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RESEARCH ARTICLE

Möbius inversion formula for monoids with zero

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Abstract The Möbius inversion formula, introduced during the 19th century in number theory, was generalized to a wide class of monoids called locally finite such as the free partially commutative, plactic and hypoplactic monoids for instance. In this contribution are developed and used some topological and algebraic notions for monoids with zero, similar to ordinary objects such as the (total) algebra of a monoid, the augmentation ideal or the star operation on proper series. The main concern is to extend the study of the Möbius function to some monoids with zero, *i.e.*, with an absorbing element, in particular the so-called Rees quotients of locally finite monoids. Some relations between the Möbius functions of a monoid and its Rees quotient are also provided.

Keywords Möbius function · Monoid with zero · Locally finite monoid · Rees quotient · Contracted algebra

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1 Introduction

The classic Möbius inversion formula from number theory, introduced during the 19th century, states that, for any complex or real-valued functions f, g defined on the positive integers $\mathbb{N} \setminus \{0\}$, the following assertions are equivalent:

- For all n, $g(n) = \sum_{d|n} f(d)$.
- For all n, $f(n) = \sum_{d|n} \mu(n/d) f(d)$.

In both formulae the sums are extended over all positive divisors d of n, and μ is the classical Möbius function. This result actually uses the fact that μ and ζ are inverse one from the other with respect to the usual Dirichlet convolution, where ζ is the characteristic function of positive integers (see for instance [1]).

This classic version of the Möbius inversion formula was generalized in different ways by different authors. P. Doubilet, G.-C. Rota, and R.P. Stanley proposed a systematic treatment of this problem for locally finite posets in [13, 29], while P. Cartier and D. Foata in [8] proved such a formula holds in a wide class of monoids called *locally finite* [16], and the Möbius function was even explicitly computed for some of them. This paper is a contribution to the study of the Möbius inversion formula, still in the context of locally finite monoids but for the particular case of monoids with zero. For instance, let M be the set $\{0, 1, a, b, c, ab, ac, ba, bc, ca, cb, abc, acb, bac, bca, cab, cba\}$. It becomes a monoid with zero when equipped with concatenation of words without common letters, also called *standard words*; the other products give 0. Let ζ_0 be the characteristic function of $M_0 = M \setminus \{0\}$. Then, ζ_0 is invertible—with respect to convolution—in the algebra $\mathbb{Z}_0[M]$ of all functions that annihilate the zero 0 of M, which is, in a first approximation, the $\mathbb{Z}$-algebra of polynomials in the noncommutative variables $\{a, b, c\}$ with only standard words as monomials. Indeed, $\zeta_0 = 1 + \zeta_0^+$, where 1 is the characteristic function of the singleton $\{1\}$ and since ζ_0^+ has no constant term, as a noncommutative polynomial (that is $\zeta_0^+(1) = 0$), ζ_0 is invertible, with inverse $\mu_0 = \sum_{n \geq 0} (-\zeta_0^+)^n$. Due to the particular multiplication in M, the "proper part" ζ_0^+ of ζ_0 is actually nilpotent, and the previous summation stops after four steps. Therefore μ_0 can be computed by hand, and we obtain $\mu_0 = 1 - a - b - c$.

Rather surprisingly, μ_0—interpreted as the Möbius function of the monoid with zero M—is the same as the Möbius function of the free noncommutative monoid $\{a, b, c\}^*$. Moreover such a phenomenon also appears for less tractable monoids with zero: for instance, let us consider a monoid similar to M but on an infinite alphabet X: it is the set of all words on X without multiple occurrences of any letter, and with product $\omega \times \omega'$ equal to the usual concatenation $\omega\omega'$ when each letter appears at most one time in the resulting word, and 0 otherwise. Contrary to M, this monoid is found infinite. Nevertheless we can prove its characteristic function to be invertible, and its inverse is still equal to the usual Möbius function of the free monoid X^*. In this case, it is not as easy to compute because the corresponding "proper part" is no more nilpotent, and the sum of a series needs to be evaluated in some relevant topology.

The explanation of this general phenomenon is given in the present paper whose main concern is the development of an algebraic and topological toolbox for a systematic and rigorous treatment of the Möbius inversion formula for locally finite monoids with zero.

2 Monoids with zero

A monoid with zero is an ordinary monoid with a two-sided absorbing element, called the *zero*. Such structures obviously occur in ring theory (the multiplicative monoid of an associative ring with unit is a monoid with zero), but they are also used to solve some (co)homological problems [25, 26], and mainly in the study of ideal extensions of semigroups [2, 9, 10].

These structures are defined as follows: let M be an ordinary monoid (with 1_M as its identity element) such that $|M| \geq 2$. Then, M is called a *monoid with zero* if, and only if, there is a two-sided absorbing element 0_M, *i.e.*, $x0_M = 0_M = 0_M x$ for every $x \in M$, with[1] $0_M \neq 1_M$. The distinguished element 0_M is called the *zero* of M (uniqueness is obvious). If in addition M is commutative, then M is called a *commutative monoid with zero*. In the sequel, for any monoid M with zero 0_M, M_0 stands for $M \setminus \{0_M\}$.

Example 1

1. The set of all natural numbers $\mathbb{N}$ with the ordinary multiplication is a commutative monoid with zero;
2. The multiplicative monoid of any (associative) ring R with a unit 1_R is a monoid with zero 0_R;
3. If M is any usual monoid (with or without zero), then for every $0 \notin M$ (take $0 = \{M\}$ for instance, in presence of the axiom of foundation), $M^0 = M \cup \{0\}$ is a monoid with zero 0: $x0 = 0 = 0x$ for every $x \in M^0$ extending the operation of M. It is commutative if, and only if, the same holds for M. The transformation of M into M^0 is called an *adjunction of a zero*, and M^0 is a monoid with a *(two-sided) adjoined zero*. Note that M^0 is obviously isomorphic with M^{z} for every $\mathsf{z} \notin M$, where z plays the same role as 0;
4. The set $\aleph_0 \cup \{\aleph_0\}$ of all cardinal numbers less or equal to $\aleph_0$ (that is, the closed initial segment $[0, \aleph_0]$), with the usual cardinal addition (recall that $\aleph_0 = [0, \aleph_0[= \mathbb{N}$ and $n + \aleph_0 = \aleph_0 = \aleph_0 + n$ for every $n \leq \aleph_0$) is a commutative monoid with $\aleph_0$ as zero. More generally given any transfinite cardinal number κ, the set $[0, \kappa]$ of all cardinal numbers smaller than κ, with addition, is also a commutative monoid with κ as zero;
5. Let C be a small category [22]. Then its set of arrows $\mathcal{A}(\mathsf{C})$, together with adjoined zero 0 and identity 1, is a monoid with zero when arrows composition is extended using $f \circ g = 0$ whenever $\mathsf{dom}(f) \neq \mathsf{codom}(g)$ for every $f \in \mathcal{A}(\mathsf{C})$, and $f \circ 1 = f = 1 \circ f$, $f \circ 0 = 0 = 0 \circ f$ for every $f \in \mathcal{A}(\mathsf{C}) \cup \{0, 1\}$. Now suppose that P is a poset, and $\mathsf{Int}(P)$ is the set of its intervals $[x, y] = \{z \in P : x \leq z \leq y\}$ for all $x \leq y$ in P (see [13, 29]). An interval $[x, y]$ may be seen as an arrow from x to y, and a composition may be defined: $[x, z] \circ [z, y] = [x, y]$. It follows that P turns to be a small category, and $\mathsf{Int}(P) \cup \{0, 1\}$, where $0, 1 \notin \mathsf{Int}(P)$ and $0 \neq 1$, becomes a

[1] It is easy to see that our monoids with zero together with the trivial monoid—with $0 = 1$—are exactly the monoid objects in the monoidal category of sets and partial functions, or, equivalently, of pointed sets and mappings that carry base point to base point (the zero of a monoid being its base point), where the monoidal structure is the usual set-theoretical Cartesian product.

monoid with zero. Another specialization is possible: let $n \in \mathbb{N} \setminus \{0\}$ be fixed, and consider the set I of all pairs (i, j) of integers such that $1 \leq i, j \leq n$. Any usual n-by-n *matrix unit* $E_{(i,j)}$ may be seen as an arrow from i to j, and such arrows are composed by $E_{(i,k)} \circ E_{(k,j)} = E_{(i,j)}$. Then I becomes a small category, and the set of all matrix units, with adjoined 0 and 1, may be interpreted as a monoid with zero which is also quite similar to A. Connes's groupoids [12].

A major class of monoids with zero, that deserves a short paragraph on its own, is given by the so-called Rees quotients (see [2, 10, 19]). Let M be a monoid and I be a two-sided ideal of M, that is $IM \subseteq I \supseteq MI$, which is proper ($I$ is *proper* if, and only if, $I \neq M$, or, in other terms, $1_M \notin I$). A congruence θ_I on M is defined as follows: $(x, y) \in \theta_I$ if, and only if, $x, y \in I$ or $x = y$. The equivalence class of $x \in M$ modulo θ_I is

$$\begin{cases} \{x\} & \text{if } x \notin I; \\ I & \text{if } x \in I. \end{cases}$$

Therefore I plays the role of a zero in the quotient monoid M/θ_I, in such a way that it is isomorphic with the monoid with zero $(M \setminus I) \cup \{0\}$, where $0 \notin M \setminus I$, and with operation

$$x \times y = \begin{cases} xy & \text{for } xy \notin I, \\ 0 & \text{for } xy \in I \end{cases} \tag{1}$$

for every $x, y \in M \setminus I$, and $x \times 0 = 0 = 0 \times x$ for every $x \in (M \setminus I) \cup \{0\}$. This monoid, unique up to isomorphism (the choice of the adjoined zero), is called the *Rees quotient of M by I*, and denoted M/I. In what follows, we identify the carrier sets of both isomorphic monoids M/θ_I and $(M \setminus I) \cup \{0\}$, and we use juxtaposition for products in M/I and in M.

Remark 2 The fact that I is proper guarantees that $1_M \in M \setminus I$, and therefore $1_M \neq 0$.

Example 3 Let $X = \{a, b, c\}$ and $I = \{\omega \in X^* : \exists x \in X,\ \text{such that } |\omega|_x \geq 2\}$, where $|\omega|_x$ denotes the number of occurrences of the letter x in the word ω. Then X^*/I is the monoid with zero M described in the Introduction (see Sect. 1).

3 Contracted monoid algebra

Convention In the present paper, a ring is assumed to be associative, commutative and with a unit 1_R; the zero of a ring is denoted by 0_R. An R-algebra A is assumed to be associative (but non necessarily commutative) and has a unit 1. Its zero is denoted by 0.

The main objective of this section is to recall the relevant version of the monoid algebra of a monoid with zero over some given ring: in brief, the zeros of the monoid and the ring are identified. Let R be a ring, and X be any set. The *support* of $f \in R^X$ is the set $\{x \in X : f(x) \neq 0_R\}$. Now let M be a monoid with zero 0_M. Let us consider the usual monoid algebra $R[M]$ of M over R, which is, as an R-module, the set $R^{(M)}$ of all maps from M to R with finite support, endowed with the usual Cauchy product [4]. By *contracted monoid algebra* of M over R (see [10, 27]), we mean the factor algebra $R_0[M] = R[M]/R0_M$, where $R0_M$ is the two-sided ideal $R[(0_M)] = \{\alpha 0_M : \alpha \in R\}$. Thus, $R_0[M]$ may be identified with the set of all finite sums $\sum_{x\in M_0} \alpha_x x$, subject to the multiplication table given by the rule

$$x \times y = \begin{cases} xy & \text{if } xy \neq 0_M, \\ 0 & \text{if } xy = 0_M \end{cases} \tag{2}$$

defined on basis M_0 (formula (2) gives the constants of structure, see [4], of the algebra $R_0[M]$). In what follows we use juxtaposition rather than "$\times$" for the products. From the definition, it follows directly that for any ordinary monoid M, $R_0[M^0] \cong R[M^0]/R0 \cong R[M]$. This fact is extended to the Rees quotients as follows.

Lemma 4 [10, 27] *Let M be a monoid and I be a proper two-sided ideal of M. Then $R_0[M/I] \cong R[M]/R[I]$. (Note that $R[I]$ is the semigroup algebra of the ideal I.)*

Example 5 Let X be any non empty set and $n \in \mathbb{N} \setminus \{0, 1\}$. Let I be the proper ideal of X^* of all words ω of length $|\omega| \geq n$. Then $R_0[X^*/I]$ consists in noncommutative polynomials truncated at length n.

The notion of contracted monoid algebra is sufficient to treat the problem of the Möbius formula for finite and locally finite (see Sect. 5) monoids with zero. Nevertheless infinite monoids with zero also occur, and formal series must be considered in those cases.

4 Total contracted algebra of a finite decomposition monoid with zero

Let R be a ring, and M be a usual monoid. The set of all functions R^M has a natural structure of R-module. By abuse of notation,[2] any function $f \in R^M$ may be denoted by $\sum_{x\in M} \langle f, x\rangle x$, where[3] $\langle f, x\rangle = f(x) = \pi_x(f)$ (π_x is the projection onto Rx). The carrier structure of the algebra $R[M]$ of the monoid M is then seen as a submodule of R^M. Now taking M to be a monoid with zero, we can also construct $R[M]$, however we would like to identify 0_M with 0 of R^M in the same way as $R_0[M]$. Let us consider the set $R0_M = \{f \in R^M : \forall x \neq 0_M,\ \langle f, x\rangle = 0_R\}$, *i.e.*, $R0_M$ is the cyclic submodule generated by 0_M. Then the quotient module $R^M/R0_M$ may be identified with the R-module R^{M_0} of all "infinite" sums[4] $\sum_{x\in M_0} \langle f, x\rangle x$, or more likely the

[2] When R^M is endowed with the topology of simple convergence, R being discrete, the family $(f(x)x)_{x\in M}$ is summable, and $f = \sum_{x\in M} f(x)x$.

[3] The notation "$\langle f, x\rangle$" is commonly referred to as a "Dirac bracket". It was successfully used by Schützenberger to develop his theory of automata [3].

[4] As in the previous note 2, it can be easily proved that such sums are actually the sums of summable series in the product topology on $R^M/R0_M$, with R discrete.

space of all functions from M_0 to R, *i.e.*, $R^{M_0} = \{f \in R^M : f(0_M) = 0_R\}$. This quotient module is the completion $\widehat{R_0[M]}$ of the topological module $R_0[M]$ equipped with the product topology (R is given the discrete topology), also called "topology of simple convergence" or "finite topology".[5] It should be noticed that the quotient topology of R^{M_0} induced by R^M is equivalent to its product topology.

Recall that an ordinary semigroup (resp. monoid) M is said to be a *finite decomposition semigroup* (resp. *finite decomposition monoid*), or to have the *finite decomposition property*, if, and only if, it satisfies the following condition

$$\forall x \in M, \quad |\{(y,z) \in M \times M : yz = x\}| < +\infty. \tag{3}$$

This condition is called the (D) *condition* in [4]. If (3) holds, then R^M can be equipped with the usual Cauchy or convolution product: therefore the R-algebra $R[[M]]$ of all formal power series over M with coefficients in R is obtained, which is also called the *total algebra of the semigroup* (resp. *monoid*) M over R. This notion is now adapted to the case of monoids with zero.

Definition 6 *A monoid M with zero 0_M is said to be a* finite decomposition monoid with zero *if, and only if, it satisfies the following condition*

$$\forall x \in M_0 = M \setminus \{0_M\}, \quad |\{(y,z) \in M \times M : yz = x\}| < +\infty. \tag{4}$$

Example 7 Let P be a locally finite poset ([13, 29]), *i.e.*, such that every interval $[x,y] \in \mathsf{Int}(P)$ is finite. Then the monoid $\mathsf{Int}(P) \cup \{0,1\}$ of Example 1.5 is a finite decomposition monoid with zero.

Some obvious results are given below without proofs.

Lemma 8

1. *Let M be a monoid with zero which has the finite decomposition property as an ordinary monoid. Then M is finite.*
2. *Suppose that M is a finite decomposition monoid. Then M^0 is a finite decomposition monoid with zero.*
3. *Suppose that M is a finite decomposition monoid and I is a two-sided proper ideal of M. Then the Rees quotient monoid M/I is a finite decomposition monoid with zero.*

Let us suppose that M is a finite decomposition monoid with zero. Let $f, g \in R^M/R0_M$. Then we can define the corresponding Cauchy product:

$$fg = \sum_{x \in M_0} \left(\sum_{yz = x} \langle f, y\rangle \langle g, z\rangle \right) x. \tag{5}$$

[5] Actually $R_0[M]$ is equipped with the initial topology with respect to the projections which coincides with the subspace topology induced on $R_0[M]$ by the product topology for R^{M_0}; we shall call this the *product topology* on $R_0[M]$.

The algebra $R^M/R0_M$ is then denoted $R_0[[M]]$ and called the *total contracted algebra of the monoid M over R*. The R-module $R_0[[M]]$ is the completion of $R_0[M]$ and because the Cauchy product of "formal series" in $R_0[[M]]$ is the continuous extension of its polynomial version in $R_0[M]$ (this product is separately continuous and continuous at zero [5]), the following lemma holds.

Lemma 9 *Let M be a finite decomposition monoid with zero. Then $R_0[[M]]$ is the completion of the contracted algebra $R_0[M]$, and, in particular, $R_0[[M]]$ is a topological algebra.*

Let M be an ordinary monoid and I be a two-sided proper ideal of M. Then the R-module $R^{M/I}/R0$ is isomorphic to the set of all formal infinite R-linear combinations $\sum_{x\notin I}\langle f,x\rangle x$, where $f\in R^M$. Now suppose that M is a finite decomposition monoid. According to Lemma 8, M/I is a finite decomposition monoid with zero. We can define both total algebras $R[[M]]$ and $R_0[[M/I]]$, with respectively R^M and $R^{M/I}/R0$ as carrier sets. The product on $R^{M/I}/R0$ is therefore given by

$$\left(\sum_{x\notin I}\langle f,x\rangle x\right)\left(\sum_{x\notin I}\langle g,x\rangle x\right)=\sum_{x\notin I}\left(\sum_{yz=x}\langle f,y\rangle\langle g,z\rangle\right)x. \tag{6}$$

Let define

$$\begin{aligned}\Phi:\ R[[M]]&\to R_0[[M/I]],\\ \sum_{x\in M}\langle f,x\rangle x&\mapsto\sum_{x\notin I}\langle f,x\rangle x.\end{aligned} \tag{7}$$

Then Φ is an R-algebra homomorphism, which is onto and obviously continuous (for the topologies of simple convergence). Moreover $\ker(\Phi)=R[[I]]$, then $R_0[[M/I]]\cong R[[M]]/R[[I]]$. According to Lemma 9, $R_0[[M/I]]$ is complete (as an R-algebra) for the product topology. In summary we obtain:

Proposition 10 *Let M be a finite decomposition monoid and I be a proper two-sided ideal of M. Then,*

$$\begin{aligned}R_0[[M/I]]&\cong\widehat{R_0[M/I]}\\&\cong R[[M]]/R[[I]].\end{aligned} \tag{8}$$

5 Locally finite monoids with zero

In order to study the Möbius inversion formula for monoids with zero, we need to characterize invertible series in the total contracted algebra. This can be done by exploiting a star operation on series without constant terms (*i.e.*, for which $\langle f,1_M\rangle=0$). This star operation is easily defined when a topology on the algebra of series is given by some filtration which generalizes the ordinary valuation. A particular class of monoids with zero satisfies this requirement. First we recall some classic results, and then we mimic them in the context of monoids with zero.

A locally finite monoid M [8, 16] is a monoid such that

$$\forall x \in M, \quad |\{(n, x_1, \ldots, x_n) : x = x_1 \ldots x_n,\ x_i \neq 1_M\}| < +\infty. \tag{9}$$

For instance, any free partially commutative monoid [8, 14] is locally finite. A locally finite monoid is obviously a finite decomposition monoid, but the converse is false since every non trivial finite group has the finite decomposition property, but is not locally finite because it has torsion. Furthermore, in a locally finite monoid, $xy = 1_M \Rightarrow x = y = 1_M$, or in other terms, $M \setminus \{1_M\}$ is a semigroup (and actually a locally finite semigroup in a natural sense), or, equivalently, the only invertible element of M is the identity (such monoids are sometimes called *conical* [11]).

Remark 11 In [7, 28] the authors—L.N. Shevrin and T.C. Brown—used another notion, well-known in universal algebra. They called *locally finite* any semigroup in which every finitely-generated sub-semigroup is finite. This concept is really different and not comparable from the one used in this paper which follows [16].

When M is locally finite, the R-algebra $R[[M]]$ may be equipped with a *star operation* defined for every *proper series* f (*i.e.* such that $\langle f, 1_M \rangle = 0_R$) by $f^* = \sum_{x \in M} (\sum_{n \geq 0} \sum_{x_1 \cdots x_n = x} \langle f, x_1 \rangle \cdots \langle f, x_n \rangle) x$ (*i.e.* by $f^* = \sum_{n \in \mathbb{N}} f^n$). It follows that the *augmentation ideal* $\mathfrak{M} = \{f \in R[[M]] : f$ is proper$\}$, kernel of the usual augmentation map $\epsilon(f) = \langle f, 1_M \rangle$ for every $f \in R[[M]]$, has the property that $1 + \mathfrak{M}$ is a group (under multiplication; the inverse of $1 - f \in 1 + \mathfrak{M}$, when f is proper, is precisely f^*), called the *Magnus group* (see [14] for instance). For this kind of monoids, we can define a natural notion of *order function*. Let $x \in M$, then $\omega_M(x) = \max\{n \in \mathbb{N} : \exists x_1, \ldots, x_n \in M \setminus \{1_M\},\ x = x_1 \ldots x_n\}$. For instance if M is a free partially commutative monoid $M(X, C)$, then $\omega_M(w)$ is the length $|w'|$ of any element $w' \in X^*$ in the class w.

Let us adapt this situation to the case of monoids with zero. In what follows, if M is any monoid (ordinary or with zero), then $M^+ = M \setminus \{1_M\}$. A *locally finite monoid with zero* (see [17] for a similar notion) is a monoid with zero M such that

$$\forall x \in M_0, \quad |\{(n, x_1, \cdots, x_n) : x = x_1 \ldots x_n,\ x_i \neq 1_M\}| < +\infty. \tag{10}$$

A locally finite monoid with zero obviously is also a finite decomposition monoid with zero. As in the case of usual monoids, the converse is false. Besides, if M is a locally finite monoid, and I is a two-sided proper ideal of M, then the Rees quotient M/I is a locally finite monoid with zero.

Example 12 Let $M = X^*/I$. Then $\omega_{M/I}(w) = |w|$ for every $w \in X^* \setminus I$.

Counter-example 13 The monoid with zero $\mathsf{Int}(P) \cup \{0, 1\}$ of a non void locally finite poset is not a locally finite monoid with zero, since for every $x \in P$, $1 \neq [x, x] = [x, x] \cdot [x, x]$ holds.

As in the classical case, we can define a natural notion of *order function* in a locally finite monoid with zero: let $x \in M_0$, then $\omega_M(x) = \max\{n \in \mathbb{N} : \exists x_1, \dots, x_n \in M^+,\ x = x_1 \cdots x_n\}$ (we use the notation " $\omega(x)$" when no confusion arises). Therefore $\omega(x) = 0$ if, and only if, $x = 1_M$. Moreover for every $x \in M_0$, if $x = yz$, then $\omega(x) \geq \omega(y) + \omega(z)$. If M is a locally finite monoid and I is a two-sided proper ideal of M, then we already know that M/I is a locally finite monoid with zero, and more precisely for every $x \in M \setminus I$, $\omega_{M/I}(x) = \omega_M(x)$.

Now let, $f \in R_0[[M]]$ (the total contracted algebra exists because M is a finite decomposition monoid with zero since it is a locally finite monoid with zero). We define an *order function* or *pseudo-valuation* (that extends the order function ω_M of M): $\omega(f) = \inf\{\omega_M(x) : x \in M_0,\ \langle f, x\rangle \neq 0_R\}$, where the infimum is taken in $\mathbb{N} \cup \{+\infty\}$. In particular, $\omega(f) = +\infty$ if, and only if, $f = 0$. The following holds:

1. $\omega(1) = 0$;
2. $\omega(f + g) \geq \min\{\omega(f), \omega(g)\}$;
3. $\omega(fg) \geq \omega(f) + \omega(g)$.

Let us introduce $\mathfrak{M} = \{f \in R_0[[M]] : \langle f, 1_M\rangle = 0_R\} = \{f \in R_0[[M]] : \omega(f) \geq 1\}$. This set obviously is a two-sided ideal of $R_0[[M]]$, called—as in the ordinary case—the *augmentation ideal*.[6] For each $n \in \mathbb{N}$, let $\mathfrak{M}_{\geq n} = \{f \in R_0[[M]] : \omega(f) \geq n\}$, in such a way that $\mathfrak{M}_{\geq 0} = R_0[[M]]$, and $\mathfrak{M}_{\geq 1} = \mathfrak{M}$. The following lemma holds trivially.

Lemma 14 *For every n, $\mathfrak{M}_{\geq n}$ is a two-sided ideal of $R_0[[M]]$, and the sequence $(\mathfrak{M}_{\geq n})_n$ is an exhaustive and separated decreasing filtration on $R_0[[M]]$, i.e., $\mathfrak{M}_{\geq n+1} \subseteq \mathfrak{M}_{\geq n}$, $\bigcup_{n\geq 0} \mathfrak{M}_{\geq n} = R_0[[M]]$, and $\bigcap_{n\geq 0} \mathfrak{M}_{\geq n} = (0)$.*

According to Lemma 14, $R_0[[M]]$ with the topology $\mathcal{F}$ defined by the filtration $(\mathfrak{M}_{\geq n})_n$ is an Hausdorff topological ring (note also that this topology is metrizable [6]), and even an Hausdorff topological R-algebra when R is discrete.

Remark 15 It can be proved that if for every $n \in \mathbb{N}$, $M(n) = \{x \in M_0 : \omega_M(x) = n\}$ is finite, then the topology of simple convergence and the topology $\mathcal{F}$ on $R_0[[M]]$ are equivalent. In all cases, the topology defined by the filtration is always finer than the product topology (in particular, each projection $\pi_x : R_0[[M]] \to R$ is continuous for the filtration), and it can be even strictly finer as it is shown in the following example.

Example 16 Let us consider a countable set $X = \{x_i\}_{i\in\mathbb{N}}$ (that is $x_i \neq x_j$ for every $i \neq j$). We consider M as the monoid X^* with some zero 0 adjoined. It is obviously a locally finite monoid with zero but the number of elements of a given order is not finite (for instance the number of elements of order 1 is $\aleph_0$). We denote by $|\omega|$ the usual length of a word in X^*. Now let us consider the sequence of series $f_n = \sum_{k=0}^{n} x_k \in R_0[M] \subset R_0[[M]]$ which converges to the sum $f = \sum_{k=0}^{\infty} x_k$ in $R_0[[M]]$ endowed with the product topology (f is the characteristic function of the alphabet X). But this series does not converge in $R_0[[M]]$ with the topology defined

[6] It is the kernel of the character $\epsilon : R_0[[M]] \to R$ given by $\epsilon(f) = \langle f, 1_M\rangle = \pi_{1_M}(f)$, for $f \in R_0[[M]]$.

by the filtration, because $\omega(f - f_n) = 1$ for all n. Nevertheless f belongs to $R_0[[M]]$ since it is the completion of $R_0[M]$ in the product topology.

Without technical difficulties the lemma below is obtained.

Lemma 17 *The algebra* $R_0[[M]]$ *with the topology* $\mathcal{F}$ *is complete.*

Remark 18 Suppose that M is a locally finite monoid (with or without zero) which is also finite, then there exists $N \in \mathbb{N}$ such that for every $n \geq N$, $\mathfrak{M}_{\geq n} = (0)$. In this case, the topology defined by the filtration coincides with the discrete topology on $R[[M]] = R[M]$ (or $R_0[[M]] = R_0[M]$). So no topology is needed in this case as explained in Introduction (Sect. 1).

6 Star operation and the Möbius inversion formula

In this section, M is assumed to be a locally finite monoid with zero.

Lemma 19 *For every* $f \in \mathfrak{M}$, $(1 - f)$ *is invertible and* $(1 - f)^{-1} = \sum_{n\geq 0} f^n$.

Proof First of all, $\sum_{n=0}^{+\infty} f^n$ is convergent in $R_0[[M]]$ (in the topology defined by the filtration), and is even summable, because $\omega(f^n) \to \infty$ when $n \to +\infty$ (see [6]). Now for every $N \in \mathbb{N}$, $(1 - f)\sum_{n=0}^{N} f^n = 1 - f^{N+1} \to 1$ when $N \to +\infty$. Since $\sum_{n\geq 0} f^n$ is summable, and $R_0[[M]]$ is a topological algebra, we obtain asymptotically $(1 - f)\sum_{n\geq 0} f^n = 1$. □

According to Lemma 19, for every element $f \in \mathfrak{M}$, we can define, as in the ordinary case, the *star operation* $f^* = \sum_{n\geq 0} f^n$.

Remark 20 Suppose that M is a locally finite monoid with zero which is also finite. Then for every $f \in \mathfrak{M}$, f is nilpotent (since $(f^n)_{n\in\mathbb{N}}$ is summable in the discrete topology). So in this particular case, there is no need of topology to compute f^*, as the example given in the Introduction.

Lemma 21 *The set* $1 + \mathfrak{M}$ *is a group under multiplication.*

Proof It is sufficient to prove that $\langle f^*, 1_M\rangle = 1_R$ for every $f \in \mathfrak{M}$. For every $n > 0$, $\langle f^n, 1_M\rangle = 0$. Since the projection π_{1_M} is continuous, we have

$$\langle f^*, 1_M\rangle = \left\langle 1 + \sum_{n\geq 1} f^n, 1_M\right\rangle = \langle 1, 1_M\rangle + \sum_{n\geq 0}\langle f^n, 1_M\rangle = 1_R. \tag{11}$$

□

If M is an ordinary locally finite monoid, the *characteristic series* of M is define as the series $\zeta = \sum_{x\in X} x \in R[[M]]$. If $X \subseteq M$, then $\underline{X} = \sum_{x\in X} x$ is the *characteristic series* of X. More generally, if M is a locally finite monoid with zero, then we also define the *characteristic series* of M by $\zeta_0 = \sum_{x\in M_0} x \in R_0[[M]]$, and if $X \subseteq M$, then its characteristic series is $\underline{X}_0 = \sum_{x \in X_0} x$ where $X_0 = X \setminus \{0_M\}$. We are now in a position to state the Möbius inversion formula in the setting of (locally finite) monoids with zero.

Proposition 22 (Möbius inversion formula) *The characteristic series ζ_0 is invertible.*

Proof It is sufficient to prove that $\zeta_0 \in 1 + \mathfrak{M}$, which is obviously the case since $\zeta_0 = 1 + \zeta_0^+$, where $\zeta_0^+ = \underline{M^+}_0 = \sum_{x \in M_0 \setminus \{1_M\}} x \in \mathfrak{M}$. □

We now apply several of the previous results on Rees quotients. So let M be a locally finite monoid and I be a two-sided proper ideal of M in such a way that M/I is a locally finite monoid with zero. Let us denote by $\mathfrak{M}_I$ (resp. $\mathfrak{M}$) the augmentation ideal of M/I (resp. M). Let $\Phi : R[[M]] \to R_0[[M/I]]$ be the R-algebra epimorphism defined in (7). We know that it is continuous when both $R[[M]]$ and $R_0[[M/I]]$ have their topology of simple convergence. It is also continuous for the topologies defined by the filtrations $(\mathfrak{M}_{\geq n})_n$ and $((\mathfrak{M}_I)_{\geq n})_n$. Indeed, let $n \in \mathbb{N}$, then for every $f \in \mathfrak{M}_{\geq n}$, $\Phi(f) \in (\mathfrak{M}_I)_{\geq n}$. It admits a section s from $R_0[[M/I]]$ into $R[[M]]$ (so $\Phi(s(f)) = f$ for every $f \in R_0[[M/I]]$) defined by

$$\langle s(f), x\rangle = \begin{cases} \langle f, x\rangle & \text{if } x \notin I, \\ 0_R & \text{otherwise.} \end{cases}$$

This map is easily seen as an R-module morphism but in general not a ring homomorphism.

Lemma 23 *Let $f \in 1 + \mathfrak{M}_I$, then $s(f) \in 1 + \mathfrak{M}$, and $f^{-1} = \Phi(s(f))^{-1}$.*

Proof Since $\langle f, 1_{M/I}\rangle = 1_R$, then $\langle s(f), 1_M\rangle = 1_R$ (because $1_{M/I} = 1_M$). Therefore $s(f) \in 1 + \mathfrak{M}$. Thus $s(f)^{-1} \in 1 + \mathfrak{M}$, and $\Phi(s(f)^{-1}) = \Phi(s(f))^{-1}$ (because Φ is a ring homomorphism). Finally, $f\Phi(s(f))^{-1} = \Phi(s(f))\Phi(s(f))^{-1} = 1$ and $\Phi(s(f))^{-1}$ is a right inverse of f. The same holds for the left-side. □

In the ordinary case, *i.e.*, when M is a (locally finite) monoid, the inverse $(-\zeta^+)^*$ of the characteristic series $\zeta = 1 + \zeta^+$ is called the *Möbius series*, and denoted by $\mu(M)$. By analogy, we define the *Möbius series* of a locally finite monoid with zero M as the series $\mu_0(M) = (-\zeta_0^+)^*$, inverse of $\zeta_0 = 1 + \zeta_0^+$ in $R_0[[M]]$. Therefore it satisfies $\mu_0(M)\zeta_0 = \zeta_0\mu_0(M) = 1$.

Lemma 24 *Let M be a locally finite monoid and I be a two-sided proper ideal of M. Then, $\mu_0(M/I) = \Phi(\mu(M))$. Moreover if $\langle \mu(M), x\rangle = 0_R$ for every $x \in I$, then $\mu_0(M/I) = \mu(M)$.*

Proof The Rees quotient M/I is a locally finite monoid with zero, and so its Möbius series exists. Moreover $\zeta_0 = \underline{M/I}_0 \in 1 + \mathfrak{M}_I$, and according to Lemma 23, $s(\zeta_0) \in 1 + \mathfrak{M}$, and $(\zeta_0)^{-1} = \Phi(s(\zeta_0))^{-1}$. We have

$$s(\zeta_0) = \sum_{x \notin I} x$$

$$\begin{aligned} &= \underline{M \setminus I} \\ &= \underline{M} - \underline{I} \\ &= \zeta - \underline{I}. \end{aligned} \tag{12}$$

The series $\zeta^+ - \underline{I} \in R[[M]]$ belongs to the augmentation ideal $\mathfrak{M}$ of $R[[M]]$ (as we already know), so the series $\zeta - \underline{I} = 1 + \zeta^+ - \underline{I}$ is invertible in $R[[M]]$ with inverse $(\underline{I} - \zeta^+)^*$. Therefore, according to Lemma 23,

$$\begin{aligned} \mu_0(M/I) &= \Phi(s(\zeta_0))^{-1} \\ &= \Phi(s(\underline{M/I}_0))^{-1} \\ &= \Phi(s(\underline{M/I}_0)^{-1}) \\ &= \Phi((\underline{I} - \underline{M^+})^*) \\ &= \Phi((\underline{I} - \zeta^+)^*) \\ &= (\Phi(\underline{I} - \zeta^+))^* \\ &\qquad \text{(because } \Phi \text{ is a continuous—for the filtrations—algebra} \\ &\qquad \text{homomorphism)} \\ &= (\underbrace{\Phi(\underline{I})}_{=0} - \Phi(\zeta^+))^* \\ &= \Phi((-\zeta^+)^*) \\ &= \Phi((1 + \zeta^+)^{-1}) \\ &= \Phi(\zeta^{-1}) \\ &= \Phi(\mu(M)). \end{aligned} \tag{13}$$

Now, if $\langle \mu(M), x\rangle = 0_R$ for every $x \in I$, then $\mu_0(M) = \Phi(\mu(M)) = \mu(M)$. □

Corollary 25 *Let X be any nonempty set. Let I be a proper two-sided ideal of X^*. Then,*

$$\mu_0(X^*/I) = \begin{cases} \mu(X^*) & \text{if } X \cap I = \emptyset, \\ \mu((X \setminus I)^*) & \text{if } X \cap I \neq \emptyset. \end{cases} \tag{14}$$

Proof We can apply Lemma 24 to obtain $\mu_0(X^*/I) = \Phi(\mu(X^*))$. According to [8], $\mu(X^*) = 1 - \sum_{x\in X} x$. If $X \cap I = \emptyset$, then $\mu_0(X^*/I) = \Phi(\mu(X^*)) = \mu(X^*)$, and if $X \cap I \neq \emptyset$, then let $Y = X \setminus I$. We have $\Phi(\mu(X^*)) = 1 - \sum_{y\in Y} y = \mu(Y^*)$. □

Example 26

1. Let X be any nonempty set. Let $I = \{\omega \in X^* : \exists x \in X,\ |\omega|_x \geq 2\}$. The set X^*/I consists of all standard words, *i.e.*, word without repetition of any letter. Then

according to Corollary 25, $\mu_0(X^*/I) = \mu(X^*) = 1 - \underline{X}$ as announced in Sect. 1 Introduction.

2. Let X be any set. A congruence $\equiv$ on X^* is said to be *multihomogeneous* [14, 15] if, and only if, $\omega \equiv \omega'$ implies $|\omega|_x = |\omega'|_x$ for every $x \in X$. A quotient monoid $X^*/\equiv$ of X^* by a multihomogeneous congruence is called a *multihomogeneous monoid*. For instance, any free partially commutative monoid, the plactic [23], hypoplactic [21, 24], Chinese [15] and sylvester [20] monoids are multihomogeneous. Such a monoid $M = X^*/\equiv$ is locally finite and therefore admits a Möbius function μ with $\mu(1_M) = 1$ and $\mu(x) = -1$ for every $x \in X$. An epimorphism Ev from M onto the free commutative monoid $X^{\oplus}$, the *commutative image*, is given by $\mathsf{Ev}(\omega) = \sum_{x\in X} |\omega|_x \delta_x$, where δ_x is the indicator function of x. Any proper ideal I of $X^{\oplus}$ gives rise to a proper two-sided ideal $\mathsf{Ev}^{-1}(I)$ of M. Let $I = \{f \in X^{\oplus} : \sum_{x\in X} f(x) \geq 2\}$. Then, as sets, $M/\mathsf{Ev}^{-1}(I) = \{0, 1_M\} \cup X$ and $\mu_0(M/\mathsf{Ev}^{-1}(I)) = 1 - \underline{X}$.

7 Some remarks about Hilbert series

Now, let X be a finite set, and I be a proper two-sided ideal of X^*. For any $S \subseteq X^*$ or $S \subseteq X^*/I$, we define $S(n) = \{w \in S : |w| = n\}$ for any $n \in \mathbb{N}$. (Note that the notation $M(n)$ is consistent with the given one in Remark 15 for $M = X^*/I$.) Let $\mathbb{K}$ be a field. Let define $A_n = \mathbb{K}X^*(n)$ (the $\mathbb{K}$-vector space spanned by $X^*(n)$), and $B_n = \mathbb{K}(X^*/I)(n)$ for every $n \in \mathbb{N}$, in such a way that $\mathbb{K}[X^*] = \bigoplus_{n\geq 0} A_n$ and $\mathbb{K}_0[X^*/I] = \bigoplus_{n\geq 0} B_n$. (Note that for every $w, w' \in X^*/I$, we have $|ww'| = |w| + |w'|$ if $ww' \neq 0$, in such a way that $B_m B_n \subseteq B_{m+n}$ since $0 \in B_i$ for every i.) Since X is finite, for every integer n, $X^*(n)$ and $(X^*/I)(n)$ are finite, and therefore A_n and B_n are finite-dimensional $\mathbb{K}$-vector spaces. Moreover $\dim(B_n) = \dim(A_n) - |I(n)|$ because $(X^*/I)(n) = X^*(n) \setminus I(n)$. So in particular respective Hilbert series[7] are related by

$$\mathcal{Hilb}_{\mathbb{K}_0[X^*/I]}(t) = \mathcal{Hilb}_{\mathbb{K}[X]}(t) - \sum_{n\geq 0} |I(n)| t^n = \frac{1}{1 - |X| t} - \sum_{n\geq 0} |I(n)| t^n. \tag{15}$$

Note that since I is a proper ideal, $I(0) = \emptyset$, and $\sum_{n\geq 1} |I(n)| t^n$ may be interpreted rather naturally as the Hilbert series of the ideal $\mathbb{K}[I] = \bigoplus_{n\geq 1} \mathbb{K}I(n)$ (it also follows that $\mathcal{Hilb}_{\mathbb{K}[I]}(t)$ is not invertible in $\mathbb{Z}[[t]]$). We have

$$\mathcal{Hilb}_{\mathbb{K}_0[X^*/I]}(t) = \mathcal{Hilb}_{\mathbb{K}[X]}(t) - \mathcal{Hilb}_{\mathbb{K}[I]}(t). \tag{16}$$

This equation may be recovered from (12), namely $s(\zeta_0) = \zeta - \underline{I}$, using an evaluation map. Suppose now that $\mathbb{K}$ is a field of characteristic zero, and t is a variable. Let $e : X^* \to \{t^i\}_{i\in\mathbb{N}}$ be the unique morphism of monoids such that $e(x) = t$ for every $x \in X$. We extend it to a $\mathbb{Z}$-linear map from $\mathbb{Z}[X^*]$ to $\mathbb{Z}[t]$ by

[7] Let $\mathcal{A} = \bigoplus_{n\geq 0} A_n$ be a graded algebra, where for every n, A_n is finite dimensional. The *Hilbert series of* $\mathcal{A}$ (in the variable t) is defined by $\mathcal{Hilb}_{\mathcal{A}}(t) = \sum_{n\geq 0} \dim(A_n) t^n$.

$e(\sum_{w\in X^*} n_w w) = \sum_{n\in\mathbb{N}}(\sum_{w\in X^*,\ |w|=n} n_w)t^n$ from . Moreover since X is finite, for every $n \in \mathbb{N}$, $X(n)$ is also finite (of cardinality $|X|^n$), and therefore for every series $f = \sum_{w\in X^*} n_w w \in \mathbb{Z}[[X^*]]$, by summability, we have $f = \sum_{n\in\mathbb{N}} f_n$, where $f_n = \sum_{w\in X^*(n)} n_w w \in \mathbb{Z}[X^*]$ for every $n \in \mathbb{N}$, and we extend e (by continuity) as a linear map from $\mathbb{Z}[[X^*]]$ to $\mathbb{Z}[[t]]$ by $e(f) = \sum_{n\in\mathbb{N}} e(f_n) = \sum_{n\in\mathbb{N}}(\sum_{w\in X^*(n)} n_w)t^n$. Now, applying e on both side of (12), we obtain (note that $s(\zeta_0)$, ζ and $\underline{I}$ belong to $\mathbb{Z}[[X^*]]$)

$$\begin{aligned}
&e(s(\zeta_0)) = e(\zeta) - e(\underline{I})\\
&\iff e\left(\sum_{w\notin I} w\right) = e\left(\sum_{w\in X^*} w\right) - e\left(\sum_{w\in I} w\right)\\
&\iff e\left(\sum_{n\in\mathbb{N}} \underline{X(n)\setminus I(n)}\right) = e\left(\sum_{n\in\mathbb{N}} \underline{X(n)}\right) - e\left(\sum_{n\in\mathbb{N}} \underline{I(n)}\right)\\
&\iff \sum_{n\in\mathbb{N}}(|X(n)| - |I(n)|)t^n = \sum_{n\in\mathbb{N}} |X(n)|t^n - \sum_{n\in\mathbb{N}} |I(n)|t^n\\
&\iff \mathcal{H}ilb_{\mathbb{K}_0[X^*/I]}(t) = \mathcal{H}ilb_{\mathbb{K}[X]}(t) - \mathcal{H}ilb_{\mathbb{K}[I]}(t). \qquad (17)
\end{aligned}$$

Last equality is nothing else than the obvious relation between the ordinary generating functions of the combinatorial class $X^* \setminus I$, X^* and I, where the notion of size is the length of words (see [18], Theorem I.5 “implicit specifications”).

Example 27

1. Suppose that $I = \{\omega \in X^* : \exists x \in X,\ \ |\omega|_x \geq 2\}$. It is clear that for every $n > |X|$, $I(n) = X(n)$. For every $n \leq |X|$, $|(X^*/I)(n)| = \prod_{i=0}^{n-1}(|X| - i) = |X|^{\underline{n}}$ (in particular, $|(X^*/I)(0)| = |\{\epsilon\}| = 1$, and $|(X^*/I)(1)| = |X|$). If follows that $\mathcal{H}ilb_{\mathbb{K}_0[X^*/I]}(t) = \sum_{n=0}^{|X|} |X|^{\underline{n}} t^n$, and therefore $\mathcal{H}ilb_{\mathbb{K}[I]}(t) = \sum_{n\geq 2}(|X|^n - |X|^{\underline{n}})t^n$.
2. Let $n_0 \in \mathbb{N}$ such that $n_0 \geq 1$. Let $I = \{w \in X^* : |w| > n_0\}$. Then $\mathcal{H}ilb_{\mathbb{K}_0[X^*/I]}(t) = \sum_{n=0}^{n_0} |X|^n t^n$, $\mathcal{H}ilb_{\mathbb{K}[I]}(t) = \sum_{n\geq n_0+1} |X|^n t^n$.

References

1. Apostol, T.M.: Introduction to Analytic Number Theory. Undergraduate Texts in Mathematics. Springer, Berlin (1976)
2. Arendt, B.D., Stuth, C.J.: On partial homomorphisms of semigroups. Pac. J. Math. **35**(1), 7–9 (1970)
3. Berstel, J., Reutenauer, C.: Rational Series and Their Languages. EATCS Monographs in Theoretical Computer Science, vol. 12. Springer, Berlin (1988)
4. Bourbaki, N.: Elements of Mathematics—Algebra. Springer, Berlin (1989), Chaps. 1–3
5. Bourbaki, N.: Elements of Mathematics—General Topology. Springer, Berlin (2006), Chaps. 5–10
6. Bourbarki, N.: Elements of Mathematics—Commutative Algebra. Springer, Berlin (2006), Chaps. 1–4
7. Brown, T.C.: On locally finite semigroups. Ukr. Math. J. **20**, 631–636 (1968)
8. Cartier, P., Foata, D.: Problèmes Combinatoires de Commutation et Réarrangements. Lecture Notes in Mathematics, vol. 85. Springer, Berlin (1969)

9. Clifford, A.H.: Partial homomorphic images of Brandt groupoids. Proc. Am. Math. Soc. **16**(3), 538–544 (1965)
10. Clifford, A.H., Preston, G.B.: The Algebraic Theory of Semigroups. Mathematical Surveys, vol. 7. American Mathematical Society, Providence (1961)
11. Cohn, P.M.: Further Algebra and Applications. Springer, Berlin (2003)
12. Connes, A.: Noncommutative Geometry. Academic Press, San Diego (1994)
13. Doubilet, P., Rota, G.-C., Stanley, R.P.: The idea of generating function. In: Rota, G.-C. (ed.) Finite Operator Calculus, pp. 83–134. Academic Press, San Diego (1975)
14. Duchamp, G.H.E., Krob, D.: Partially commutative formal power series. In: Proceedings of the LITP Spring School on Theoretical Computer Science on Semantics of Systems of Concurrent Processes, pp. 256–276 (1990)
15. Duchamp, G.H.E., Krob, D.: Plactic-growth-like monoids. In: Ito, M., Jürgensen, H. (eds.) Words, Languages and Combinatorics II, Kyoto, Japan 1992, pp. 124–142 (1994)
16. Eilenberg, S.: Automata, Languages and Machines, vol. A. Academic Press, San Diego (1974)
17. Ésik, Z.: Iteration semirings. In: Proceedings of the 12th International Conference on Developments in Language Theory, Kyoto, Japan. Lecture Notes in Computer Science, vol. 5257, pp. 1–20. Springer, Berlin (2008)
18. Flajolet, P., Sedgewick, R.: Analytic Combinatorics. Cambridge University Press, Cambridge (2009)
19. Grillet, P.A.: Semigroups: An Introduction to the Structure Theory. Monographs and Textbooks in Pure and Applied Mathematics. Dekker, New York (1995)
20. Hivert, F., Novelli, J.-C., Thibon, J.-Y.: Un analogue du monoïde plaxique pour les arbres binaires de recherche. C. R. Acad. Sci. Paris Sr. I Math. **332**, 577–580 (2002)
21. Krob, D., Thibon, J.-Y.: Noncommutative symmetric functions IV: Quantum linear groups and Hecke algebras at $q = 0$. J. Algorithms Comb. **6**(4), 339–376 (1997)
22. Mc Lane, S.: Categories for the Working Mathematician, 2nd edn. Graduate Texts in Mathematics, vol. 5. Springer, Berlin (1998)
23. Lascoux, A., Schützenberger, M.P.: Le monoïde plaxique. In: Luca, De, A. (ed.) Non-commutative Structures in Algebra and Geometric Combinatorics, Quaderni de la ricerca scientifica, vol. 109, pp. 129–156. C.N.R., Rome (1981)
24. Novelli, J.-C.: On the hypoplactic monoid. Discrete Math. **217**(1), 315–336 (2000)
25. Novikov, B.V.: 0-cohomology of semigroups. In: Hazewinkel, M. (ed.) Handbook of Algebra, vol. 5, pp. 189–210. Elsevier, Amsterdam (2008)
26. Novikov, B.V., Polyakova, L.Y.: On 0-homology of categorical at zero semigroups. Cent. Eur. J. Math. **7**(2), 165–175 (2009)
27. Okniński, J.: Semigroup Algebras. Pure and Applied Algebra: A Series of Monographs and Textbooks, vol. 138. Dekker, Berlin (1990)
28. Shevrin, L.N.: On locally finite semigroups. Sov. Math. Dokl. **6**, 769–772 (1965)
29. Stanley, R.P.: Enumerative Combinatorics, vol. 1. University Press, Cambridge (1996)