}.$$

Proof.

$$\begin{aligned}
\mathbf{V}[X] &\simeq \mathbf{V}[\mathbf{W}[X]]_\mathbf{W} \\
&\quad \text{(by composition of left adjoints)} \\
&\simeq \mathbf{V}[E_U(\mathbf{Sp}_U(\mathbf{W}[X]))]_\mathbf{W} \\
&\quad \text{(according to the paragraph following the proof of Lemma 31)} \\
&= \mathbf{V}[E_U(\mathbf{W}_U[X])]_\mathbf{W} \\
&\quad \text{(by definition of } \mathbf{W}_U[X]\text{)}
\end{aligned} \tag{15}$$

□

Finally it may be useful to recall the explicit (or rather inductive) construction of $\langle X\rangle_\Omega = \mathbf{W}_U[X]$ (see [8, Proposition 5.1, p. 79]). Let $X_0 := X$, and for each $n \in \mathbb{N}$, let $X_{n+1} := X_n \cup \{ F_k(f)(t_1, \cdots, t_k)\colon k \in \mathbb{N},$

$f \in \Omega(k),\ t_1, \cdots, t_k \in X_n\}$ (where $(F_n)_n$ is the Ω-algebra structure on $U(\mathbf{V}[X])$). Then,

$$\langle X \rangle_\Omega = \bigcup_{k \geq 0} X_k.$$

Remark 34. Every Ω-algebra in $\mathbf{W}$ which embeds into its $\mathbf{V}$-envelope is a quotient of some free algebra $\mathbf{W}_U[X]$.

3.5. Free (differential) Lie algebras of vector fields

The constructions from the previous sections apply without fail to the case of Lie algebras. More precisely, one first has the following commutative diagram which consists in three particular instances of Diagram (6), where the rôle of U is played by the functors W and C, which transforms a differential commutative algebra (respectively, an associative algebra) into a (differential) Lie algebra under the Wronskian bracket (respectively, commutator bracket, see Example 25.2). The unnamed arrows are the evident forgetful functors.

$$\begin{tikzcd}
\mathbf{DiffCom} \arrow[r, "W"] \arrow[d, "W"'] \arrow[ddr] & \mathbf{Lie} \arrow[dd] & \mathbf{Ass} \arrow[l, "C"'] \arrow[ddl] \\
\mathbf{DiffLie} \arrow[dr] & & \\
& \mathbf{Set} &
\end{tikzcd} \tag{16}$$

As we proceeded to pass from Diagram (6) to Diagram (11), one obtains the following diagram from Diagram 16, completed with some already known functors.

$$\begin{tikzcd}
\mathbf{DiffVect} \arrow[dr, hook, "E_W"] \arrow[ddd, bend right=60, "dLie"'] & & \\
& \mathbf{DiffLie} \arrow[d, "dLie"] & \\
\mathbf{DiffCom} \arrow[uu, "W_0"] \arrow[ur, "W"] \arrow[r, "W"] \arrow[d, "W_0"] & \mathbf{Lie} & \mathbf{Ass} \arrow[l, "C"'] \arrow[d, "C_0"] \\
\mathbf{Vect} \arrow[ur, hook, "E_W"'] \arrow[rr, hook, "E"'] & & \mathbf{Lie}_C \arrow[ul, hook', "E_C"']
\end{tikzcd} \tag{17}$$

In Diagram (17), $\mathbf{DiffVect} = \mathbf{DiffLie}_W$ and $\mathbf{Vect} = \mathbf{Lie}_W$, according to Theorem 17. Moreover, the full inclusion functor E from $\mathbf{Vect}$ and $\mathbf{Lie}_C$ is just a translation of Corollary 19. By Theorem 29, both functors denoted by E_W have a left adjoint $\mathbf{Sp}_W$. E_C also has a left adjoint $\mathbf{Sp}_C$ for the same reason.

Applying Lemma 31 one time on each of the three triangles, with vertices **DiffCom**, **DiffVect**, **DiffLie**, with vertices **DiffCom**, **Vect**, **Lie** and with vertices **Ass**, $\mathbf{Lie}_C$, **Lie**, provides a left adjoint, respectively, for the two W_0's and for C_0. Another application of Lemma 31 on the triangle with vertices **Vect**, **Lie**, $\mathbf{Lie}_C$ provides the existence of a left adjoint for E. The fact that the leftmost occurrence of $dLie$ has a left adjoint also follows from an application of Lemma 31 on the triangle with vertices **DiffVect**, **Vect** and **Lie** (since $E_W \colon \mathbf{Vect} \to \mathbf{Lie}$ is fully faithful, and $dLie \circ E_W \colon \mathbf{DiffVect} \to \mathbf{Lie}$ has a left adjoint). In conclusion, every functor in this diagram has a left adjoint. In consequence of that, there is a corresponding commutative diagram (up to natural isomorphisms) of left adjoints (where $\mathcal{U}$ is the usual universal enveloping algebra functor).

$$\begin{tikzcd}
\mathbf{DiffVect} \arrow[dd, "\mathcal{W}\circ E_W"] & & \\
& \mathbf{DiffLie} \arrow[ul, "\mathbf{Sp}_W"'] \arrow[dl, "\mathcal{W}"'] & \\
\mathbf{DiffCom} & \mathbf{Lie} \arrow[l, "\mathcal{W}"'] \arrow[u, "\mathcal{D}iff"'] \arrow[r, "\mathcal{U}"] \arrow[dl, "\mathbf{Sp}_W"] & \mathbf{Ass} \\
\mathbf{Vect} \arrow[u, "\mathcal{W}\circ E_W"'] \arrow[uuu, bend left=60, "\mathbf{Sp}_W\circ\mathcal{D}iff\circ E_W"] & & \mathbf{Lie}_C \arrow[ll, "\mathbf{Sp}_W\circ E_C"] \arrow[u, "\mathcal{U}\circ E_C"'] \arrow[ul, "\mathbf{Sp}_C"']
\end{tikzcd} \tag{18}$$

Diagram (18) makes it possible to exhibit a major difference between the universal enveloping algebra and the Wronskian envelope. Given a Lie algebra $\mathfrak{g}$, its image, by the canonical map, into $\mathcal{U}(\mathfrak{g})$, seen as a Lie algebra under its commutator bracket, is $\mathbf{Sp}_C(\mathfrak{g})$. In particular, if one considers the free Lie algebra $\mathbf{Lie}[X]$ on a set X, then $\mathcal{U}(\mathbf{Lie}[X])$ is the free associative algebra $R\langle X\rangle$ on X, and thus $\mathbf{Sp}_C(\mathbf{Lie}[X])$ is the Lie algebra $\mathbf{LiePol}\langle X\rangle = \langle X\rangle_{\mathbf{Lie}}$ of *Lie polynomials* in $R\langle X\rangle$ (cf. [27]). But, as is well known [5], $\mathbf{Lie}[X]$ is free as a module, whence, by Poincaré-Birkhoff-Witt theorem, embeds into its universal enveloping algebra, so that $\mathbf{Lie}[X] \simeq \mathbf{LiePol}\langle X\rangle$. In this situation there is no difference between free Lie algebras and free Lie algebras which embed into their universal enveloping algebras.

Now, given a (differential) Lie algebra $\mathfrak{g}$, its image, by the canonical map, into $\mathcal{W}(\mathfrak{g})$, seen as a Lie algebra under its Wronskian bracket, of course is $\mathbf{Sp}_W(\mathfrak{g})$. Let X be a set. Then, the Wronskian envelope $\mathcal{W}(\mathbf{Lie}[X])$ (respectively, $\mathcal{W}(\mathbf{DiffLie}[X])$) of the free (respectively, free differential) Lie algebra $\mathbf{Lie}[X]$ (respectively, $\mathbf{DiffLie}[X]$) on X, is the algebra $R\{X\} := \mathbf{DiffCom}[X]$ of differential polynomials (see e.g. [15]). (This follows by composing some left adjoints of the functors occurring in Diagram (16).) As a commutative algebra $R\{X\} = R[X \times \mathbb{N}]$, and it has a derivation d determined by the relations $d(x^{(i)}) = x^{(i+1)}$, $x \in X$, $i \in \mathbb{N}$ (the notation $x^{(i)}$ is usually preferred than (x, i) and also $x = x^{(0)}$). The derivation $d(P)$ of a differential polynomials $P \in R\{X\}$ is denoted by P'.

Let us consider the canonical images. Let $\mathbf{Vect}[X] = \mathbf{Sp}_W(\mathbf{Lie}[X])$ and $\mathbf{DiffVect}[X] = \mathbf{Sp}_W(\mathbf{DiffLie}[X])$ be, respectively, the *free Lie* and the *free differential Lie algebras of vector fields on a line*. These Lie algebras thus are (differential) Lie sub-algebras of $R\{X\}$.

Definition 35. The members of $\mathbf{DiffVect}[X]$ will be referred to as *differential Lie polynomials*.

In detail, let $X_0 := X$ and let $X_{n+1} = X_n \cup \{\alpha P\colon \alpha \in R,\ P \in X_n\} \cup \{P + Q\colon P, Q \in X_n\} \cup \{PQ' - P'Q\colon P, Q \in X_n\}$, $n \in \mathbb{N}$, and then $\mathbf{Vect}[X] = \bigcup_{n\geq 0} X_n$, and let $Y_0 = X$, $Y_{n+1} = Y_n \cup \{\alpha P\colon \alpha \in R,\ P \in Y_n\} \cup \{P + Q\colon P, Q \in Y_n\} \cup \{PQ' - P'Q\colon P, Q \in Y_n\} \cup \{P'\colon P \in Y_n\}$, $n \in \mathbb{N}$, and then $\mathbf{DiffVect}[X] = \bigcup_{n\geq 0} Y_n$.

Remark 36. Every (differential) Lie algebra of vector fields on a line is a quotient a free (differential) Lie algebra of vector fields on a line.

Of course, $\mathbf{Lie}[\emptyset] \simeq (0)$ so that, as an abelian Lie algebra, it embeds into its Wronskian envelope $R\{\emptyset\} \simeq \mathcal{W}(0) \simeq (R, 0)$ (Corollary 10), whence $\mathbf{Lie}[\emptyset] \simeq \mathbf{Vect}[\emptyset]$. Similarly, $\mathbf{Lie}[x] \simeq Rx$ (abelian Lie algebra) and thus by Corollary 10 again, it embeds into its Wronskian envelope $R\{x\}$, so that $\mathbf{Lie}[x] \simeq \mathbf{Vect}[x]$. The crucial argument of a major difference with the case of Lie polynomials appears now. One already knows that $\mathbf{Lie}[x, y]$ does not embed into its Wronskian envelope $R\{x, y\}$ (because it does not satisfy the T_4 identity). Therefore, $\mathbf{Lie}[x, y]$ is not isomorphic to the Lie algebra $\mathbf{Vect}[x, y]$ (Remark 27). Moreover, since for every set X with $|X| \geq 2$, $\mathbf{Lie}[X]$ contains a free Lie algebra on 2 generators, it cannot be isomorphic to the free Lie algebra **Vect**$[X]$ of vector fields on a line. The following result thus is proved.

Theorem 37. *No free Lie algebra on at least two generators embeds into its Wronskian envelope, nor is isomorphic to its canonical image.*

In order to get a similar result for the free differential Lie algebras, one first needs to analyze them. A *magma* is a pair $(M, *)$ consisting of a set M and a binary operation $*\colon M \times M \to M$ satisfying no laws. Let **Mag**$[X \times \mathbb{N}]$ be the free magma (i.e., "non-associative" words or trees) on $X \times \mathbb{N}$ [5]. One writes $x^{(i)}$ rather than (x, i), $x \in X$, $i \geq 0$, following the same convention as used for $R\{X\}$, and, of course, $x^{(i)}$ will be interpreted as the ith-derivative of x. The (non-associative) multiplication in **Mag**$[X \times \mathbb{N}]$ is denoted by $*$. Recall that **Mag**$[X \times \mathbb{N}]$ admits a canonical $\mathbb{N}$-gradation defined as the *number of leaves* of a tree: $\ell(x^{(i)}) = 1$, $(x, i) \in X \times \mathbb{N}$, and $\ell(s * t) = \ell(s) + \ell(t)$. Let $M_n := \ell^{-1}(\{n\})$, $n \in \mathbb{N}$. Then, **Mag**$[X \times \mathbb{N}] = \bigsqcup_{n \geq 0} M_n$ (disjoint union), $M_0 = \emptyset$, $M_1 = X \times \mathbb{N}$, and $M_m * M_n \subseteq M_{m+n}$, $m, n \in \mathbb{N}$. Actually, $M_n = \bigsqcup_{k+\ell=n} M_k * M_\ell$, $n \geq 1$.

The free non-associative R-algebra **Alg**$[X \times \mathbb{N}]$ on $X \times \mathbb{N}$ is the free R-module on **Mag**$[X \times \mathbb{N}]$ with multiplication, again denoted by $*$, inherited from the product law of **Mag**$[X \times \mathbb{N}]$.

Let us define a derivation d on **Alg**$[X \times \mathbb{N}]$ as follows: $d(x^{(i)}) = x^{(i+1)}$ on $X \times \mathbb{N}$, and $d(s * t) = d(s) * t + s * d(t)$, $s, t \in$ **Mag**$[X \times \mathbb{N}]$ (this is a well-defined construction as shows an easy induction on the number of leaves of a tree), and then extend d by linearity to the whole **Alg**$[X \times \mathbb{N}]$. Let **DiffAlg**$[X] :=$ **Alg**$[X \times \mathbb{N}]$.

Lemma 38. *The pair* (**DiffAlg**$[X], d$) *is the free non-associative differential algebra generated by X.*

Proof. Let $((A, *), e)$ be a not necessarily associative differential algebra. Let $f\colon X \to A$ be a map. Define $f_1(x^{(i)}) = e^i(f(x))$ (with $e^{(0)} = id$), $(x, i) \in X \times \mathbb{N}$, and let $f_2\colon$ **Mag**$[X \times \mathbb{N}] \to (A, *)$ be the unique homomorphism extension of f_1, i.e., $f_2 = f_1$ on $X \times \mathbb{N}$, and $f_2(s * t) = f_2(s) * f_2(t)$, $s, t \in$ **Mag**$[X \times \mathbb{N}]$. Let finally $f_3\colon$ **Alg**$[X \times \mathbb{N}] \to (A, *)$ be the extension by linearity of f_2. Because f_2 is a homomorphism of magmas, f_3 is an algebra map. It is easy to see by induction on the degree that f_3 commutes to the derivations, whence provides a differential algebra map $f_3\colon$ (**DiffAlg**$[X], d) \to ((A, *), e)$. Uniqueness follows because f_3 is determined by its values on X. $\square$

Let us now consider the two-sided ideal L of **Alg**$[X \times \mathbb{N}]$ generated by $t * t$, and by $s * (t * u) + t * (s * u) + u * (s * t)$, $s, t, u \in$ **Alg**$[X \times \mathbb{N}]$. From [5, Proposition 1, p. 18], **Alg**$[X \times \mathbb{N}]/L =$ **Lie**$[X \times \mathbb{N}]$, where the Lie bracket is the quotient multiplication on **Alg**$[X \times \mathbb{N}]/L$.

Lemma 39. *L is a differential ideal of* (**DiffAlg**$[X], d$).

Proof. Let $s, t \in$ **DiffAlg**$[X]$. Then, $(s + t) * (s + t) = s * s + t * t + s * t + t * s$ so that $s * t + t * s = (s + t) * (s + t) - s * s - t * t \in L$. Thus, $d(t * t) = d(t) * t + t * d(t) \in L$. Furthermore, let $s, t, u \in$ **DiffAlg**$[X]$. Then,

$$\begin{aligned} d(s * (t * u)) &= d(s) * (t * u) + s * (d(t) * u) + s * (t * d(u)), \\ d(t * (u * s)) &= d(t) * (u * s) + t * (d(u) * s) + t * (u * d(s)), \\ d(u * (s * t)) &= d(u) * (s * t) + u * (d(s) * t) + u * (s * d(t)). \end{aligned} \tag{19}$$

Therefore, $d(s * (t * u) + t * (u * s) + u * (s * t)) \in L$. Whence $d(L) \subseteq L$. $\square$

By Lemma 39, the Lie algebra **Lie**$[X \times \mathbb{N}]$ thus admits the quotient derivation $\tilde{d}$.

Lemma 40. *The Lie algebra* **Lie**$[X \times \mathbb{N}]$*, with the derivation $\tilde{d}$, is the free differential Lie algebra* **DiffLie**$[X]$ *on X.*

Proof. Let $(\mathfrak{g}, e)$ be a differential Lie algebra, and let $f\colon X \to \mathfrak{g}$ be a map. According to Lemma 38 there is a unique homomorphism of differential algebras $\hat{f}\colon (\mathbf{DiffAlg}[X], d) \to (\mathfrak{g}, e)$ (since $(\mathfrak{g}, e)$ is a particular kind of a not necessarily associative differential algebra) such that $\hat{f}(x^{(0)}) = f(x)$, $x \in X$. Since the multiplication in $\mathfrak{g}$ is alternating and satisfies the Jacobi identity, it follows that $L \subseteq \ker \hat{f}$, which completes the proof. □

According to Lemma 40, one finally observes that $(\mathbf{DiffLie}[\emptyset], \tilde{d}) = ((0), 0)$, whence embeds into its Wronskian envelope (see Corollary 10). Therefore the following corollary to Theorem 37 holds.

Corollary 41. *No free differential Lie algebra on at least one generator embeds into its Wronskian envelope, nor is isomorphic to its canonical image.*

Proof. This follows by an application of Theorem 37 from the fact that $\mathcal{W}(\mathbf{DiffLie}[X]) = \mathcal{W}(\mathbf{Lie}[X]) = R\{X\}$ and because the underlying Lie algebra of $\mathbf{DiffLie}[X]$ is the free Lie algebra on $X \times \mathbb{N}$. □

Remark 42. Corollary 41 implies in particular that it is generally not true that if $\mathfrak{g}$ is an object of **Vect**, then its differential envelope $\mathcal{Diff}(\mathfrak{g})$ is an object of **DiffVect** (consider $\mathfrak{g} = \mathbf{Lie}[x]$, $\mathcal{Diff}(\mathbf{Lie}[x]) = \mathbf{DiffLie}[x] = \mathbf{Lie}[\{x\} \times \mathbb{N}]$).

4. Other universal envelopes

In this final section are introduced some other universal envelopes related to differential commutative and to Lie algebras. Some results about embedding conditions into these envelopes are provided.

4.1. Lie-Rinehart algebras

Given a differential commutative algebra $((A, *), d)$, the algebra structure $(A, *)$ and the Lie structure (A, W_d) interact rather nicely. Indeed, the triple $((A, *), (A, W_d), \partial_d)$ is a Lie-Rinehart algebra over R ([11, 28] and see below), with $\partial_d\colon (A, W_d) \to \mathfrak{Der}_R(A, *)$, $a \mapsto a \cdot d$ (where as in Section 2.3, $(a \cdot \delta)(x) = a * \delta(x)$, $a, x \in A$, $\delta \in \mathfrak{Der}_R(A, *)$, provides the structure of left $(A, *)$-module on $\mathfrak{Der}_R(A, *)$).

Definition 43. A *Lie-Rinehart R-algebra* is a pair $((A, *), \mathfrak{g}, \partial)$, where $(A, *)$ is a commutative algebra with a unit, $\mathfrak{g}$ is a Lie R-algebra which is also a left $(A, *)$-module (the left $(A, *)$-action is denoted by $a \cdot x$, $a \in A$, $x \in \mathfrak{g}$), $\partial\colon \mathfrak{g} \to \mathfrak{Der}_R(A, *)$ is a Lie R-algebra map, called the *anchor*, such that
1. $\partial(a \cdot x) = a \cdot \partial(x)$, i.e., ∂ is left $(A, *)$-linear,
2. $[x, a \cdot y] = a \cdot [x, y] + \partial(x)(a) \cdot y$

for all $a \in A$, $x, y \in \mathfrak{g}$.

Remark 44. The algebraic laws of a Lie-Rinehart algebra are modeled on those of the pair $(C^\infty(V), \mathfrak{X}(V))$ where V is a finite-dimensional smooth manifold, $C^\infty(V)$ is the ring of smooth functions on V and $\mathfrak{X}(V)$ is the Lie algebra of smooth vector fields on V, and the underlying Lie algebra $\mathfrak{g}$ of a Lie-Rinehart algebra $(\mathsf{A}, \mathfrak{g}, \partial)$ is the algebraic counterpart of a Lie algebroid [21] from differential geometry.

Given two Lie-Rinehart R-algebras $(\mathsf{A}, \mathfrak{g}, \partial)$ and $(\mathsf{B}, \mathfrak{h}, \eth)$, a morphism between them is a pair (f, g) such that $f\colon \mathsf{A} \to \mathsf{B}$ is a (unit-preserving) algebra map, $g\colon \mathfrak{g} \to \mathfrak{h}$ is a Lie map such that
1. $g(a \cdot x) = f(a) \cdot g(x)$, and
2. $f(\partial(x)(a)) = \eth(g(x))(f(a))$

$a \in \mathsf{A}$, $x \in \mathfrak{g}$. All of this provides a category **LieRin**.

There are two obvious forgetful functors $\mathbf{Com} \xleftarrow{Com} \mathbf{LieRin} \xrightarrow{Lie} \mathbf{Lie}$. As explained above at the beginning of this section, there is also a functor $LR\colon \mathbf{DiffCom} \to \mathbf{LieRin}$ with $LR((A,*),d) = ((A,*),(A,W_d),\partial_d)$. At the level of morphisms, it maps a differential algebra map $\phi\colon ((A,*),d) \to ((B,*),e)$ to the Lie-Rinehart map $(\phi, W(\phi))$. These functors interact as in the following commutative diagram (where the unnamed arrows are the usual forgetful functors, and $dCom$ is as in Diagram (1)). ("LR" of course stands for "Lie-Rinehart".)

$$\begin{tikzcd}
\mathbf{DiffCom} \arrow[rr, "W"] \arrow[dd, "dCom"'] \arrow[dr, "LR"] \arrow[ddrr, bend left, "W"] \arrow[ddddr, bend right=60] & & \mathbf{DiffLie} \arrow[dd, "dLie"] \arrow[ddddl, bend left=60] \\
& \mathbf{LieRin} \arrow[dl, "Com"] \arrow[dr, "Lie"'] & \\
\mathbf{Com} \arrow[dr] \arrow[ddr] & & \mathbf{Lie} \arrow[dl] \arrow[ddl] \\
& \mathbf{Mod} \arrow[d] & \\
& \mathbf{Set} &
\end{tikzcd} \tag{20}$$

LieRin is not (at least in an obvious way) a variety of universal algebras as introduced in Section 3.1 (it could be however described as a variety of *heterogeneous algebras* [4]), because it has two carrier sets (that of the underlying commutative and that of the underlying Lie algebras). In consequence of what one cannot rely entirely on the theory of algebraic functors between varieties to deduce the existence of left adjoints for those functors with domain or codomain **LieRin** in the above diagram, namely, LR, Com and Lie. Nevertheless these adjunctions do exist as we now observe.

Let $F\colon \mathbf{Com} \to \mathbf{LieRin}$ be given by $F(\mathsf{A}) := (\mathsf{A},(0),i_{\mathfrak{Der}_R(A)})$, where, as in Section 2.2, $i_{\mathfrak{g}}\colon (0) \hookrightarrow \mathfrak{g}$ is the initial map of a Lie algebra $\mathfrak{g}$, and for $f\colon \mathsf{A} \to \mathsf{B}$, an algebra map, $F(f) := (f, id_{(0)})\colon F(\mathsf{A}) \to F(\mathsf{B})$. It is not difficult to check that F provides a left adjoint to Com. (Given an algebra map $f\colon \mathsf{A} \to \mathsf{B} = Com(\mathsf{B},\mathfrak{g},\partial)$, there is a unique Lie-Rinehart algebra map $\hat{f}\colon (\mathsf{A},(0),i_{\mathfrak{Der}_R(\mathsf{A})}) \to (\mathsf{B},\mathfrak{g},\partial)$ such that $Com(\hat{f}) = f$, namely, $\hat{f} = (f, i_{\mathfrak{g}})$.)

Let $G\colon \mathbf{Lie} \to \mathbf{LieRin}$ be given by $G(\mathfrak{g}) = (R,\mathfrak{g},t_{\mathfrak{g}})$, where, again as in Section 2.2, $t_{\mathfrak{g}}\colon \mathfrak{g} \to (0)$ is the terminal map of the Lie algebra $\mathfrak{g}$. (Since if $d \in \mathfrak{Der}_R(R)$, $d(1) = 0$, it follows that $\mathfrak{Der}_R(R) = (0)$.) G is a left adjoint to Lie. (Given a Lie map $g\colon \mathfrak{g} \to \mathfrak{h} = Lie(\mathsf{A},\mathfrak{h},\partial)$ there is a unique Lie-Rinehart algebra map $\tilde{g}\colon (R,\mathfrak{g},t_{\mathfrak{g}}) \to (\mathsf{A},\mathfrak{h},\partial)$ such that $Lie(\tilde{g}) = g$, namely $\tilde{g} = (\eta_{\mathsf{A}}, g)$, where $\eta_{\mathsf{A}}\colon R \to \mathsf{A}$ is given by $\eta_{\mathsf{A}}(r) = r1$.) One may call $(R,\mathfrak{g},t_{\mathfrak{g}})$ the *free Lie-Rinehart algebra* generated by $\mathfrak{g}$ or the *universal Lie-Rinehart envelope* of $\mathfrak{g}$ (this terminology is inspired from Section 3).

Let us finally prove directly, by constructing a universal object, that LR also has a left adjoint, enabling the definition of the *Wronskian envelope* of a Lie-Rinehart algebra. Let $(\mathsf{A},\mathfrak{g},\partial)$ be a Lie-Rinehart algebra.

Let us consider $\mathcal{D}(\mathsf{A},\mathfrak{g})$ as the free commutative differential R-algebra $R\{|\mathsf{A}| \sqcup |\mathfrak{g}|\}$ generated by the disjoint sum $|\mathsf{A}| \sqcup |\mathfrak{g}|$ (where, $|\mathsf{A}|$, and respectively $|\mathfrak{g}|$, denotes the underlying set of the algebra A, respectively, Lie algebra $\mathfrak{g}$). Let $q_{|\mathsf{A}|\sqcup|\mathfrak{g}|}\colon |\mathsf{A}| \sqcup |\mathfrak{g}| \hookrightarrow \mathcal{D}(\mathsf{A},\mathfrak{g})$ be the unit of the adjunction, i.e., the insertion of variables in the algebra of differential polynomials. Let $q_{\mathsf{A}}\colon |\mathsf{A}| \xrightarrow{i_{|\mathsf{A}|}} |\mathsf{A}|\sqcup|\mathfrak{g}| \xrightarrow{q_{|\mathsf{A}|\sqcup|\mathfrak{g}|}} \mathcal{D}(\mathsf{A},\mathfrak{g})$, and $q_{\mathfrak{g}}\colon |\mathfrak{g}| \xrightarrow{i_{|\mathfrak{g}|}} |\mathsf{A}|\sqcup|\mathfrak{g}| \xrightarrow{q_{|\mathsf{A}|\sqcup|\mathfrak{g}|}} \mathcal{D}(\mathsf{A},\mathfrak{g})$ (where by $i_X\colon X \hookrightarrow X \sqcup Y$ and $i_Y\colon Y \hookrightarrow X \sqcup Y$ are the canonical injections).

The differential algebra $\mathcal{D}(\mathsf{A},\mathfrak{g})$, by definition, satisfies the following universal property: given a differential commutative R-algebra (B,e) and a set-theoretic map $h\colon |\mathsf{A}|\sqcup|\mathfrak{g}| \to |\mathsf{B}|$ there exists a unique homomorphism of differential algebras $\bar{h}\colon \mathcal{D}(\mathsf{A},\mathfrak{g}) \to (\mathsf{B},e)$ such that $|\bar{h}| \circ q_{|\mathsf{A}|\sqcup|\mathfrak{g}|} = h$.

One denotes by D the R-derivation of the differential algebra $\mathcal{D}(\mathsf{A},\mathfrak{g})$, and by 1 its identity element. The multiplication, and scalar action, on $\mathcal{D}(\mathsf{A},\mathfrak{g})$ are denoted by juxtaposition. Now, let us consider the differential ideal $I(\mathsf{A},\mathfrak{g},\partial)$ of the differential algebra $(\mathcal{D}(\mathsf{A},\mathfrak{g}),D)$ generated by $q_{\mathsf{A}}(a+b)-q_{\mathsf{A}}(a)-q_{\mathsf{A}}(b)$, $q_{\mathsf{A}}(ab)-q_{\mathsf{A}}(a)q_{\mathsf{A}}(b)$, $q_{\mathsf{A}}(1_{\mathsf{A}})-1$, $q_{\mathsf{A}}(ra)-rq_{\mathsf{A}}(a)$, $q_{\mathfrak{g}}(x+y)-q_{\mathfrak{g}}(x)-q_{\mathfrak{g}}(y)$, $q_{\mathfrak{g}}(rx)-rq_{\mathfrak{g}}(x)$, $q_{\mathfrak{g}}([x,y])=q_{\mathfrak{g}}(x)D(q_{\mathfrak{g}}(y))-D(q_{\mathfrak{g}}(x))q_{\mathfrak{g}}(y)$, $q_{\mathfrak{g}}(a\cdot x)-q_{\mathsf{A}}(a)q_{\mathfrak{g}}(x)$, and $q_{\mathsf{A}}(\partial(x)(a))-q_{\mathfrak{g}}(x)D(q_{\mathsf{A}}(a))$ for all $a\in\mathsf{A}$, $x\in\mathfrak{g}$ and $r\in R$. Let $\pi\colon\mathcal{D}(\mathsf{A},\mathfrak{g})\to\mathcal{D}(A,\mathfrak{g})/I(\mathsf{A},\mathfrak{g})$ be the canonical epimorphism (which is an homomorphism of differential algebras, where $\mathcal{D}(A,\mathfrak{g})/I(\mathsf{A},\mathfrak{g},\partial)$ is equipped with the quotient derivation $\tilde{D}\circ\pi=\pi\circ D$). By definition of the differential ideal $I(\mathsf{A},\mathfrak{g},\partial)$ it is clear that $\pi\circ q_{\mathsf{A}}$ defines an algebra map from A to $\mathcal{D}(\mathsf{A},\mathfrak{g})/I(\mathsf{A},\mathfrak{g},\partial)$, and that $\pi\circ q_{\mathfrak{g}}$ defines a Lie map from $\mathfrak{g}$ to $(\mathcal{D}(\mathsf{A},\mathfrak{g})/I(\mathsf{A},\mathfrak{g},\partial),W_{\tilde{D}})$.

Let (B,e) be a commutative differential R-algebra (and B be the underlying module of B), and let $(f,g)\colon(\mathsf{A},\mathfrak{g},\partial)\to LR(\mathsf{B},e)=(\mathsf{B},(B,W_e),\partial_e)$ be a homomorphism of Lie-Rinehart algebras. Let $h\colon|\mathsf{A}|\sqcup|\mathfrak{g}|\to|\mathsf{B}|$ be the unique set-theoretic map such that $h\circ i_{|\mathsf{A}|}=\lfloor f\rfloor$ and $h\circ i_{|\mathfrak{g}|}=|g|$. By the universal property of $\mathcal{D}(\mathsf{A},\mathfrak{g})$ there is a unique differential algebra map $\bar{h}\colon(\mathcal{D}(\mathsf{A},\mathfrak{g}),D)\to(\mathsf{B},e)$ such that $|\bar{h}|\circ q_{|\mathsf{A}|\sqcup|\mathfrak{g}|}=h$. In particular, $|\bar{h}|\circ q_{\mathsf{A}}=\lfloor f\rfloor$ and $|\bar{h}|\circ q_{\mathfrak{g}}=|g|$. It is easily checked that the map $\bar{h}$ passes to the quotient by $I(\mathsf{A},\mathfrak{g},\partial)$. Hence, there is a unique well-defined homomorphism $\tilde{h}\colon(\mathcal{D}(\mathsf{A},\mathfrak{g})/I(\mathsf{A},\mathfrak{g},\partial),\tilde{D})\to(\mathsf{B},e)$ of differential algebras such that $\tilde{h}\circ\pi=\bar{h}$. In particular, $\tilde{h}\circ\pi\circ q_{\mathsf{A}}=\bar{h}\circ q_{\mathsf{A}}=f$ and $\tilde{h}\circ\pi\circ q_{\mathfrak{g}}=\bar{h}\circ q_{\mathfrak{g}}=g$. It is clearly the unique differential algebra map with those two properties.

Remark 45. The map $\pi\circ q_{\mathsf{A}}\colon\mathsf{A}\to\mathcal{D}(\mathsf{A},\mathfrak{g})/I(\mathsf{A},\mathfrak{g},\partial)$ is an algebra map, while $\pi\circ q_{\mathfrak{g}}\colon\mathfrak{g}\to(\mathcal{D}(\mathsf{A},\mathfrak{g})/I(\mathsf{A},\mathfrak{g},\partial),W_{\tilde{D}})$ is a Lie map by definition of the ideal $I(\mathsf{A},\mathfrak{g},\partial)$. Moreover,

$$\pi(q_{\mathsf{A}}(a\cdot x))=\pi(q_{\mathsf{A}}(a)q_{\mathfrak{g}}(x))=\pi(q_{\mathsf{A}}(a))\pi(q_{\mathfrak{g}}(x))$$

and

$$\begin{aligned}\pi(q_{\mathsf{A}}(\partial(x)(a)))&=\pi(q_{\mathfrak{g}}(x)D(q_{\mathsf{A}}(a)))\\&=\pi(q_{\mathfrak{g}}(x))\pi(D(q_{\mathsf{A}}(a)))\\&=\pi(q_{\mathfrak{g}}(x))\tilde{D}(\pi(q_{\mathsf{A}}(a))),\end{aligned}\tag{21}$$

so that $(\pi\circ q_{\mathsf{A}},\pi\circ q_{\mathfrak{g}})$ is a morphism of Lie-Rinehart algebras from $(\mathsf{A},\mathfrak{g},\partial)$ to $LR(\mathcal{D}(\mathsf{A},\mathfrak{g})/I(\mathsf{A},\mathfrak{g},\partial))$.

In conclusion, $(\mathcal{W}(\mathsf{A},\mathfrak{g},\partial),\tilde{D}):=(\mathcal{D}(\mathsf{A},\mathfrak{g})/I(\mathsf{A},\mathfrak{g},\partial),\tilde{D})$ is the *Wronskian envelope* of the Lie-Rinehart algebra $(\mathsf{A},\mathfrak{g},\partial)$. It follows from Diagram (20) (more precisely, by using the corresponding diagram for left adjoints obtained by reversing the direction of the arrows) that the Wronskian envelope $(\mathcal{W}(\mathfrak{g}),D)$ of a Lie algebra $\mathfrak{g}$ is (isomorphic to) the Wronskian envelope $(\mathcal{W}(R,\mathfrak{g},t_{\mathfrak{g}}),\tilde{D})$ of the universal Lie-Rinehart algebra $(R,\mathfrak{g},t_{\mathfrak{g}})$ of $\mathfrak{g}$. The following result thus is immediate.

Lemma 46. *A Lie algebra $\mathfrak{g}$ embeds into its Wronskian envelope (as a Lie sub-algebra for the Wronskian bracket) if, and only if, the map $\pi\circ q_{\mathfrak{g}}\colon\mathfrak{g}\to(|\mathcal{W}(R,\mathfrak{g},t_{\mathfrak{g}})|,W_{\tilde{D}})=Lie(LR(\mathcal{W}(R,\mathfrak{g},t_{\mathfrak{g}}),\tilde{D}))$ is a one-to-one Lie map.*

Remark 47. Herz in [11] and Rinehart in [28] both provide a *universal enveloping algebra* construction for a Lie-Rinehart algebra $(\mathsf{A},\mathfrak{g},\partial)$ (see also [21]) which is rather different from our. It is given as an associative algebra with a unit $\mathcal{U}(\mathsf{A},\mathfrak{g},\partial)$ together with an algebra map $\iota_{\mathsf{A}}\colon\mathsf{A}\to\mathcal{U}(\mathsf{A},\mathfrak{g},\partial)$ and a Lie map $\iota_{\mathfrak{g}}\colon\mathfrak{g}\to C(\mathcal{U}(\mathsf{A},\mathfrak{g},\partial))$ (recall from Section 3.5 that C turns an associative algebra into a Lie algebra under the commutator bracket) which interact nicely: $\iota_{\mathsf{A}}(a)\iota_{\mathfrak{g}}(x)=\iota_{\mathfrak{g}}(a\cdot x)$ and $\iota_{\mathfrak{g}}(x)\iota_{\mathsf{A}}(a)-\iota_{\mathsf{A}}(a)\iota_{\mathfrak{g}}(x)=\iota_{\mathsf{A}}(\partial(x)(a))$, $a\in\mathsf{A}$, $x\in\mathfrak{g}$. All of this satisfies a universal property: given an associative (not necessarily commutative) R-algebra B, an algebra map $f\colon\mathsf{A}\to\mathsf{B}$ and a Lie map $g\colon\mathfrak{g}\to C(\mathsf{B})$ such that $f(a)g(x)=f(a\cdot x)$ and $f(\partial(x)(a))=[g(x),f(a)]$, $a\in\mathsf{A}$, $x\in\mathfrak{g}$, there is a unique algebra map $h\colon \mathcal{U}(\mathsf{A}, \mathfrak{g}, \partial) \to \mathsf{B}$ with $h \circ \iota_{\mathsf{A}} = f$ and $h \circ \iota_{\mathfrak{g}} = g$. This notion is thus different from our, and does not provide a left adjoint construction.

4.2. Jacobi algebras: The associative side

Definition 48. A *Jacobi algebra* over R [10, 29] is a unital commutative R-algebra A (the unit is denoted by 1) together with a R-linear Lie bracket $[\cdot,\cdot]$ (called *Jacobi bracket*) satisfying the following version of Leibniz rule, called *Jacobi-Leibniz rule*,

$$[ab, c] = a[b, c] + b[a, c] - ab[1, c],\ a, b, c \in \mathsf{A}.$$

One obtains the category **Jac** of Jacobi R-algebras (a morphism between Jacobi algebras is a morphism between both the underlying unital R-algebras and the underlying Lie R-algebras).

Remark 49. Jacobi algebras have been introduced as the algebraic counterpart of Jacobi manifolds or local Lie algebras [13, 18] from differential geometry.

Such Jacobi algebras may be seen as a common generalization of differential commutative algebras and Poisson algebras. Recall that a *Poisson algebra* is a pair $(\mathsf{A}, \{\cdot,\cdot\})$, where A is a commutative (unital) algebra, $\{\cdot,\cdot\}$ is a Lie bracket on the module $|\mathsf{A}|$, and for each $a \in \mathsf{A}$, $\{a,\cdot\} \in \mathfrak{Der}_R(\mathsf{A})$, or, in other terms, $\{a, bc\} = \{a, b\}c + b\{a, c\}$, $a, b, c \in \mathsf{A}$. By a morphism of Poisson algebras is meant an algebra map which is also a Lie algebra map. It is clear that one has an obvious full embedding functor $P\colon \mathbf{Poi} \hookrightarrow \mathbf{Jac}$.

Likewise, let (A, d) be a differential commutative algebra. Then, (A, W_d) turns to be a Jacobi algebra since one has $W_d(ab, c) = abd(c) - d(ab)c = abd(c) - d(a)bc - ad(b)c$, $aW_d(b, c) = abd(c) - ad(b)c$, $bW(a, c) = bad(c) - bd(a)c = abd(c) - d(a)bc$, and $abW(1, c) = abd(c)$, $a, b, c \in \mathsf{A}$, which shows that the Wronskian bracket is a Jacobi bracket. This clearly provides a functor $DC\colon \mathbf{DiffCom} \to \mathbf{Jac}$, $(\mathsf{A}, d) \mapsto (\mathsf{A}, W_d)$, which is full (since any differential algebra map is automatically a Lie algebra map for the corresponding Wronskian brackets), faithful and injective on objects (the derivation d of A is recovered from the Jacobi algebra (A, W_d) as $W_d(1,\cdot)$).

So **Poi** and **DiffCom** may be seen as full sub-categories of **Jac**. In particular, a Poisson algebra is a Jacobi algebra with $[1, x] = 0$ for every x (as is easily checked from Jacobi-Leibniz rule).

Whence one has $P\colon \mathbf{Poi} \hookrightarrow \mathbf{Jac} \hookleftarrow \mathbf{DiffCom}\colon DC$, and the following diagram is commutative (as usually the unnamed arrows are the forgetful functors).

$$\begin{tikzcd}
\mathbf{Poi} \arrow[r, hook, "P"] \arrow[dr] & \mathbf{Jac} \arrow[d] & \mathbf{DiffCom} \arrow[l, hook', "DC"'] \arrow[dl] \\
& \mathbf{Set} &
\end{tikzcd} \tag{22}$$

It follows from Diagram (22), because the categories here are varieties of universal algebras, that both P and DC have a left adjoint, which makes possible to define the *universal Poisson envelope* and the *universal differential commutative algebra envelope* of a Jacobi algebra.

These constructions are easily provided.

1. Given a Jacobi algebra $(\mathsf{A}, [\cdot,\cdot])$, let us consider its Jacobi ideal I_P (i.e., both an ideal and a Lie ideal) generated by $[1, x]$, $x \in \mathsf{A}$. Then, A/I_P, with its quotient Jacobi bracket, is the universal Poisson envelope of $(\mathsf{A}, [\cdot,\cdot])$ [1]. The left adjoint functor is denoted by $\mathbf{Poi}[\cdot]_{\mathbf{Jac}}\colon \mathbf{Jac} \to \mathbf{Poi}$.
2. Given a Jacobi algebra $(\mathsf{A}, [\cdot,\cdot])$, let $(\mathcal{Diff}[\mathsf{A}], d)$ be the differential envelope of A (i.e., obtained using the left adjoint of the algebraic functor $dCom\colon \mathbf{DiffCom} \to \mathbf{Com}$), and let I be the differential ideal of $(\mathcal{Diff}[\mathsf{A}], d)$ generated by $W_d(x, y) - [x, y]$, $x, y \in \mathsf{A}$. Then, $(\mathcal{Diff}[\mathsf{A}]/I, \tilde{d})$ is the universal differential commutative envelope of $(\mathsf{A}, [\cdot,\cdot])$. In what follows, $(\mathbf{DiffCom}[\mathsf{A}, [\cdot,\cdot]], d)$ denotes this universal object.

Contrary to what it seems, **Jac** does not play a symmetric rôle for **Poi** and for **DiffCom**. It follows from the Jacobi-Leibniz rule, with $c = 1$, that $ad_1 \colon |\mathsf{A}| \to |\mathsf{A}|$, $a \mapsto [1, a]$, provides a R-derivation on A (since $[1, ab] = a[1, b] + b[1, a]$). It is then possible to turn such a Jacobi algebra $(\mathsf{A}, [\cdot, \cdot])$ into a differential commutative algebra (A, ad_1). Moreover, let $f \colon (\mathsf{A}, [\cdot, \cdot]) \to (\mathsf{B}, [\cdot, \cdot])$ be a morphism of Jacobi algebras. Then, f is a (unital) algebra map from A to B, and, furthermore, $f(ad_1(x)) = f([1, x]) = [f(1), f(x)] = [1, f(x)] = ad_1(f(x))$, so that $f \colon (\mathsf{A}, ad_1) \to (\mathsf{B}, ad_1)$ is a differential algebra map. This readily provides an algebraic functor $AD_1 \colon \mathbf{Jac} \to \mathbf{DiffCom}$ which makes it possible to consider the *universal Jacobi algebra* $\mathbf{Jac}[\mathsf{A}, d]_{\mathbf{DiffCom}}$ of a differential commutative algebra (A, d) as a left adjoint to AD_1. One observes that $AD_1 \circ DC = id_{\mathbf{DiffCom}}$, whence $id_{(\mathsf{A},d)} \colon (\mathsf{A}, d) \to AD_1(DC(\mathsf{A}, d))$ provides an embedding of a differential commutative algebra into a Jacobi algebra (seen as a differential commutative algebra under the derivation ad_1). From these observations the following result is obtained.

Lemma 50. *Any differential commutative algebra* (A, d) *embeds into its universal Jacobi envelope* $\mathbf{Jac}[\mathsf{A}, d]_{\mathbf{DiffCom}}$.

4.3. Jacobi algebras: The Lie side

Clearly, $(\mathsf{A}, [\cdot, \cdot]) \to (|\mathsf{A}|, [\cdot, \cdot])$ provides an algebraic functor $L \colon \mathbf{Jac} \to \mathbf{Lie}$. One observes that the following diagram commutes.

$$\begin{tikzcd} \mathbf{DiffCom} \arrow[r, hook, "DC"] \arrow[d, "W"'] & \mathbf{Jac} \arrow[dl, "L"] \\ \mathbf{Lie} & \end{tikzcd} \tag{23}$$

Note also that one has another algebraic functor $L' \colon \mathbf{Jac} \to \mathbf{Lie}$ given by $L' := W \circ AD_1$, i.e., $L'(\mathsf{A}, [\cdot, \cdot]) = W(\mathsf{A}, ad_1) = (|\mathsf{A}|, W_{ad_1})$. In general $L \neq L'$, i.e., $L'(\mathsf{A}, [\cdot, \cdot]) = (|\mathsf{A}|, W_{ad_1}) \neq (|\mathsf{A}|, [\cdot, \cdot])$, or, in other terms, $W_{ad_1} \neq [\cdot, \cdot]$. Indeed, in general, $[\cdot, \cdot] - W_{ad_1}$ is a non-zero alternating biderivation (i.e., alternating bilinear map which is a derivation of A in both of its variables), i.e., $[x, y] \neq x[1, y] - [1, x]y$ (it suffices to consider a Poisson algebra with a non-zero bracket).

So we are left with two different algebraic functors $L, L' \colon \mathbf{Jac} \to \mathbf{Lie}$. Each of them admitting a left adjoint, one has two different kinds of universal Jacobi algebra of a Lie algebra. Let $\mathbf{Jac}[\cdot]_{\mathbf{Lie}}, \mathbf{Jac}'[\cdot]_{\mathbf{Lie}} \colon \mathbf{Lie} \to \mathbf{Jac}$ be the corresponding left adjoints.

According to Diagram (23), one has a commutative (up to isomorphisms) diagram of left adjoint functors.

$$\begin{tikzcd} \mathbf{DiffCom} & \mathbf{Jac} \arrow[l, "\mathbf{DiffCom}[\cdot]"'] \\ \mathbf{Lie} \arrow[u, "\mathcal{W}"] \arrow[ur, "\mathbf{Jac}[\cdot]_{\mathbf{Lie}}"'] & \end{tikzcd} \tag{24}$$

In other terms, for every Lie algebra $\mathfrak{g}$, $(\mathcal{W}(\mathfrak{g}), D) \simeq (\mathbf{DiffCom}[\mathbf{Jac}[\mathfrak{g}]_{\mathbf{Lie}}], d)$ (as differential Lie algebras).

From the definition of L', one gets another commutative diagram of left adjoints.

$$\begin{tikzcd} \mathbf{DiffCom} \arrow[r, "\mathbf{Jac}[\cdot]_{\mathbf{DiffCom}}"] & \mathbf{Jac} \\ \mathbf{Lie} \arrow[u, "\mathcal{W}"] \arrow[ur, "\mathbf{Jac}'[\cdot]_{\mathbf{Lie}}"'] & \end{tikzcd} \tag{25}$$

In other terms, for every Lie algebra $\mathfrak{g}$, $\mathbf{Jac}[\mathcal{W}(\mathfrak{g}), D]_{\mathbf{DiffCom}} \simeq \mathbf{Jac}'[\mathfrak{g}]_{\mathbf{Lie}}$ (as Jacobi algebras).

Theorem 51. *Let $\mathfrak{g}$ be a Lie algebra. The following assertions 1 and 2 are equivalent, and both of them imply 3.*

1. *$\mathfrak{g}$ embeds into $W(\mathcal{W}(\mathfrak{g}), D)$.*
2. *$\mathfrak{g}$ embeds into $L'(\mathbf{Jac}'[\mathfrak{g}]_{\mathbf{Lie}})$.*
3. *$\mathfrak{g}$ embeds into $L(\mathbf{Jac}[\mathfrak{g}]_{\mathbf{Lie}})$.*

Proof.

1⇒3: By Lemma 22 since $\mathfrak{g} \hookrightarrow W(\mathcal{W}(\mathfrak{g}), D) \simeq W(\mathbf{DiffCom}[\mathbf{Jac}[\mathfrak{g}]_{\mathbf{Lie}}], d) = L(DC(\mathbf{DiffCom}[\mathbf{Jac}[\mathfrak{g}]_{\mathbf{Lie}}], d))$.

1⇒2: If $\mathfrak{g}$ embeds into $W(\mathcal{W}(\mathfrak{g}), D)$, then, since by Lemma 50 $\mathcal{W}(\mathfrak{g}, D) \hookrightarrow AD_1(\mathbf{Jac}[\mathcal{W}(\mathfrak{g}), D]_{\mathbf{DiffCom}})$, it follows that

$$\begin{aligned} \mathfrak{g} &\hookrightarrow W(AD_1(\mathbf{Jac}[\mathcal{W}(\mathfrak{g}), D]_{\mathbf{DiffCom}})) \\ &= L'(\mathbf{Jac}[\mathcal{W}(\mathfrak{g}), D]_{\mathbf{DiffCom}}) \\ &\simeq L'(\mathbf{Jac}'[\mathfrak{g}]_{\mathbf{Lie}}). \end{aligned} \tag{26}$$

2⇒1: By Lemma 22 since $\mathfrak{g} \hookrightarrow L'(\mathbf{Jac}'[\mathfrak{g}]_{\mathbf{Lie}}) \simeq L'(\mathbf{Jac}[\mathcal{W}(\mathfrak{g}), D]_{\mathbf{DiffCom}}) = W(AD_1(\mathbf{Jac}[\mathcal{W}(\mathfrak{g}), D]_{\mathbf{DiffCom}}))$. □

Theorem 17 together with Theorem 51 provides a sufficient condition for the embedding problem of Lie algebras into Jacobi algebras with respect to the functor L.

Corollary 52. *Every Lie algebra $\mathfrak{g}$ of vector fields on a line embeds into its universal Jacobi envelope $L(\mathbf{Jac}[\mathfrak{g}])$ seen as a Lie algebra under L.*

There is another sufficient condition for the same embedding problem. Let us consider the algebraic functor $L \circ P\colon \mathbf{Poi} \to \mathbf{Lie}$ (this is also the obvious forgetful functor, whence an algebraic functor, from **Poi** to **Lie**, and it has a left adjoint denoted by $\mathbf{Poi}[\cdot]_{\mathbf{Lie}}$). One has a corresponding commutative diagram of left adjoints.

$$\begin{tikzcd} \mathbf{Jac} \arrow[d, "{\mathbf{Poi}[\cdot]_{\mathbf{Jac}}}"'] & \mathbf{Lie} \arrow[l, "{\mathbf{Jac}[\cdot]_{\mathbf{Lie}}}"'] \arrow[dl, "{\mathbf{Poi}[\cdot]_{\mathbf{Lie}}}"] \\ \mathbf{Poi} & \end{tikzcd} \tag{27}$$

Corollary 53. *Let $\mathfrak{g}$ be a Lie algebra. If $\mathfrak{g}$ embeds into its universal Poisson envelope $L(P(\mathbf{Poi}[\mathfrak{g}]_{\mathbf{Lie}}))$, seen as a Lie algebra under $L \circ P$, then $\mathfrak{g}$ embeds into $L(\mathbf{Jac}[\mathfrak{g}]_{\mathbf{Lie}})$.*

Proof. It is obvious according to Diagram (27). □

A. Standard Lie identity of degree 5

The computation of $T_4(x_1, x_2, x_3, x_4, y)$ in $\mathbb{Z}\{x_1, x_2, x_3, x_4, y\}$ is displayed in the following Equation (28) (see pages 1663–1666). After factorization and simplification, it gives the value 0. Since $T_4(x_1, x_2, x_3, x_4, y)$ is a member of $\mathbf{Vect}[x_1, x_2, x_3, x_4, y]$ (over the integers), which is the free Lie algebra of vector fields on 5 generators (see Section 3.5), it is satisfied by all Lie algebras of vector fields on a line.

$$\begin{aligned} &T4(x_1, x_2, x_3, x_4, y) \\ &\quad = (((yx_4^{(2)} - x_4y^{(2)})x_1 - (yx_4^{(1)} - x_4y^{(1)})x_1^{(1)})x_3^{(2)} + ((yx_4^{(2)} - x_4y^{(2)})x_1^{(2)} + (yx_4^{(1)} - x_4y^{(1)})x_1^{(3)} \end{aligned}$$

$$
\begin{aligned}
&- (yx_4^{(4)} - x_4y^{(4)} + 2y^{(1)}x_4^{(3)} - 2x_4^{(1)}y^{(3)})x_1 - (yx_4^{(3)} - x_4y^{(3)} + y^{(1)}x_4^{(2)} - x_4^{(1)}y^{(2)})x_1^{(1)})x_3)x_2\\
&- (((yx_4^{(2)} - x_4y^{(2)})x_1 - (yx_4^{(1)} - x_4y^{(1)})x_1^{(1)})x_2^{(2)} + ((yx_4^{(2)} - x_4y^{(2)})x_1^{(2)} + (yx_4^{(1)} - x_4y^{(1)})x_1^{(3)}\\
&- (yx_4^{(4)} - x_4y^{(4)} + 2y^{(1)}x_4^{(3)} - 2x_4^{(1)}y^{(3)})x_1 - (yx_4^{(3)} - x_4y^{(3)} + y^{(1)}x_4^{(2)} - x_4^{(1)}y^{(2)})x_1^{(1)})x_2)x_3\\
&- (((yx_4^{(2)} - x_4y^{(2)})x_2 - (yx_4^{(1)} - x_4y^{(1)})x_2^{(1)})x_3^{(2)} + ((yx_4^{(2)} - x_4y^{(2)})x_2^{(2)} + (yx_4^{(1)} - x_4y^{(1)})x_2^{(3)}\\
&- (yx_4^{(4)} - x_4y^{(4)} + 2y^{(1)}x_4^{(3)} - 2x_4^{(1)}y^{(3)})x_2 - (yx_4^{(3)} - x_4y^{(3)} + y^{(1)}x_4^{(2)} - x_4^{(1)}y^{(2)})x_2^{(1)})x_3)x_1\\
&+ (((yx_4^{(2)} - x_4y^{(2)})x_2 - (yx_4^{(1)} - x_4y^{(1)})x_2^{(1)})x_1^{(2)} + ((yx_4^{(2)} - x_4y^{(2)})x_2^{(2)} + (yx_4^{(1)} - x_4y^{(1)})x_2^{(3)}\\
&- (yx_4^{(4)} - x_4y^{(4)} + 2y^{(1)}x_4^{(3)} - 2x_4^{(1)}y^{(3)})x_2 - (yx_4^{(3)} - x_4y^{(3)} + y^{(1)}x_4^{(2)} - x_4^{(1)}y^{(2)})x_2^{(1)})x_1)x_3\\
&+ (((yx_4^{(2)} - x_4y^{(2)})x_3 - (yx_4^{(1)} - x_4y^{(1)})x_3^{(1)})x_2^{(2)} + ((yx_4^{(2)} - x_4y^{(2)})x_3^{(2)} + (yx_4^{(1)} - x_4y^{(1)})x_3^{(3)}\\
&- (yx_4^{(4)} - x_4y^{(4)} + 2y^{(1)}x_4^{(3)} - 2x_4^{(1)}y^{(3)})x_3 - (yx_4^{(3)} - x_4y^{(3)} + y^{(1)}x_4^{(2)} - x_4^{(1)}y^{(2)})x_3^{(1)})x_2)x_1\\
&- (((yx_4^{(2)} - x_4y^{(2)})x_3 - (yx_4^{(1)} - x_4y^{(1)})x_3^{(1)})x_1^{(2)} + ((yx_4^{(2)} - x_4y^{(2)})x_3^{(2)} + (yx_4^{(1)} - x_4y^{(1)})x_3^{(3)}\\
&- (yx_4^{(4)} - x_4y^{(4)} + 2y^{(1)}x_4^{(3)} - 2x_4^{(1)}y^{(3)})x_3 - (yx_4^{(3)} - x_4y^{(3)} + y^{(1)}x_4^{(2)} - x_4^{(1)}y^{(2)})x_3^{(1)})x_1)x_2\\
&- (((yx_3^{(2)} - x_3y^{(2)})x_1 - (yx_3^{(1)} - x_3y^{(1)})x_1^{(1)})x_4^{(2)} + ((yx_3^{(2)} - x_3y^{(2)})x_1^{(2)} + (yx_3^{(1)} - x_3y^{(1)})x_1^{(3)}\\
&- (yx_3^{(4)} - x_3y^{(4)} + 2y^{(1)}x_3^{(3)} - 2x_3^{(1)}y^{(3)})x_1 - (yx_3^{(3)} - x_3y^{(3)} + y^{(1)}x_3^{(2)} - x_3^{(1)}y^{(2)})x_1^{(1)})x_4)x_2\\
&+ (((yx_3^{(2)} - x_3y^{(2)})x_1 - (yx_3^{(1)} - x_3y^{(1)})x_1^{(1)})x_2^{(2)} + ((yx_3^{(2)} - x_3y^{(2)})x_1^{(2)} + (yx_3^{(1)} - x_3y^{(1)})x_1^{(3)}\\
&- (yx_3^{(4)} - x_3y^{(4)} + 2y^{(1)}x_3^{(3)} - 2x_3^{(1)}y^{(3)})x_1 - (yx_3^{(3)} - x_3y^{(3)} + y^{(1)}x_3^{(2)} - x_3^{(1)}y^{(2)})x_1^{(1)})x_2)x_4\\
&+ (((yx_3^{(2)} - x_3y^{(2)})x_2 - (yx_3^{(1)} - x_3y^{(1)})x_2^{(1)})x_4^{(2)} + ((yx_3^{(2)} - x_3y^{(2)})x_2^{(2)} + (yx_3^{(1)} - x_3y^{(1)})x_2^{(3)}\\
&- (yx_3^{(4)} - x_3y^{(4)} + 2y^{(1)}x_3^{(3)} - 2x_3^{(1)}y^{(3)})x_2 - (yx_3^{(3)} - x_3y^{(3)} + y^{(1)}x_3^{(2)} - x_3^{(1)}y^{(2)})x_2^{(1)})x_4)x_1\\
&- (((yx_3^{(2)} - x_3y^{(2)})x_2 - (yx_3^{(1)} - x_3y^{(1)})x_2^{(1)})x_1^{(2)} + ((yx_3^{(2)} - x_3y^{(2)})x_2^{(2)} + (yx_3^{(1)} - x_3y^{(1)})x_2^{(3)}\\
&- (yx_3^{(4)} - x_3y^{(4)} + 2y^{(1)}x_3^{(3)} - 2x_3^{(1)}y^{(3)})x_2 - (yx_3^{(3)} - x_3y^{(3)} + y^{(1)}x_3^{(2)} - x_3^{(1)}y^{(2)})x_2^{(1)})x_1)x_4\\
&- (((yx_3^{(2)} - x_3y^{(2)})x_4 - (yx_3^{(1)} - x_3y^{(1)})x_4^{(1)})x_2^{(2)} + ((yx_3^{(2)} - x_3y^{(2)})x_4^{(2)} + (yx_3^{(1)} - x_3y^{(1)})x_4^{(3)}\\
&- (yx_3^{(4)} - x_3y^{(4)} + 2y^{(1)}x_3^{(3)} - 2x_3^{(1)}y^{(3)})x_4 - (yx_3^{(3)} - x_3y^{(3)} + y^{(1)}x_3^{(2)} - x_3^{(1)}y^{(2)})x_4^{(1)})x_2)x_1\\
&+ (((yx_3^{(2)} - x_3y^{(2)})x_4 - (yx_3^{(1)} - x_3y^{(1)})x_4^{(1)})x_1^{(2)} + ((yx_3^{(2)} - x_3y^{(2)})x_4^{(2)} + (yx_3^{(1)} - x_3y^{(1)})x_4^{(3)}\\
&- (yx_3^{(4)} - x_3y^{(4)} + 2y^{(1)}x_3^{(3)} - 2x_3^{(1)}y^{(3)})x_4 - (yx_3^{(3)} - x_3y^{(3)} + y^{(1)}x_3^{(2)} - x_3^{(1)}y^{(2)})x_4^{(1)})x_1)x_2\\
&+ (((yx_2^{(2)} - x_2y^{(2)})x_1 - (yx_2^{(1)} - x_2y^{(1)})x_1^{(1)})x_4^{(2)} + ((yx_2^{(2)} - x_2y^{(2)})x_1^{(2)} + (yx_2^{(1)} - x_2y^{(1)})x_1^{(3)}\\
&- (yx_2^{(4)} - x_2y^{(4)} + 2y^{(1)}x_2^{(3)} - 2x_2^{(1)}y^{(3)})x_1 - (yx_2^{(3)} - x_2y^{(3)} + y^{(1)}x_2^{(2)} - x_2^{(1)}y^{(2)})x_1^{(1)})x_4)x_3\\
&- (((yx_2^{(2)} - x_2y^{(2)})x_1 - (yx_2^{(1)} - x_2y^{(1)})x_1^{(1)})x_3^{(2)} + ((yx_2^{(2)} - x_2y^{(2)})x_1^{(2)} + (yx_2^{(1)} - x_2y^{(1)})x_1^{(3)}\\
&- (yx_2^{(4)} - x_2y^{(4)} + 2y^{(1)}x_2^{(3)} - 2x_2^{(1)}y^{(3)})x_1 - (yx_2^{(3)} - x_2y^{(3)} + y^{(1)}x_2^{(2)} - x_2^{(1)}y^{(2)})x_1^{(1)})x_3)x_4\\
&- (((yx_2^{(2)} - x_2y^{(2)})x_3 - (yx_2^{(1)} - x_2y^{(1)})x_3^{(1)})x_4^{(2)} + ((yx_2^{(2)} - x_2y^{(2)})x_3^{(2)} + (yx_2^{(1)} - x_2y^{(1)})x_3^{(3)}\\
&- (yx_2^{(4)} - x_2y^{(4)} + 2y^{(1)}x_2^{(3)} - 2x_2^{(1)}y^{(3)})x_3 - (yx_2^{(3)} - x_2y^{(3)} + y^{(1)}x_2^{(2)} - x_2^{(1)}y^{(2)})x_3^{(1)})x_4)x_1\\
&+ (((yx_2^{(2)} - x_2y^{(2)})x_3 - (yx_2^{(1)} - x_2y^{(1)})x_3^{(1)})x_1^{(2)} + ((yx_2^{(2)} - x_2y^{(2)})x_3^{(2)} + (yx_2^{(1)} - x_2y^{(1)})x_3^{(3)}\\
&- (yx_2^{(4)} - x_2y^{(4)} + 2y^{(1)}x_2^{(3)} - 2x_2^{(1)}y^{(3)})x_3 - (yx_2^{(3)} - x_2y^{(3)} + y^{(1)}x_2^{(2)} - x_2^{(1)}y^{(2)})x_3^{(1)})x_1)x_4\\
&+ (((yx_2^{(2)} - x_2y^{(2)})x_4 - (yx_2^{(1)} - x_2y^{(1)})x_4^{(1)})x_3^{(2)} + ((yx_2^{(2)} - x_2y^{(2)})x_4^{(2)} + (yx_2^{(1)} - x_2y^{(1)})x_4^{(3)}\\
&- (yx_2^{(4)} - x_2y^{(4)} + 2y^{(1)}x_2^{(3)} - 2x_2^{(1)}y^{(3)})x_4 - (yx_2^{(3)} - x_2y^{(3)} + y^{(1)}x_2^{(2)} - x_2^{(1)}y^{(2)})x_4^{(1)})x_3)x_1\\
&- (((yx_2^{(2)} - x_2y^{(2)})x_4 - (yx_2^{(1)} - x_2y^{(1)})x_4^{(1)})x_1^{(2)} + ((yx_2^{(2)} - x_2y^{(2)})x_4^{(2)} + (yx_2^{(1)} - x_2y^{(1)})x_4^{(3)}\\
&- (yx_2^{(4)} - x_2y^{(4)} + 2y^{(1)}x_2^{(3)} - 2x_2^{(1)}y^{(3)})x_4 - (yx_2^{(3)} - x_2y^{(3)} + y^{(1)}x_2^{(2)} - x_2^{(1)}y^{(2)})x_4^{(1)})x_1)x_3\\
&- (((yx_1^{(2)} - x_1y^{(2)})x_2 - (yx_1^{(1)} - x_1y^{(1)})x_2^{(1)})x_4^{(2)} + ((yx_1^{(2)} - x_1y^{(2)})x_2^{(2)} + (yx_1^{(1)} - x_1y^{(1)})x_2^{(3)}
\end{aligned}
$$

$$- (yx_1^{(4)} - x_1y^{(4)} + 2y^{(1)}x_1^{(3)} - 2x_1^{(1)}y^{(3)})x_2 - (yx_1^{(3)} - x_1y^{(3)} + y^{(1)}x_1^{(2)} - x_1^{(1)}y^{(2)})x_2^{(1)})x_4)x_3$$
$$+ (((yx_1^{(2)} - x_1y^{(2)})x_2 - (yx_1^{(1)} - x_1y^{(1)})x_2^{(1)})x_3^{(2)} + ((yx_1^{(2)} - x_1y^{(2)})x_2^{(2)} + (yx_1^{(1)} - x_1y^{(1)})x_2^{(3)}$$
$$- (yx_1^{(4)} - x_1y^{(4)} + 2y^{(1)}x_1^{(3)} - 2x_1^{(1)}y^{(3)})x_2 - (yx_1^{(3)} - x_1y^{(3)} + y^{(1)}x_1^{(2)} - x_1^{(1)}y^{(2)})x_2^{(1)})x_3)x_4$$
$$+ (((yx_1^{(2)} - x_1y^{(2)})x_3 - (yx_1^{(1)} - x_1y^{(1)})x_3^{(1)})x_4^{(2)} + ((yx_1^{(2)} - x_1y^{(2)})x_3^{(2)} + (yx_1^{(1)} - x_1y^{(1)})x_3^{(3)}$$
$$- (yx_1^{(4)} - x_1y^{(4)} + 2y^{(1)}x_1^{(3)} - 2x_1^{(1)}y^{(3)})x_3 - (yx_1^{(3)} - x_1y^{(3)} + y^{(1)}x_1^{(2)} - x_1^{(1)}y^{(2)})x_3^{(1)})x_4)x_2$$
$$- (((yx_1^{(2)} - x_1y^{(2)})x_3 - (yx_1^{(1)} - x_1y^{(1)})x_3^{(1)})x_2^{(2)} + ((yx_1^{(2)} - x_1y^{(2)})x_3^{(2)} + (yx_1^{(1)} - x_1y^{(1)})x_3^{(3)}$$
$$- (yx_1^{(4)} - x_1y^{(4)} + 2y^{(1)}x_1^{(3)} - 2x_1^{(1)}y^{(3)})x_3 - (yx_1^{(3)} - x_1y^{(3)} + y^{(1)}x_1^{(2)} - x_1^{(1)}y^{(2)})x_3^{(1)})x_2)x_4$$
$$- (((yx_1^{(2)} - x_1y^{(2)})x_4 - (yx_1^{(1)} - x_1y^{(1)})x_4^{(1)})x_3^{(2)} + ((yx_1^{(2)} - x_1y^{(2)})x_4^{(2)} + (yx_1^{(1)} - x_1y^{(1)})x_4^{(3)}$$
$$- (yx_1^{(4)} - x_1y^{(4)} + 2y^{(1)}x_1^{(3)} - 2x_1^{(1)}y^{(3)})x_4 - (yx_1^{(3)} - x_1y^{(3)} + y^{(1)}x_1^{(2)} - x_1^{(1)}y^{(2)})x_4^{(1)})x_3)x_2$$
$$+ (((yx_1^{(2)} - x_1y^{(2)})x_4 - (yx_1^{(1)} - x_1y^{(1)})x_4^{(1)})x_2^{(2)} + ((yx_1^{(2)} - x_1y^{(2)})x_4^{(2)} + (yx_1^{(1)} - x_1y^{(1)})x_4^{(3)}$$
$$- (yx_1^{(4)} - x_1y^{(4)} + 2y^{(1)}x_1^{(3)} - 2x_1^{(1)}y^{(3)})x_4 - (yx_1^{(3)} - x_1y^{(3)} + y^{(1)}x_1^{(2)} - x_1^{(1)}y^{(2)})x_4^{(1)})x_2)x_3$$
$$- (((yx_4^{(1)} - x_4y^{(1)})x_1^{(2)} - (yx_4^{(3)} - x_4y^{(3)} + y^{(1)}x_4^{(2)} - x_4^{(1)}y^{(2)})x_1)x_3$$
$$+ ((yx_4^{(2)} - x_4y^{(2)})x_1 - (yx_4^{(1)} - x_4y^{(1)})x_1^{(1)})x_3^{(1)})x_2^{(1)}$$
$$+ (((yx_4^{(1)} - x_4y^{(1)})x_1^{(2)} - (yx_4^{(3)} - x_4y^{(3)} + y^{(1)}x_4^{(2)} - x_4^{(1)}y^{(2)})x_1)x_2$$
$$+ ((yx_4^{(2)} - x_4y^{(2)})x_1 - (yx_4^{(1)} - x_4y^{(1)})x_1^{(1)})x_2^{(1)})x_3^{(1)}$$
$$+ (((yx_4^{(1)} - x_4y^{(1)})x_2^{(2)} - (yx_4^{(3)} - x_4y^{(3)} + y^{(1)}x_4^{(2)} - x_4^{(1)}y^{(2)})x_2)x_3$$
$$+ ((yx_4^{(2)} - x_4y^{(2)})x_2 - (yx_4^{(1)} - x_4y^{(1)})x_2^{(1)})x_3^{(1)})x_1^{(1)}$$
$$- (((yx_4^{(1)} - x_4y^{(1)})x_2^{(2)} - (yx_4^{(3)} - x_4y^{(3)} + y^{(1)}x_4^{(2)} - x_4^{(1)}y^{(2)})x_2)x_1$$
$$+ ((yx_4^{(2)} - x_4y^{(2)})x_2 - (yx_4^{(1)} - x_4y^{(1)})x_2^{(1)})x_1^{(1)})x_3^{(1)}$$
$$- (((yx_4^{(1)} - x_4y^{(1)})x_3^{(2)} - (yx_4^{(3)} - x_4y^{(3)} + y^{(1)}x_4^{(2)} - x_4^{(1)}y^{(2)})x_3)x_2$$
$$+ ((yx_4^{(2)} - x_4y^{(2)})x_3 - (yx_4^{(1)} - x_4y^{(1)})x_3^{(1)})x_2^{(1)})x_1^{(1)}$$
$$+ (((yx_4^{(1)} - x_4y^{(1)})x_3^{(2)} - (yx_4^{(3)} - x_4y^{(3)} + y^{(1)}x_4^{(2)} - x_4^{(1)}y^{(2)})x_3)x_1$$
$$+ ((yx_4^{(2)} - x_4y^{(2)})x_3 - (yx_4^{(1)} - x_4y^{(1)})x_3^{(1)})x_1^{(1)})x_2^{(1)}$$
$$+ (((yx_3^{(1)} - x_3y^{(1)})x_1^{(2)} - (yx_3^{(3)} - x_3y^{(3)} + y^{(1)}x_3^{(2)} - x_3^{(1)}y^{(2)})x_1)x_4$$
$$+ ((yx_3^{(2)} - x_3y^{(2)})x_1 - (yx_3^{(1)} - x_3y^{(1)})x_1^{(1)})x_4^{(1)})x_2^{(1)}$$
$$- (((yx_3^{(1)} - x_3y^{(1)})x_1^{(2)} - (yx_3^{(3)} - x_3y^{(3)} + y^{(1)}x_3^{(2)} - x_3^{(1)}y^{(2)})x_1)x_2$$
$$+ ((yx_3^{(2)} - x_3y^{(2)})x_1 - (yx_3^{(1)} - x_3y^{(1)})x_1^{(1)})x_2^{(1)})x_4^{(1)}$$
$$- (((yx_3^{(1)} - x_3y^{(1)})x_2^{(2)} - (yx_3^{(3)} - x_3y^{(3)} + y^{(1)}x_3^{(2)} - x_3^{(1)}y^{(2)})x_2)x_4$$
$$+ ((yx_3^{(2)} - x_3y^{(2)})x_2 - (yx_3^{(1)} - x_3y^{(1)})x_2^{(1)})x_4^{(1)})x_1^{(1)}$$
$$+ (((yx_3^{(1)} - x_3y^{(1)})x_2^{(2)} - (yx_3^{(3)} - x_3y^{(3)} + y^{(1)}x_3^{(2)} - x_3^{(1)}y^{(2)})x_2)x_1$$
$$+ ((yx_3^{(2)} - x_3y^{(2)})x_2 - (yx_3^{(1)} - x_3y^{(1)})x_2^{(1)})x_1^{(1)})x_4^{(1)}$$
$$+ (((yx_3^{(1)} - x_3y^{(1)})x_4^{(2)} - (yx_3^{(3)} - x_3y^{(3)} + y^{(1)}x_3^{(2)} - x_3^{(1)}y^{(2)})x_4)x_2$$
$$+ ((yx_3^{(2)} - x_3y^{(2)})x_4 - (yx_3^{(1)} - x_3y^{(1)})x_4^{(1)})x_2^{(1)})x_1^{(1)}$$
$$- (((yx_3^{(1)} - x_3y^{(1)})x_4^{(2)} - (yx_3^{(3)} - x_3y^{(3)} + y^{(1)}x_3^{(2)} - x_3^{(1)}y^{(2)})x_4)x_1$$
$$+ ((yx_3^{(2)} - x_3y^{(2)})x_4 - (yx_3^{(1)} - x_3y^{(1)})x_4^{(1)})x_1^{(1)})x_2^{(1)}$$
$$- (((yx_2^{(1)} - x_2y^{(1)})x_1^{(2)} - (yx_2^{(3)} - x_2y^{(3)} + y^{(1)}x_2^{(2)} - x_2^{(1)}y^{(2)})x_1)x_4$$

$$
\begin{aligned}
&+ ((yx_2^{(2)} - x_2y^{(2)})x_1 - (yx_2^{(1)} - x_2y^{(1)})x_1^{(1)})x_4^{(1)})x_3^{(1)}\\
&+ (((yx_2^{(1)} - x_2y^{(1)})x_1^{(2)} - (yx_2^{(3)} - x_2y^{(3)} + y^{(1)}x_2^{(2)} - x_2^{(1)}y^{(2)})x_1)x_3\\
&+ ((yx_2^{(2)} - x_2y^{(2)})x_1 - (yx_2^{(1)} - x_2y^{(1)})x_1^{(1)})x_3^{(1)})x_4^{(1)}\\
&+ (((yx_2^{(1)} - x_2y^{(1)})x_3^{(2)} - (yx_2^{(3)} - x_2y^{(3)} + y^{(1)}x_2^{(2)} - x_2^{(1)}y^{(2)})x_3)x_4\\
&+ ((yx_2^{(2)} - x_2y^{(2)})x_3 - (yx_2^{(1)} - x_2y^{(1)})x_3^{(1)})x_4^{(1)})x_1^{(1)}\\
&- (((yx_2^{(1)} - x_2y^{(1)})x_3^{(2)} - (yx_2^{(3)} - x_2y^{(3)} + y^{(1)}x_2^{(2)} - x_2^{(1)}y^{(2)})x_3)x_1\\
&+ ((yx_2^{(2)} - x_2y^{(2)})x_3 - (yx_2^{(1)} - x_2y^{(1)})x_3^{(1)})x_1^{(1)})x_4^{(1)}\\
&- (((yx_2^{(1)} - x_2y^{(1)})x_4^{(2)} - (yx_2^{(3)} - x_2y^{(3)} + y^{(1)}x_2^{(2)} - x_2^{(1)}y^{(2)})x_4)x_3\\
&+ ((yx_2^{(2)} - x_2y^{(2)})x_4 - (yx_2^{(1)} - x_2y^{(1)})x_4^{(1)})x_3^{(1)})x_1^{(1)}\\
&+ (((yx_2^{(1)} - x_2y^{(1)})x_4^{(2)} - (yx_2^{(3)} - x_2y^{(3)} + y^{(1)}x_2^{(2)} - x_2^{(1)}y^{(2)})x_4)x_1\\
&+ ((yx_2^{(2)} - x_2y^{(2)})x_4 - (yx_2^{(1)} - x_2y^{(1)})x_4^{(1)})x_1^{(1)})x_3^{(1)}\\
&+ (((yx_1^{(1)} - x_1y^{(1)})x_2^{(2)} - (yx_1^{(3)} - x_1y^{(3)} + y^{(1)}x_1^{(2)} - x_1^{(1)}y^{(2)})x_2)x_4\\
&+ ((yx_1^{(2)} - x_1y^{(2)})x_2 - (yx_1^{(1)} - x_1y^{(1)})x_2^{(1)})x_4^{(1)})x_3^{(1)}\\
&- (((yx_1^{(1)} - x_1y^{(1)})x_2^{(2)} - (yx_1^{(3)} - x_1y^{(3)} + y^{(1)}x_1^{(2)} - x_1^{(1)}y^{(2)})x_2)x_3\\
&+ ((yx_1^{(2)} - x_1y^{(2)})x_2 - (yx_1^{(1)} - x_1y^{(1)})x_2^{(1)})x_3^{(1)})x_4^{(1)}\\
&- (((yx_1^{(1)} - x_1y^{(1)})x_3^{(2)} - (yx_1^{(3)} - x_1y^{(3)} + y^{(1)}x_1^{(2)} - x_1^{(1)}y^{(2)})x_3)x_4\\
&+ ((yx_1^{(2)} - x_1y^{(2)})x_3 - (yx_1^{(1)} - x_1y^{(1)})x_3^{(1)})x_4^{(1)})x_2^{(1)}\\
&+ (((yx_1^{(1)} - x_1y^{(1)})x_3^{(2)} - (yx_1^{(3)} - x_1y^{(3)} + y^{(1)}x_1^{(2)} - x_1^{(1)}y^{(2)})x_3)x_2\\
&+ ((yx_1^{(2)} - x_1y^{(2)})x_3 - (yx_1^{(1)} - x_1y^{(1)})x_3^{(1)})x_2^{(1)})x_4^{(1)}\\
&+ (((yx_1^{(1)} - x_1y^{(1)})x_4^{(2)} - (yx_1^{(3)} - x_1y^{(3)} + y^{(1)}x_1^{(2)} - x_1^{(1)}y^{(2)})x_4)x_3\\
&+ ((yx_1^{(2)} - x_1y^{(2)})x_4 - (yx_1^{(1)} - x_1y^{(1)})x_4^{(1)})x_3^{(1)})x_2^{(1)}\\
&- (((yx_1^{(1)} - x_1y^{(1)})x_4^{(2)} - (yx_1^{(3)} - x_1y^{(3)} + y^{(1)}x_1^{(2)} - x_1^{(1)}y^{(2)})x_4)x_2\\
&+ ((yx_1^{(2)} - x_1y^{(2)})x_4 - (yx_1^{(1)} - x_1y^{(1)})x_4^{(1)})x_2^{(1)})x_3^{(1)}.
\end{aligned}
\tag{28}
$$

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