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Research Article

Wronskian Envelope of a Lie Algebra

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The famous Poincaré-Birkhoff-Witt theorem states that a Lie algebra, free as a module, embeds into its associative envelope—its universal enveloping algebra—as a sub-Lie algebra for the usual commutator Lie bracket. However, there is another functorial way—less known—to associate a Lie algebra to an associative algebra and inversely. Any commutative algebra equipped with a derivation $a \mapsto a'$, that is, a commutative differential algebra, admits a Wronskian bracket $W(a,b) = ab' - a'b$ under which it becomes a Lie algebra. Conversely, to any Lie algebra a commutative differential algebra is universally associated, its Wronskian envelope, in a way similar to the associative envelope. This contribution is the beginning of an investigation of these relations between Lie algebras and differential algebras which is parallel to the classical theory. In particular, we give a sufficient condition under which a Lie algebra may be embedded into its Wronskian envelope, and we present the construction of the free Lie algebra with this property.

1. Introduction

Any (associative) algebra (and more generally a Lie-admissible algebra), say A, admits a derived structure of Lie algebra under the commutator bracket $[a,b] = ab - ba$, $a, b \in A$. This actually describes a forgetful functor, more precisely an algebraic functor, from associative to Lie algebras. This functor admits a left adjoint that enables to associate to any Lie algebra its universal associative envelope. In this way the theory of Lie algebras may be explored through (but not reduced to) that of associative algebras. A Lie algebra which embeds into its universal enveloping algebra is referred to as special. The famous Poincaré-Birkhoff-Witt theorem states that any Lie algebra which is free as a module (and therefore, any Lie algebra over a field) is special. When the Lie algebra is Abelian, then its universal enveloping algebra reduces to the symmetric algebra of its underlying module structure, and thus any commutative Lie algebra is trivially special.

However, there is another way to associate an associative algebra to a Lie algebra, and reciprocally, in a functorial way. The idea does not consist anymore to consider non-commutative algebras under commutators but differential commutative algebras together with the so-called Wronskian determinant. A derivation of an algebra A is a linear map $\partial : A \to A$ that satisfies the usual Leibniz rule. An algebra with a distinguished derivation is said to be a differential algebra. For any such pair (A, ∂) may be defined a bilinear map $W : A^2 \to A$ by $W(a,b) = a\partial(b) - \partial(a)b$. When the algebra A is commutative, then W is alternating and satisfies the Jacobi identity so that it defines a Lie bracket on A. The definition of the Lie algebra (A, W) from (A, ∂) is functorial (homomorphisms of differential algebras are transformed into homomorphism of Lie algebras) as is the definition of the Lie algebra $(A, [\cdot,\cdot])$ associated to A from the classical theory. Also as in the classical case, this functor admits a left adjoint that allows us to define a Wronskian envelope for a Lie algebra, that is, a universal commutative and differential algebra.

This paper contributes to the study of this functorial relation (an adjointness) between Lie and differential commutative algebras in a way parallel to the classical theory of Lie and associative algebras. In particular, we describe the construction of the Wronskian envelope, and we present a sufficient condition (Theorem 16) over a field of characteristic zero for certain Lie algebras to be special, that is, to embed into their Wronskian envelopes. Contrary to the usual case, freeness of the underlying R-module is no more sufficient: there are Lie algebras, free as modules, which do not embed into their Wronksian envelopes. Indeed, special Lie algebras, in this new setting, satisfy a nontrivial relation similar to a relation that holds in Lie algebras of vector fields.

2. Differential Algebra

Because the notion of Wronskian envelope is based upon commutative differential algebras, we start by a basic and brief presentation of this theory that is inspired from [1]. Besides we show that any algebra (commutative or not) may be embedded into a differential algebra in a functorial way.

Let R be a commutative ring with a unit (it is assumed hereafter to be nonzero), and let A be an R-algebra. An R-*derivation* ∂ of A is an endomorphism of the underlying R-module structure of A that satisfies Leibniz's rule

$$\partial(ab) = \partial(a)b + a\partial(b) \tag{1}$$

for every $a, b \in R$. We remark that $\partial(1_A) = 0$ (1_A denoting the unit of A) for if $\partial(1_A) = \partial(1_A 1_A) = \partial(1_A) + \partial(1_A)$. A pair (A, ∂) is called a *differential R-algebra*. It is said to be *commutative* when A is so. We note that any R-algebra is actually a differential R-algebra with the zero derivation. Let (A, ∂_A) and (B, ∂_B) be two differential R-algebras, and $\phi : A \to B$ be a (unit-preserving) homomorphism of R-algebras. It is a *homomorphism of differential R-algebras* when $\phi \circ \partial_A = \partial_B \circ \phi$. Differential (resp., commutative) R-algebras and their homomorphisms form a category denoted by R-$\mathtt{DiffAlg}$ (resp., R-$\mathscr{C}\mathtt{DiffAlg}$). A *differential (two-sided) ideal I* of a differential R-algebra (A, ∂_A) is an (two-sided) ideal of the carrier R-algebra A such that $\partial_A(I) \subseteq I$. It is clear that the quotient-algebra A/I becomes a differential R-algebra in a natural way and that the canonical epimorphism $\pi_I : A \to A/I$ is a homomorphism of differential R-algebras. Let $S \subseteq A$ be a subset. Then, the intersection of all differential ideals of (A, ∂_A) that contain S is again a differential ideal, called the *differential ideal generated by S*. We may observe that this differential ideal is the same as the (algebraic) ideal generated by $\{\partial_A^i(x) : x \in S, i \geq 0\}$.

The obvious forgetful functor $\mathscr{A}$ (resp., $\mathscr{C}\mathscr{A}$) from R-$\mathtt{DiffAlg}$ (resp., R-$\mathscr{C}\mathtt{DiffAlg}$) to the category R-$\mathtt{Alg}$ (resp., R-$\mathscr{C}\mathtt{Alg}$) of R-algebras (resp., commutative R-algebras), that forgets the derivation, has a left adjoint (see [2] for usual category-theoretic definitions). To prove this fact, let us introduce some notations. Let X be any set and M be any monoid with unit 0. Let $f \in M^X$. Its *support* $\mathrm{supp}(f)$ is the set of all $i \in X$ such that $f(i) \neq 0$. *Finitely supported maps* are those maps with a finite support, and $M^{(X)}$ is the set of all such maps. Let V be a R-module. We denote by $V^{(\mathbb{N})}$ the R-module of all finitely supported maps from $\mathbb{N}$ to V (with point-wise addition and scalar multiplication), namely, $\bigoplus_{n\in\mathbb{N}} V$. For every $x \in V$ and $i \in \mathbb{N}$, let $x_{(i)} \in V^{(\mathbb{N})}$ be defined by $x_{(i)}(j) = x$ if $i = j$ and 0_V otherwise; the maps $q_i : x \mapsto x_{(i)}$ are the canonical injections of the coproduct $V^{(\mathbb{N})}$. We also denote by $\mathscr{T}(V)$ (resp., $\mathscr{S}(V)$) the tensor (resp., symmetric) R-algebra of V (see [3]). The natural injection from V to $\mathscr{T}(V)$ (resp., $\mathscr{S}(V)$) is denoted by n_V. Now, let A be an R-algebra (resp., commutative R-algebra), and let us denote by $\mathfrak{mod}(A)$ the underlying R-module structure of A. Let $\partial : \mathfrak{mod}(A)^{(\mathbb{N})} \to \mathfrak{mod}(A)^{(\mathbb{N})}$ be the R-linear endomorphism defined by $\partial(x_{(i)}) = x_{(i+1)}$ for every $x \in A$ and $i \in \mathbb{N}$. According to [3, Lemma 4] there exists a unique R-derivation of $\mathscr{T}(\mathfrak{mod}(A)^{(\mathbb{N})})$, again denoted by ∂, that extends ∂. Let C be the two-sided ideal of $\mathscr{T}(\mathfrak{mod}(A)^{(\mathbb{N})})$ generated by $x \otimes y - y \otimes x$ for every $x, y \in \mathfrak{mod}(A)^{(\mathbb{N})}$. Since $\partial(C) \subseteq C$, $(\mathscr{S}(\mathfrak{mod}(A)^{(\mathbb{N})}), \partial)$ is a commutative differential R-algebra (by abuse of notations ∂ is the derivation on the quotient algebra). Now, let us consider the (usual) ideal I of $\mathscr{T}(\mathfrak{mod}(A)^{(\mathbb{N})})$ (resp., $\mathscr{S}(\mathfrak{mod}(A)^{(\mathbb{N})})$) generated by $(xy)_{(i)} - \sum_{j=0}^{i} \binom{i}{j} x_{(j)} \otimes y_{(i-j)}$ for every $x, y \in A$ and $i \in \mathbb{N}$ (where the product of elements in A is denoted by a juxtaposition), by $(1_A)_{(0)} - 1$ (where 1 is the unit of $\mathscr{T}(\mathfrak{mod}(A)^{(\mathbb{N})})$ and, resp., of $\mathscr{S}(\mathfrak{mod}(A)^{(\mathbb{N})})$) and by $(1_A)_{(i)}$ for every $i > 0$. It is clear that $\pi_I \circ \partial : \mathscr{T}(\mathfrak{mod}(A)^{(\mathbb{N})}) \to \mathscr{T}(\mathfrak{mod}(A)^{(\mathbb{N})})/I$ (resp., $\pi_I \circ \partial : \mathscr{S}(\mathfrak{mod}(A)^{(\mathbb{N})}) \to \mathscr{S}(\mathfrak{mod}(A)^{(\mathbb{N})})/I$) factors through the quotient (where π_I is the canonical epimorphism from $\mathscr{T}(\mathfrak{mod}(A)^{(\mathbb{N})})$ to $\mathscr{T}(\mathfrak{mod}(A)^{(\mathbb{N})})/I$ and, resp., from $\mathscr{S}(\mathfrak{mod}(A)^{(\mathbb{N})})$ to $\mathscr{S}(\mathfrak{mod}(A)^{(\mathbb{N})})/I$), and defines an R-derivation $\overline{\partial}$ on $\mathscr{T}(\mathfrak{mod}(A)^{(\mathbb{N})})/I$ (resp., $\mathscr{S}(\mathfrak{mod}(A)^{(\mathbb{N})})/I$). This differential R-algebra is denoted thereafter by $(\mathscr{D}(A), \overline{\partial})$ (resp., $(\mathscr{C}\mathscr{D}(A), \overline{\partial})$).

Theorem 1. *Let A be an R-algebra (resp., commutative R-algebra). Let (B, ∂_B) be a differential R-algebra (resp., commutative differential R-algebra), and let $\phi : A \to B$ be a homomorphism of R-algebras. Then, there is a unique homomorphism $\overline{\phi} : (\mathscr{D}(A), \overline{\partial}) \to (B, \partial_B)$ of differential R-algebras (resp., $\overline{\phi} : (\mathscr{C}\mathscr{D}(A), \overline{\partial}) \to (B, \partial_B)$ commutative differential R-algebras) such that $\overline{\phi}(\pi_I(x_{(0)})) = \phi(x)$ for every $x \in A$.*

Proof. Let $\phi_1 : \mathfrak{mod}(A)^{(\mathbb{N})} \to \mathfrak{mod}(B)$ be the unique R-linear map such that $\phi_1(x_{(i)}) = \partial_B^i(\phi(x))$ for every $x \in A$, $i \geq 0$. Then, we may define an algebra map $\widehat{\phi}_1 : \mathscr{T}(\mathfrak{mod}(A)^{(\mathbb{N})}) \to B$ (resp., $\widehat{\phi}_1 : \mathscr{S}(\mathfrak{mod}(A)^{(\mathbb{N})}) \to B$) using the universal property of the tensor algebra (resp., symmetric algebra). We have $\widehat{\phi}_1(x_{(i)}) = \partial_B^i(\phi(x))$. This map factors through π_I. Indeed, let $x, y \in A$ and $i \geq 0$. We have $\widehat{\phi}_1((xy)_{(i)}) = \partial_B^i(\phi(xy)) = \partial_B^i(\phi(x)\phi(y)) = \sum_{j=0}^{i} \binom{i}{j} \partial_B^j(\phi(x))\partial_B^{i-j}(\phi(x)) = \sum_{j=0}^{i} \binom{i}{j} \widehat{\phi}_1(x_{(j)})\widehat{\phi}_1(y_{(i-j)}) = \widehat{\phi}_1(\sum_{j=0}^{i} \binom{i}{j} x_{(j)} \otimes y_{(i-j)})$, also $\widehat{\phi}_1((1_A)_{(0)}) = \phi(1_A) = 1_B = \widehat{\phi}_1(1)$, and $\widehat{\phi}_1((1_A)_{(i)}) = \partial_B^i(\phi(1_A)) = \partial_B^i(1_B) = 0$ for every $i > 0$. Therefore, there is a unique homomorphism $\overline{\phi}$ of R-algebras (resp., commutative R-algebras) from $\mathscr{D}(A)$ (resp., $\mathscr{C}\mathscr{D}(A)$) to B such that $\overline{\phi} \circ \pi_I(x_{(0)}) = \phi(x)$ for every $x \in A$. It is easily seen to be a homomorphism of differential (resp., commutative differential) R-algebras. □

Example 2. Let $R[X] = \mathscr{S}(RX)$ be the free commutative R-algebra over X. Therefore $(\mathscr{C}\mathscr{D}(R[X]), \overline{\partial})$ is recovered from the usual algebra of differential polynomials $R\{X\}$ over R (see [4, 5] for instance), that is, the free commutative R-algebra $R[X \times \mathbb{N}]$ with the R-derivation $\partial(x, i) = (x, i+1)$ for all $x \in X$, $i \geq 0$. Now, if $R\langle X\rangle = \mathscr{T}(RX)$ is the free R-algebra over X, then $(\mathscr{D}(R\langle X\rangle), \overline{\partial})$ is the (not so wellknown, see [6] however) noncommutative counterpart of $R\{X\}$, that is, the free R-algebra $R\langle X \times \mathbb{N}\rangle$ with derivation

$\partial((x_1,i_1)(x_2,i_2)\cdots(x,i_n)) = (x_1,i_1+1)(x_2,i_2)\cdots(x,i_n) + (x_1,i_1)(x_2,i_2+1)\cdots(x_n,i_n)+\cdots+(x_1,i_1)(x_2,i_2)\cdots(x_n,i_n+1)$.

Remark 3. It is clear that (commutative) differential algebras form a variety (in the sense of universal algebra, see [7]), and therefore, we may define the free (commutative) differential algebra over a set X. It is not difficult to check that $R\{X\}$ is the free commutative differential algebra over X and $\mathscr{D}(R\langle X\rangle)$ is the free differential algebra over X. Moreover X embeds into these algebras: $X \subseteq R\{X\}$ and $X \subseteq \mathscr{D}(R\langle X\rangle)$.

Corollary 4. *The algebra A (resp., commutative algebra A) embeds into $\mathscr{D}(A)$ (resp., $\mathscr{CD}(A)$) as a subalgebra, more precisely the map $j_A : A \to \mathscr{D}(A)$ (resp., $j_A : A \to \mathscr{CD}(A)$), such that $j_A(x) = \pi_I(x_{(0)})$ for every $x \in A$ is a one-to-one algebra homomorphism.*

Proof. The map $j_A : A \to \mathscr{D}(A)$ (resp., $j_A : A \to \mathscr{CD}(A)$) is easily seen to be an algebra map. Since $(A, 0)$ is a differential R-algebra (resp., commutative differential R-algebra) and according to Theorem 1, the identity algebra map id_A extends uniquely to a homomorphism $\overline{id}_A : (\mathscr{D}(A), \overline{\partial}) \to (A, 0)$ (resp., $\overline{id}_A : (\mathscr{CD}(A), \overline{\partial}) \to (A, 0)$) of differential R-algebras (resp., commutative differential R-algebras) such that $\overline{id}_A \circ j_A = id_A$, so that j_A is one-to-one. □

From these results (Theorem 1 and Corollary 4), we may deduce a Poincaré-Birkhoff-Witt-like theorem. Let us denote by $\mathscr{U}(\mathfrak{g})$ the universal enveloping algebra of a Lie algebra $\mathfrak{g}$, and let $i_{\mathfrak{g}} : \mathfrak{g} \to \mathscr{U}(\mathfrak{g})$ denote the canonical map which is one-to-one when $\mathfrak{g}$ is a free module (Poincaré-Birkhoff-Witt theorem; see [8], e.g.). The underlying Lie algebra structure, denoted by $\mathfrak{l}(A)$, of an associative algebra A is given by the usual commutation bracket.

Corollary 5. *Let $\mathfrak{g}$ be a Lie algebra (resp., commutative Lie algebra) over R which is free as a R-module. Then, $\mathfrak{g}$ embeds into $\mathscr{D}(\mathscr{U}(\mathfrak{g}))$ (resp., $\mathscr{CD}(\mathscr{U}(\mathfrak{g}))$) as a Lie subalgebra. Moreover, if (B, ∂_B) is a differential R-algebra (resp., commutative differential R-algebra) and $\phi : \mathfrak{g} \to \mathfrak{l}(B)$ is a homomorphism of Lie algebras, then there is a unique homomorphism $\overline{\phi} : (\mathscr{D}(\mathscr{U}(\mathfrak{g})), \overline{\partial}) \to (B, \partial_B)$ (resp.,$\overline{\phi} : (\mathscr{CD}(\mathscr{U}(\mathfrak{g})), \overline{\partial}) \to (B, \partial_B)$) such that $\overline{\phi} \circ j_{\mathscr{U}(\mathfrak{g})} \circ i_{\mathfrak{g}}$.*

Proof. The proof is easy (essentially a composition of left adjoints), hence omitted. □

Corollary 5 means in particular that the forgetful functor from differential R-algebras (resp., commutative differential R-algebras) to Lie algebras (resp., commutative Lie algebras) R-`LieAlg` (resp., R-$\mathscr{C}$`LieAlg`), obtained by composition of the previous "forget-the-derivation" functor from R-`DiffAlg` to R-`Alg` (resp., R-$\mathscr{C}$`DiffAlg` to R-$\mathscr{C}$`Alg`) and $\mathfrak{l}$ from R-`Alg` to R-`LieAlg` (resp., R-$\mathscr{C}$`Alg` to R-$\mathscr{C}$`LieAlg`), has a left adjoint, also obtained by composition of the one given by Theorem 1 and the universal enveloping algebra functor. But there is another forgetful functor from R-$\mathscr{C}$`DiffAlg` to R-`LieAlg` given by the Wronskian, which is studied in what follows.

3. Wronskian Envelope

In this section the Wronskian envelope universally associated to any Lie algebra is constructed, that is, a left adjoint to the forgetful functor from commutative differential algebras to Lie algebras.

Let (A, ∂_A) be a differential commutative R-algebra. Then, it admits a (functorial) structure of a Lie R-algebra for which the bracket is defined by the usual Wronskian determinant

$$W_A(x, y) = x\partial_A(y) - \partial_A(x)\,y = (id_A \wedge \partial_A)(x \wedge y) \quad (2)$$

for every $x, y \in A$, where $\wedge$ denotes the exterior product (this kind of structure has been used to define n-Lie algebras, see [9]). Let us denote by $\mathfrak{w}(A, \partial_A)$ or more simply $\mathfrak{w}(A)$ this Lie algebra structure. Let (B, ∂_B) be another differential commutative R-algebra, and let $\phi : (A, \partial_A) \to (B, \partial_B)$ be a homomorphism of differential commutative R-algebras. Then, ϕ is also a homomorphism of Lie algebras from $\mathfrak{w}(A)$ to $\mathfrak{w}(B)$ since $\phi(W_A(x, y)) = \phi(x\partial_A(y) - \partial_A(x)y) = \phi(x)\phi(\partial_A(y)) - \phi(\partial_A(x))\phi(y) = \phi(x)\partial_B(\phi(y)) - \partial_B(\phi(x))\phi(y) = W_B(\phi(x), \phi(y))$. Hence $\mathfrak{w} : R$-`DiffAlg` $\to R$-`LieAlg` defines a (forgetful) functor.

Remark 6. We observe that the Lie algebra structure $\mathfrak{w}(A)$ associated to the differential R-algebra $(A, 0)$ is commutative. Conversely, let (A, ∂_A) be a commutative differential algebra such that $\mathfrak{w}(A, \partial_A)$ is a commutative Lie algebra; that is, $W_A(x, y) = 0$ for every $x, y \in A$. Since $W_A(1_A, x) = \partial_A(x)$, it follows that ∂_A is the zero derivative.

The functor $\mathfrak{w}$ also has a left adjoint (The existence of a left adjoint is guaranteed because $\mathfrak{w}$ is algebraic. See, e.g., [10] for the notion of algebraic functors.) that allows us to define a notion of *Wronskian universal enveloping algebra* or shortly *Wronskian envelope*. Let $\mathfrak{g}$ be a Lie algebra over R. Another time let us denote by $\mathfrak{mod}(\mathfrak{g})$ its underlying R-module structure. We now consider the symmetric R-algebra $\mathscr{S}(\mathfrak{mod}(\mathfrak{g})^{(\mathbb{N})})$ of $\mathfrak{mod}(\mathfrak{g})^{(\mathbb{N})}$. As in Section 2, let C be the two-sided ideal of $\mathscr{T}(\mathfrak{mod}(\mathfrak{g})^{(\mathbb{N})})$ generated by $x \otimes y - y \otimes x$ for every $x, y \in \mathfrak{mod}(\mathfrak{g})^{(\mathbb{N})}$. We have $\partial(C) \subseteq C$ where, as in Section 2, ∂ is the unique R-derivation of $\mathscr{T}(\mathfrak{mod}(\mathfrak{g})^{(\mathbb{N})})$ that extends the R-linear endomorphism $\partial : x_{(i)} \mapsto x_{(i+1)}$ of $\mathfrak{mod}((\mathfrak{g})^{(\mathbb{N})})$. Hence $(\mathscr{S}(\mathfrak{mod}(\mathfrak{g})^{(\mathbb{N})}), \partial)$ is a commutative differential R-algebra where, by abuse of language, ∂ is the natural derivation on the quotient algebra. Let us consider the differential ideal $J_{\mathfrak{g}}$ generated by $[x, y]_{(0)} - x_{(0)} \otimes y_{(1)} + x_{(1)} \otimes y_{(0)}$ for every $x, y \in \mathfrak{g}$. Let $\sigma : \mathscr{S}(\mathfrak{mod}(\mathfrak{g})^{(\mathbb{N})}) \to \mathscr{S}(\mathfrak{mod}(\mathfrak{g})^{(\mathbb{N})})/J_{\mathfrak{g}}$ be the canonical epimorphism, and let $\overline{\partial}$ be the (unique) derivation such that $\overline{\partial} \circ \sigma = \sigma \circ \partial$. Let $\mathscr{W}(\mathfrak{g}) = \mathscr{S}(\mathfrak{mod}(\mathfrak{g})^{(\mathbb{N})})/J_{\mathfrak{g}}$.

Remark 7. For any R-module V, we denote by $\mathscr{S}(V)_+$ the ideal of all members of $\mathscr{S}(V)$ with no constant term (relatively to the usual gradation of the symmetric algebra), and let $\mathscr{S}(V)_0 = R\cdot 1$ so that $\mathscr{S}(V) = \mathscr{S}(V)_0 \oplus \mathscr{S}(V)_+$ as R-module. We observe that $J_{\mathfrak{g}} \subseteq \mathscr{S}(\mathfrak{mod}(\mathfrak{g})^{(\mathbb{N})})_+$ (since for every $x, y \in \mathfrak{g}$, $[x, y]_{(0)} - x_{(0)} \otimes y_{(1)} + x_{(1)} \otimes y_{(0)}$ and all their derivatives

$[x,y]_{(n)} - \sum_{k=0}^{n}\binom{n}{k} x_{(n-k)} \otimes y_{(k+1)} + \sum_{k=0}^{n}\binom{n}{k} x_{(n-k+1)} \otimes y_{(k)}$ belong to $\mathcal{S}(\mathfrak{mod}(\mathfrak{g})^{(\mathbb{N})})_+)$ so that $W(\mathfrak{g})$ is not reduced to zero (except if R itself is (0)) because there exists an algebra map from $\mathcal{W}(\mathfrak{g})$ onto $\mathcal{S}(\mathfrak{mod}(\mathfrak{g})^{(\mathbb{N})})/\mathcal{S}(\mathfrak{mod}(\mathfrak{g})^{(\mathbb{N})})_+ \cong R\cdot 1$. The direct sum decomposition $\mathcal{S}(\mathfrak{mod}(\mathfrak{g})^{(\mathbb{N})}) = \mathcal{S}(\mathfrak{mod}(\mathfrak{g})^{(\mathbb{N})})_0 \oplus \mathcal{S}(\mathfrak{mod}(\mathfrak{g})^{(\mathbb{N})})_+$ induces a decomposition $\mathcal{W}(\mathfrak{g}) = \mathcal{W}(\mathfrak{g})_0 \oplus \mathcal{W}(\mathfrak{g})_+$, where $\mathcal{W}(\mathfrak{g})_+$ and $\mathcal{W}(\mathfrak{g})_0$ are the canonical images of $\mathcal{S}(\mathfrak{mod}(\mathfrak{g})^{(\mathbb{N})})_+$ and $\mathcal{S}(\mathfrak{mod}(\mathfrak{g})^{(\mathbb{N})})_0$, and $\mathcal{W}(\mathfrak{g})_0 = R\cdot 1$.

In what follows we denote simply by W the Wronskian bracket $W_{\mathcal{S}(\mathfrak{mod}(\mathfrak{g})^{(\mathbb{N})})}$ of the differential algebra $(\mathcal{S}(\mathfrak{mod}(\mathfrak{g})^{(\mathbb{N})}),\partial)$, and the tensor symbol "$\otimes$" is omitted.

Proposition 8. *For every $x,y\in\mathfrak{g}$ and every $i,j,n\geq 0$,*

$$\partial^n\left(W\left(x_{(i)},y_{(j)}\right)\right) = \sum_{k=0}^{n}\binom{n}{k} W\left(x_{(i+n-k)},y_{(j+k)}\right). \quad (3)$$

Proof. We have

$$\begin{aligned}\partial\left(W\left(x_{(i)},y_{(j)}\right)\right) &= \partial\left(x_{(i)}y_{(j+1)} - x_{(i+1)}y_{(j)}\right)\\ &= x_{(i+1)}y_{(j+1)} + x_{(i)}y_{(j+2)}\\ &\quad - x_{(i+2)}y_{(j)} - x_{(i+1)}y_{(j+1)}\\ &= x_{(i)}y_{(j+2)} - x_{(i+1)}y_{(j+1)}\\ &\quad + x_{(i+1)}y_{(j+1)} - x_{(i+2)}y_{(j)}\\ &= W\left(x_{(i)},y_{(j+1)}\right) + W\left(x_{(i+1)},y_{(j)}\right).\end{aligned} \quad (4)$$

Now, assume by induction on n that

$$\partial^n\left(W\left(x_{(i)},y_{(j)}\right)\right) = \sum_{k=0}^{n}\binom{n}{k} W\left(x_{(i+n-k)},y_{(j+k)}\right) \quad (5)$$

(the cases $n=0,1$ are checked). We have

$$\begin{aligned}&\partial^{n+1}\left(W\left(x_{(i)},y_{(j)}\right)\right)\\ &\quad= \partial\left(\partial^n\left(W\left(x_{(i)},y_{(j)}\right)\right)\right)\\ &\quad= \partial\left(\sum_{k=0}^{n}\binom{n}{k} W\left(x_{(i+n-k)},y_{(j+k)}\right)\right)\\ &\quad= \sum_{k=0}^{n}\binom{n}{k}\left(W\left(x_{(i+n+1-k)},y_{(j+k)}\right)\right.\\ &\qquad\qquad \left.+W\left(x_{(i+n-k)},y_{(j+1+k)}\right)\right)\\ &\quad= \underbrace{\binom{n}{0}}_{=1} W\left(x_{(i+n+1)},y_{(j)}\right)\\ &\qquad+ \underbrace{\binom{n}{n}}_{=1} W\left(x_{(i)},y_{(j+1+n)}\right)\\ &\qquad+ \sum_{k=1}^{n}\binom{n}{k} W\left(x_{(i+n+1-k)},y_{(j+k)}\right)\\ &\qquad+ \sum_{k=0}^{n-1}\binom{n}{k} W\left(x_{(i+n-k)},y_{(j+1+k)}\right)\\ &\quad= W\left(x_{(i+n+1)},y_{(j)}\right) + W\left(x_{(i)},y_{(j+1+n)}\right)\\ &\qquad+ \sum_{k=1}^{n}\binom{n}{k} W\left(x_{(i+n+1-k)},y_{(j+k)}\right)\\ &\qquad+ \sum_{k=1}^{n}\binom{n}{k-1} W\left(x_{(i+n+1-k)},y_{(j+k)}\right)\\ &\quad= W\left(x_{(i+n+1)},y_{(j)}\right) + W\left(x_{(i)},y_{(j+1+n)}\right)\\ &\qquad+ \sum_{k+1}^{n}\underbrace{\left(\binom{n}{k-1}+\binom{n}{k}\right)}_{=\binom{n+1}{k}} W\left(x_{(i+n+1-k)},y_{(j+k)}\right)\\ &\quad= \sum_{k=0}^{n+1}\binom{n+1}{k} W\left(x_{(i+n+1-k)},y_{(j+k)}\right).\end{aligned} \quad (6)$$

□

According to Proposition 8, $J_{\mathfrak{g}}$ is actually the ideal generated by $[x,y]_{(n)} - \sum_{k=0}^{n}\binom{n}{k} W(x_{(n-k)},y_{(k)})$ for every $x,y\in\mathfrak{g}$ and every $n\geq 0$.

We claim that $(\mathcal{W}(\mathfrak{g}),\overline{\partial})$ is the universal enveloping algebra of $\mathfrak{g}$ with respect to $\mathfrak{w}$. More precisely, the following holds.

Theorem 9. *Let $\mathfrak{g}$ be a Lie algebra over R. Let $w_{\mathfrak{g}}:\mathfrak{g}\to\mathcal{W}(\mathfrak{g})$ be the map defined by $w_{\mathfrak{g}}(x)=\sigma(x_{(0)})$ for every $x\in\mathfrak{g}$. Then, $w_{\mathfrak{g}}$ is a homomorphism of Lie algebras from $\mathfrak{g}$ to $\mathfrak{w}(\mathcal{W}(\mathfrak{g}))$. Let (A,∂_A) be a commutative differential R-algebra, and let $\phi:\mathfrak{g}\to\mathfrak{w}(A)$ be a homomorphism of Lie algebras. Then, there is a unique homomorphism of commutative differential R-algebras $\widetilde{\phi}:(\mathcal{W}(\mathfrak{g}),\overline{\partial})\to(A,\partial_A)$ such that $\widetilde{\phi}\circ w_{\mathfrak{g}}=\phi$.*

Proof. It is quite clear that $w_{\mathfrak{g}}$ is R-linear. Let $x,y\in\mathfrak{g}$. Then, we have $w_{\mathfrak{g}}([x,y]) = \sigma([x,y]_{(0)}) = \sigma(x_{(0)}\otimes y_{(1)} - x_{(1)}\otimes y_{(0)}) = \sigma(x_{(0)})\sigma(y_{(1)}) - \sigma(x_{(1)})\sigma(y_{(0)}) = \sigma(x_{(0)})\sigma(\partial(y_{(0)})) - \sigma(\partial(x_{(0)}))\sigma(y_{(0)}) = \sigma(x_{(0)})\partial_A(\sigma(y_{(1)})) - \partial_A(\sigma(x_{(0)}))\sigma(y_{(0)}) = W_B(\sigma(x_{(0)}),\sigma(y_{(0)}))$. This proves that $w_{\mathfrak{g}}$ is a homomorphism of Lie algebras. Now, let (A,∂_A) be a commutative differential R-algebra, and let $\phi:\mathfrak{g}\to\mathfrak{w}(A)$ be a homomorphism of Lie algebras. Let $\phi_1:\mathfrak{mod}(\mathfrak{g})^{(\mathbb{N})}\to\mathfrak{mod}(A)$ be the unique

R-linear map such that $\phi_1(x_{(i)}) = \partial_A^i(\phi(x))$ for every $x \in \mathfrak{g}$, $i \geq 0$. According to the universal property of the symmetric algebra, there is a unique homomorphism of algebras $\widehat{\phi}_1 : S(\mathfrak{mod}(\mathfrak{g})^{(\mathbb{N})}) \to A$ such that $\widehat{\phi}_1(x_{(i)}) = \phi_1(x_{(i)})$ for every $x \in \mathfrak{g}$, $i \geq 0$. This map factors through the quotient by $J_{\mathfrak{g}}$. Indeed, $\widehat{\phi}_1([x,y]_{(0)}) = \phi([x,y]) = W_A(\phi(x),\phi(y)) = \phi(x)\partial_A(\phi(y)) - \partial_A(\phi(x))\phi(y) = \widehat{\phi}_1(x_{(0)})\widehat{\phi}_1(y_{(1)}) - \widehat{\phi}_1(x_{(1)})\widehat{\phi}_1(y_{(0)}) = \widehat{\phi}_1(x_{(0)} \otimes y_{(1)} - x_{(1)} \otimes y_{(0)})$. Therefore, there is a unique homomorphism of algebras $\widetilde{\phi} : W(\mathfrak{g}) \to A$ such that $\widetilde{\phi} \circ \sigma = \widehat{\phi}_1$. By construction, it commutes with the derivations; hence, it is a homomorphism of differential R-algebras. Finally, $\widetilde{\phi} \circ w_{\mathfrak{g}} = \phi$ also by construction. □

The commutative differential R-algebra $(\mathscr{W}(\mathfrak{g}), \overline{\partial})$ of Theorem 9 is called the *universal enveloping algebra of* $\mathfrak{g}$ *with respect to* $\mathfrak{w}$ or the *Wronskian envelope of* $\mathfrak{g}$.

It is easy to prove that any commutative Lie algebra embeds as a sub-Lie algebra into $\mathscr{W}(\mathfrak{g})$. Moreover the universal enveloping algebra $\mathscr{U}(\mathfrak{g})$ of $\mathfrak{g}$ also embeds into $\mathscr{W}(\mathfrak{g})$ as a subalgebra but not as a differential subalgebra (when $\mathscr{U}(\mathfrak{g})$ is equipped with the zero derivative and $\mathscr{W}(\mathfrak{g})$ with its derivative $\overline{\partial}$). Indeed, let $j_{\mathfrak{g}} : \mathfrak{g} \hookrightarrow \mathscr{S}(\mathfrak{mod}(\mathfrak{g})) \cong \mathscr{U}(\mathfrak{g})$ be the canonical injection. It is clearly a homomorphism of Lie algebras from $\mathfrak{g}$ to $\mathfrak{w}(\mathscr{S}(\mathfrak{mod}(\mathfrak{g})), 0) = \mathfrak{l}(\mathscr{S}(\mathfrak{mod}(\mathfrak{g})))$. Hence by Theorem 9 there is a unique homomorphism of commutative differential algebras $\widetilde{j}_{\mathfrak{g}} : (\mathscr{W}(\mathfrak{g}), \overline{\partial}) \to (\mathscr{S}(\mathfrak{mod}(\mathfrak{g})), 0)$ such that $\widetilde{j}_{\mathfrak{g}} \circ w_{\mathfrak{g}} = j_{\mathfrak{g}}$. We then deduce that $w_{\mathfrak{g}}$ is one-to-one (let $x \in \ker w_{\mathfrak{g}}$, then $0 = \widetilde{j}_{\mathfrak{g}}(w_{\mathfrak{g}}(x)) = j_{\mathfrak{g}}(x)$ so that $x = 0$). Moreover, since $w_{\mathfrak{g}}$ is a homomorphism of Lie algebras from $\mathfrak{g}$ to $\mathfrak{w}(\mathscr{W}(\mathfrak{g}), \overline{\partial})$ and because $\mathfrak{g}$ is commutative, it is also a homomorphism of Lie algebras from $\mathfrak{g}$ to $\mathfrak{w}(\mathscr{W}(\mathfrak{g}), 0) = \mathfrak{l}(\mathscr{W}(\mathfrak{g}))$. Therefore there is a unique homomorphism of (commutative) algebras $w_{\mathfrak{g}}^{\#}$ from $\mathscr{S}(\mathfrak{mod}(\mathfrak{g}))$ to $\mathscr{W}(\mathfrak{g})$ such that $w_{\mathfrak{g}}^{\#} \circ j_{\mathfrak{g}} = w_{\mathfrak{g}}$. Thus, we obtain $w_{\mathfrak{g}}^{\#} \circ \widetilde{j}_{\mathfrak{g}} \circ w_{\mathfrak{g}} = w_{\mathfrak{g}}$ and also $\widetilde{j}_{\mathfrak{g}} \circ w_{\mathfrak{g}}^{\#} \circ j_{\mathfrak{g}} = j_{\mathfrak{g}}$. According to the universal property of the enveloping algebra, the unique homomorphism that satisfies the second equality is $id_{\mathscr{S}(\mathfrak{mod}(\mathfrak{g}))}$ so that $\widetilde{j}_{\mathfrak{g}} \circ w_{\mathfrak{g}}^{\#} = id_{\mathscr{S}(\mathfrak{mod}(\mathfrak{g}))}$. We observe, however, that we cannot deduce from the first equality that $w_{\mathfrak{g}}^{\#} \circ \widetilde{j}_{\mathfrak{g}} = id_{\mathscr{W}(\mathfrak{g})}$ because $w_{\mathfrak{g}}^{\#} \circ \widetilde{j}_{\mathfrak{g}}$ is a homomorphism of commutative differential algebras from $(\mathscr{W}(\mathfrak{g}), \overline{\partial})$ to $(\mathscr{W}(\mathfrak{g}), 0)$ and not from $(\mathscr{W}(\mathfrak{g}), \overline{\partial})$ to itself. Nevertheless from $\widetilde{j}_{\mathfrak{g}} \circ w_{\mathfrak{g}}^{\#} = id_{\mathscr{S}(\mathfrak{g})}$ it follows that $w_{\mathfrak{g}}^{\#}$ is one-to-one and $\widetilde{j}_{\mathfrak{g}}$ is onto. We observe that in general when $\mathfrak{g}$ is a commutative Lie algebra, then its universal enveloping algebra with respect to $\mathfrak{w}$ is not $(\mathscr{S}(\mathfrak{mod}(\mathfrak{g})), 0)$ that is, $\widetilde{j}_{\mathfrak{g}}$ is only onto and not one-to-one, while $w_{\mathfrak{g}}^{\#}$ is only one-to-one and not onto, and the derivative $\overline{\partial}$ on $\mathscr{W}(\mathfrak{g})$ is not the zero derivative. Indeed, for instance let $\mathfrak{g}$ be the free R-module $R \cdot x$ generated by $\{x\}$ with the zero Lie bracket. The algebra $R[x]$ of polynomials in the variable x is $\mathscr{S}(\mathfrak{mod}(\mathfrak{g})) = \mathscr{S}(R \cdot x)$. Let $\phi : R \cdot x \to R\{x\}$ be the R-module homomorphism defined by $\phi(x) = x$. It is a homomorphism of Lie algebras from $R \cdot x$ to $\mathfrak{w}(R\{x\})$ since $0 = \phi(0) = \phi([\alpha x, \beta x])$ and $\alpha\beta(\phi(x)'\phi(x) - \phi(x)\phi(x)') = 0$ for every $\alpha, \beta \in R$ while the algebra homomorphism extension $\phi^{\#} : R[x] \to R\{x\}$ of ϕ is not a homomorphism of differential algebras from $(R[x], 0)$ to $R\{x\}$ because it does not commute with the derivations ($0 = \phi^{\#}(0x)$ and $(\phi^{\#}(x))' = x' \neq 0$).

3.1. Some Remarks about the Generators of $J_{\mathfrak{g}}$. Let us denote by $P_{n+1}(x,y) = [x,y]_{(n)} - \partial^n u_{x,y}$ for every $n \geq 0$ and $x, y \in \mathfrak{g}$, where $u_{x,y} = [x,y]_{(0)} - x_{(0)}y_{(1)} + x_{(1)}y_{(0)}$. We have $P_{n+1}(x,y) = \sum_{k=0}^{n} \binom{n}{k} W(x_{(n-k)}, y_{(k)})$.

Let us introduce the following integers: $T(1,0) = -1$, $T(1,1) = 1$, and, for all $n \geq 2$ and all $k = 0, \ldots, n$,

$$T(n,k) = \begin{cases} T(n-1,0) \\ \quad \text{if } k = 0, \\ T(n-1,n-1) \\ \quad \text{if } k = n, \\ T(n-1,k-1) + T(n-1,k) \\ \quad \text{if } k = 1, \ldots, n-1. \end{cases} \tag{7}$$

Lemma 10. *For every* $n \geq 1$ *and every* $k = 0, \ldots, n$,

$$T(n,k) = \binom{n}{k} - 2\binom{n-1}{k}. \tag{8}$$

Proof. Recall that $\binom{n}{k} = 0$ for every $k > n \geq 0$. We have $T(1,0) = \binom{1}{0} - 2\binom{0}{0} - = 1 - 2 = -1$, and $T(1,1) = \binom{1}{1} - 2\binom{0}{1} = 1$. For every $n \geq 1$, $T(n,0) = \binom{n}{0} - 2\binom{n-1}{0} = 1 - 2 = -1$ and $T(n,n) = \binom{n}{n} - 2\binom{n-1}{n} = 1$. For every $n \geq 2$ and $k = 0, \ldots, n$, we have $\binom{n}{0} - 2\binom{n-1}{0} = 1 - 2 = -1 = \binom{n-1}{0} - 2\binom{n-2}{0}$, $\binom{n}{n} - 2\binom{n-1}{n} = 1 = \binom{n-1}{n-1} - 2\binom{n-2}{n-1}$. Now, let $k = 1, \ldots, n-1$. If $k \leq n-2$, then $\binom{n}{k} - 2\binom{n-1}{k} = \binom{n-1}{k-1} - 2\binom{n-2}{k-1} + \binom{n-1}{k} - 2\binom{n-2}{k}$, and if $k = n-1$, then $\binom{n}{n-1} - 2\binom{n-1}{n-1} = n-2$ while $\binom{n-1}{n-2} - 2\binom{n-2}{n-2} + \binom{n-1}{n-1} - 2\binom{n-2}{n-1} = n-1-2+1-0 = n-2$. □

Lemma 11. *For every* $n \geq 1$ *and every* $x, y \in \mathfrak{g}$, *one has*

$$P_n(x,y) = \sum_{k=0}^{n} T(n,k)\, x_{(n-k)} y_{(k)}. \tag{9}$$

Proof. We have $P_1(x,y) = x_{(0)}y_{(1)} - x_{(1)}y_{(0)} = T(1,0)x_{(1)}y_{(0)} + T(1,1)x_{(0)}y_{(1)}$. It is clear that $P_n(x,y)$ is homogeneous of degree n (in the sense that it is a sum of word $x_{(i)}y_{(j)}$ with $i + j = n$). It can be written as $P_n(x,y) = \sum_{k=0}^{n} \alpha_{n,k} x_{(n-k)} y_{(k)}$. We have $P_{n+1}(x,y) = \partial P_n(x,y) = \sum_{k=0}^{n} \alpha_{n,k} x_{(n+1-k)} y_{(k)} + \sum_{k=0}^{n} \alpha_{n,k} x_{(n-k)} y_{(k+1)} = \alpha_{n,0} x_{(n+1)} + \alpha_{n,n} y_{(n+1)} + \sum_{k=1}^{n} \alpha_{n,k} x_{(n+1-k)} y_{(k)} + \sum_{k=0}^{n-1} \alpha_{n,k} x_{(n-k)} y_{(k+1)}$ so that $\alpha_{n+1,0} = \alpha_{n,0}$, $\alpha_{n+1,n+1} = \alpha_{n,n}$ and $\alpha_{n+1,k} = \alpha_{n,k-1} + \alpha_{n,k}$. □

4. Embedding Conditions

In this section, we present a sufficient condition under which a Lie algebra embeds into its Wronskian envelope.

Adapting the terminology from [7, 11], we call *Wronskian special* those Lie R-algebras that embed into their Wronskian envelope. Thus every Abelian Lie algebra is Wronskian special (see Section 3). It is quite obvious that not all Lie algebras are Wronskian special, even when they are free as R-modules and even in the case where R is a field. This can be shown as follows. For any elements $x_1, \ldots, x_n$ of a Lie algebra, we denote by $[x_1, \ldots, x_n]$ the left-normed bracket; that is, if $n = 2$, then $[x_1, x_2]$ is the Lie bracket, and for $n > 2$, $[x_1, \ldots, x_n] = [[x_1, \ldots, x_{n-1}], x_n]$.

Let (A, ∂) be a commutative differential R-algebra (with a unit). Then, the Lie algebra $\mathfrak{w}(A, \partial)$ satisfies the nontrivial identity (called the "standard Lie identity of degree 5" in [12] and T_4 in [13])

$$\sum_{\sigma \in \mathfrak{S}_4} \epsilon(\sigma) \left[z, x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}\right], \tag{10}$$

where $\epsilon(\sigma)$ denotes the *signature* representation of the permutation σ, for every $x_1, x_2, x_3, x_4, y \in A$. (This result was noticed in [12–14], e.g.) Therefore, a necessary condition for an embedding is the following.

Lemma 12. *A Wronskian special Lie algebra satisfies T_4.*

It is not known whether the above lemma is also a sufficient condition. Nevertheless this gives a negative answer to a question of Lawvere in his Ph.D thesis [15] where he asked whether or not any Lie algebra is Wronskian special.

Following [16], let us define the Lie algebra $\mathtt{Diff}_\partial(A)$ of *special derivations* with respect to the *signature derivation* ∂ as follows: let $A \cdot \partial = \{\alpha\partial : \alpha \in A\}$ be the sub-A-module of $\mathtt{Der}_R(A)$, the A-module of all R-derivations of A, generated by ∂ (it is the image of the A-module map $\rho_\partial : A \to \mathtt{Der}_R(A)$ given by $\rho(\alpha) = \alpha\partial$). Then, $\mathtt{Diff}_\partial(A)$ is the R-module structure of $A \cdot \partial$ together with the usual bracket of derivations (inherited by the usual bracket of $\mathfrak{gl}_R(A) = \mathtt{End}_{R\text{-}\mathtt{Mod}}(A)$ the linear endomorphisms of A). Let $\mathtt{ann}(\partial) = \{a \in A : a\partial = 0\}$ be the annihilator of ∂. It is an Abelian Lie subalgebra of $\mathfrak{w}(A, \partial)$ and $\mathtt{ann}(\partial) \hookrightarrow \mathfrak{w}(A, \partial) \overset{\rho_\partial}{\twoheadrightarrow} \mathtt{Diff}_\partial(A)$ is a short exact sequence of Lie R-algebras in such a way that $\mathfrak{w}(A, \partial)$ is an extension of $\mathtt{Diff}_\partial(A)$ by $\mathtt{ann}(\partial)$. In the case where $\ker \rho_\partial = \mathtt{ann}(\partial) = 0$—for instance when A has no zero divisors and $\partial \neq 0$—then $\mathtt{Diff}_\partial(A)$ and $\mathfrak{w}(A, \partial)$ are isomorphic Lie R-algebras. The following result is thus obvious.

Lemma 13. *Let A be a commutative integral R-domain, and let ∂ be a nonzero derivation of A. Let $\mathfrak{g}$ be a Lie R-algebra such that $\mathfrak{g} \hookrightarrow \mathtt{Diff}_\partial(A)$ (as Lie R-algebras) (we note that in particular $\mathfrak{g}$ satisfies T_4 since it holds in $\mathtt{Diff}_\partial(A)$). Then, $\mathfrak{g}$ is a Wronskian special Lie algebra.*

Example 14. The Lie algebra $\mathfrak{sl}_2(\mathbb{K})$ over a field of characteristic zero is simple and embeds into $\mathtt{Diff}_\partial(\mathbb{K}[x])$ where ∂ is the usual derivation of polynomials in the variable x. Indeed, $\mathfrak{sl}_2(\mathbb{K})$ is isomorphic to the three-dimensional $\mathbb{K}$-vector space generated by $e = -x^2\partial$, $f = \partial$, $h = 2x\partial$ with the Lie algebra structure given by the usual commutator (see, e.g., [17]). Therefore, $\mathfrak{sl}_2(\mathbb{K})$ is a Wronskian special Lie algebra.

We may also remark that Wronskian speciality is preserved by direct products (even infinite). More precisely, let $(\mathfrak{g}_i)_{i \in I}$ be a collection of Lie R-algebras such that for each $i \in I$, $\mathfrak{g}_i \hookrightarrow \mathfrak{w}(\mathscr{W}(\mathfrak{g}_i))$ (as Lie R-algebras), that is, each factor $\mathfrak{g}_i$ is a Wronskian special Lie algebra. Then, $\prod_{i \in I} \mathfrak{g}_i \hookrightarrow \mathfrak{w}(\mathscr{W}(\prod_{i \in I} \mathfrak{g}_i))$ as Lie R-algebras, where the operations (Lie brackets, derivative, and product) are considered componentwise (i.e., $\prod_{i \in I} \mathfrak{g}_i$ is a Wronskian special Lie algebra). Indeed, it is clear that $\prod_{i \in I} \mathfrak{g}_i \hookrightarrow \prod_{i \in I} \mathfrak{w}(\mathscr{W}(\mathfrak{g}_i)) = \mathfrak{w}(\prod_{i \in I} \mathscr{W}(\mathfrak{g}_i))$ as a sub-Lie algebra over R. It is also clear that any sub-Lie algebra of a Wronskian special Lie algebra is itself a Wronskian special Lie algebra. Thus, Wronskian speciality is closed under product and subalgebra.

Now, we recall an important result from Razmyslov that is used hereafter to describe some basic Wronskian special Lie algebras.

Theorem 15 (see [12, 16]). *Let $\mathbb{K}$ be a field of characteristic zero. Let $\mathfrak{g}$ be a simple Lie algebra that satisfies T_4. Then, there exists a commutative integral $\mathbb{K}$-domain A and a nonzero derivation $\partial \in \mathtt{Der}_{\mathbb{K}}(A)$ such that $\mathfrak{g}$ embeds into $\mathtt{Diff}_\partial(A)$ as a Lie subalgebra.*

As a consequence of the previous theorem and Lemma 13, in characteristic zero, any simple Lie algebra satisfying T_4 is a Wronskian special Lie algebra. From this result we deduce the following.

Theorem 16. *Let $\mathbb{K}$ be a field of characteristic zero. Let $(\mathfrak{g}_i)_{i \in I}$ be a family of Lie $\mathbb{K}$-algebras such that for every $i \in I$ either $\mathfrak{g}_i$ is simple and satisfies T_4 or $\mathfrak{g}_i$ is Abelian. Let $\mathfrak{g}$ be a sub-Lie algebra of $\prod_{i \in I} \mathfrak{g}_i$. Then, $\mathfrak{g}$ is a Wronskian special Lie algebra.*

Proof. According to Razmyslov's theorem if $\mathfrak{g}_i$ is simple and satisfies T_4 then $\mathfrak{g}_i$ is Wronskian special. We also know that any Abelian Lie algebra is Wronskian special. So their product $\prod_{i \in I} \mathfrak{g}_i$ also is. Finally any sub-Lie algebra of a Wronskian special Lie algebra also is Wronskian special. □

5. The Free Wronskian Special Lie R-Algebra

Up to now it is not known if Wronskian special Lie algebras form a variety of Lie algebras (nor even if it closed under homomorphic images). In [13] it is conjectured that (finitely generated) Wronskian special Lie algebras form the variety of all (finitely generated) Lie algebras satisfying T_4 (in characteristic zero). We do not prove nor disprove this conjecture. Nevertheless we observe that Wronskian special Lie R-algebras, for some ring R, form a derived category of the variety of commutative differential algebras obtained by considering their Lie bracket as the derived operator $W(x, y) = x\partial(y) - \partial(x)y$ (see [7] for definitions of derived category and derived operator). Therefore, we may speak about the *free Wronskian special Lie R-algebra generated by a set X* (see [7, Theorem 4.4]). It is obtained as the Lie subalgebra of $\mathfrak{w}(R\{X\})$ generated by X, where we recall that $R\{X\}$ is the free commutative differential algebra over X (if $u \in R\{X\}$, then u' denotes the derivation of u in $R\{X\}$ and its Wronskian commutator is denoted by $W(u, v) = uv' - u'v$). It is constructed by induction as follows: let

$$\begin{aligned} X_0 &= X, \\ X_{k+1} &= \left\langle X_k \cup \left\{ w \in R\{X\} : \exists u, v \in X_k^2, \right.\right. \\ &\qquad \left.\left. w = W(u, v) \right\} \right\rangle \quad \forall k \geq 0, \end{aligned} \tag{11}$$

where $\langle E\rangle$ denotes the submodule of $R\{X\}$ generated by a subset E. Then, the free Wronskian special Lie algebra, denoted by $\mathtt{WspLie}(X)$, over X is the nested union $\bigcup_{k\geq 0} X_k$, and its bracket is given by the Wronskian bracket W of $R\{X\}$. (It is clear that as defined previously $\mathtt{WspLie}(X)$ is a R-module.)

Remark 17. It is clear that if $u, v \in R\{X\}$ have no nonzero constant term, then it is also the case for $W(u, v)$. Moreover every element of $\mathtt{WspLie}(X)$ has no nonzero constant term as it can be checked inductively from the previous observation and since every element of $X_0 = X$ has no nonzero constant term. Therefore, $\mathtt{WspLie}(X) \neq R\{X\}$ as sets so that $\mathtt{WspLie}(X) \neq \mathfrak{w}(R\{X\})$ as Lie algebras. (Note that when $X = \emptyset$, then $\mathtt{WspLie}(\emptyset) = 0$ while $R\{\emptyset\} \cong R$.) When X is reduced to only one element x, then $\mathtt{WspLie}(\{x\})$ is the free module Rx seen as an Abelian Lie algebra and, therefore, is isomorphic to the free Lie algebra on one generator.

We observe that since commutative differential algebras form a nontrivial variety (a nontrivial variety is a variety with algebras of cardinality >1), the natural set-theoretic map from X to $R\{X\}$ is one-to-one, and we may assume that $X \subseteq \mathtt{WspLie}(X) \subseteq R\{X\}$ (in particular, X is free over R in $\mathtt{WspLie}(X)$). According to the property of the Wronskian envelope, there is a unique differential algebra map $\phi : \mathscr{W}(\mathtt{WspLie}(X)) \to R\{X\}$ such that the following diagram commutes (where the unnamed arrows are the canonical inclusions)

$$\begin{tikzcd} \mathtt{WspLie}(X) \arrow[r, hook] \arrow[d, hook] & \mathfrak{w}(R\{X\}) \\ \mathfrak{w}(\mathscr{W}(\mathtt{WspLie}(X))) \arrow[ur, "\mathfrak{w}(\phi)"'] & \end{tikzcd} \tag{12}$$

Moreover the composition $X \hookrightarrow \mathtt{WspLie}(X) \hookrightarrow \mathfrak{w}(\mathscr{W}(\mathtt{WspLie}(X)))$ of natural embeddings gives rise to a set-theoretic map $X \hookrightarrow \mathscr{W}(\mathtt{WspLie}(X))$. Therefore, there is a unique homomorphism of commutative differential algebras $\psi : R\{X\} \to \mathscr{W}(\mathtt{WspLie}(X))$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, hook] & \mathscr{W}(\mathtt{WspLie}(X)) \\ R\{X\} \arrow[ur, "\psi"'] & \end{tikzcd} \tag{13}$$

Both compositions $\psi \circ \phi$ and $\phi \circ \psi$ are the identity on X and hence, by uniqueness, the identity everywhere. Therefore, $\mathscr{W}(\mathtt{WspLie}(X))$ and $R\{X\}$ are canonically isomorphic (as commutative differential algebras):

$$W(\mathtt{WspLie}(X)) \cong R\{X\}. \tag{14}$$

Remark 18. The previous result shares some similarity with the well-known fact that the universal enveloping algebra of the free Lie algebra generated by X is canonically isomorphic to the free associative algebra generated by X (see [3]).

Remark 19. Let us assume that R is an integral domain. Then, $R\{X\}$ is also an integral domain, and $\mathfrak{w}(R\{X\}) \cong \mathtt{Diff}_\partial(R\{X\})$ (as Lie algebras over R), where ∂ is the usual derivation on $R\{X\}$ (i.e, $\partial(u) = u'$). Since $\mathtt{Diff}_\partial(R\{X\})$ is a sub-Lie algebra of $\mathfrak{gl}_R(R\{X\})$ (under the commutator), it follows that $\mathfrak{w}(R\{X\})$ is also a special Lie algebra.

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