Statistics on Graphs, Exponential Formula and Combinatorial Physics

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Exponential Formula : Informal Version

Informally speaking, the exponential formula means that *"the exponential generating function* $\mathsf{EGF}(S;z)$ *of a class* S *of (combinatorial) structures is equal to the exponential* $e^{\mathsf{EGF}(S_c;z)}$ *of those of the connected substructures* S_c*", i.e.,*

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For a given class of structures S, it is often possible to define a subclass S_c of connected structures. They can be seen as the fundamental components used to build some "bigger" structures. Connected structures cannot be divided into simpler structures : they are themselves indecomposable.

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Examples

- The disjoint sum of two graphs with disjoint set of vertices is nothing else (in terms of pictures) than their juxtaposition. Every graph may be written as a disjoint sum of its connected components ;
- Let SFI be the set of square-free integers, *i.e.*, the integers which are the product of **distinct** prime numbers. The connected structures are the prime numbers, and every element of SFI is written as a “disjoint” product of prime numbers ;
- Every complex finite-dimensional linear representation of a finite group can be written as the direct sum of irreducible representations.

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Objective

Following the examples of graphs and square-free integers, I introduce an algebraic structure which allows the definition of connected elements and the contruction of bigger elements using simple ones.

Regarding the previous examples, I deduce the main concept : a partially defined (commutative and associative) operation of disjoint sum.

Convention : Since I will deal with a partially defined function, I adopt the following convention. If f is a partial function, then "$f(x) = f(y)$" means that $f(x)$ is defined if, and only if, $f(y)$ also is, and in this case, they have the same value.

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Partial commutative monoids

A partial commutative monoid is a (non empty) set M together with a partially defined binary operation $\oplus : D \subseteq M \times M \to M$ (D is the domain of $\oplus$), such that

1. $\oplus$ is associative : for each $x, y, z \in M$, $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;
2. $\oplus$ is commutative : for each $x, y \in M$, $x \oplus y = y \oplus x$;
3. There is a (unique) element $0 \in M$, such that for every $x \in M$, $x \oplus 0 = x = 0 \oplus x$. The element 0 is called the (total) identity of M.

If $D = M \times M$, that is, $\oplus$ is totally defined, then M is a (total) usual monoid.

Examples : The set of all graphs with vertices in some given set, with the disjoint union as operation, is a partial commutative monoid. This is also the case for square-free integers.

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Notations

A sum $x_1 \oplus x_2 \oplus \cdots \oplus x_n$ is written as $\bigoplus_{i=1}^{n} x_i$, and, $n.x := \underbrace{x \oplus \cdots \oplus x}_{n \text{ factors}}$ for an integer n.

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Indecomposables and decompositions

Let $(M, \oplus)$ be a partial commutative monoid. An indecomposable element of M is an element that cannot be written as the sum of two elements that are both not the identity of the monoid. More rigorously, $p \in M$ is indecomposable, if $p \neq 0$, and, $p = x \oplus y$ implies $x = 0$ or $y = 0$. Let $I(M)$ be the set of all indecomposable elements of M.

A decomposition of $x \in M$ is a mapping f from $I(M)$ to $\mathbb{N}$, with only finitely many non-zero values, such that $x = \bigoplus_{p \in I(M)} f(p).p$. If x has a unique decomposition, then I shall denote it by $\partial_x \in \mathbb{N}^{(I(M))}$. If every element has a unique decomposition, then we say that M has the unique decomposition property.

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A decomposition of $x \in M$ is a mapping f from $I(M)$ to $\mathbb{N}$, with only finitely many non-zero values, such that $x = \bigoplus_{p \in I(M)} f(p).p$. If x has a unique decomposition, then I shall denote it by $\partial_x \in \mathbb{N}^{(I(M))}$. If every element has a unique decomposition, then we say that M has the unique decomposition property.

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Now, a question arises : what are the properties of partial commutative monoids that characterize monoids with the unique decomposition property ?

1. Cancellation ;
2. Well-founded divisibility relation ;
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Characterization of the unique decomposition property

A partial commutative monoid M has the unique decomposition property iff

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3. If $p \in I(M)$, and $p|x \oplus y$, then $p|x$ or $p|y$ ("p is prime with respect to $|$").

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Square-free partial commutative monoids

A partial commutative monoid M with the unique decomposition property is called square-free if for every $x \in M$, and every $p \in I(M)$, then $\partial_x(p) \in \{0, 1\}$. The intuitive meaning is that no indecomposable element can appear more than one time in a decomposition. The graphs and the square-free integers are examples of square-free partial commutative monoids.

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Set-theoretical support (1/2)

Let $(M, \oplus)$ a square-free partial commutative monoid with D for the domain of $\oplus$. Now M is considered as a class of structures, *i.e.*, there exists a set X and a set-theoretical mapping $\sigma : M \to \mathcal{P}_{fin}(X)$, called support mapping, such that

$$\begin{aligned} \sigma(x) &= \emptyset && \text{iff} \quad x = 0 \, , \\ D &= \{(x, y) \in M^2 : \sigma(x) \cap \sigma(y) = \emptyset\} \, , \\ \sigma(x \oplus y) &= \sigma(x) \cup \sigma(y) \, . \end{aligned} \tag{2}$$

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Set-theoretical support (2/2)

For instance, let us consider the square-free partial commutative monoid $\mathcal{G}(\mathbb{N})$ of graphs with integer numbers as vertices. Then, the mapping $V : \mathcal{G}(\mathbb{N}) \to \mathcal{P}_{fin}(\mathbb{N})$ which maps a graph G to its set of vertices $V(G)$ is a support mapping.

A 3-tuple (M, X, σ) defined as in the previous slide is called a square-free partial commutative monoid with support in (the finite subsets of) X.

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Locally finite square-free monoids

Let (M, X, σ) be a square-free partial commutative monoid with support in X. For every $N \subseteq M$ and $Y \in \mathcal{P}_{fin}(X)$, we define

$$N_Y := \{x \in N : \sigma(x) = Y\} . \qquad (3)$$

N_Y is the set of all elements of M with support equals to Y.
We say that (M, X, σ) is locally finite if for every finite subset Y of X, M_Y is also finite, *i.e.*, there is only finitely many elements supported by Y.

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Statistics

From a combinatorial point of view, the elements of M should be "counted" or "measured" by some statistics. A statistic μ on a locally finite square-free partial commutative monoid M is a mapping from M to a (unitary) ring R of characteristic zero such that

1. μ is equivariant on sets of indecomposable elements, *i.e.*, for every finite subsets Y_1, Y_2 of X with the same cardinality n, then

$$\mu(I(M)_{Y_1}) = \mu(I(M)_{Y_2}) := \mu(I(M)[n]) . \quad (4)$$

(where $\mu(N) := \sum_{x \in N} \mu(x)$ for every finite subset N of M.)

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Proposition : Equivariance property on M_Y

Let M be a locally finite square-free partial commutative monoid and μ be a multiplicative and equivariant statistic. Let Y, Y' be two finite subsets of X of same cardinality n. Then,

$$\mu(M_Y) = \mu(M_{Y'}) := \mu(M[n]) \ . \tag{5}$$

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Exponential generating function of M and $I(M)$

Let M be a locally finite square-free partial commutative monoid and μ be a multiplicative and equivariant statistic. Let $N \in \{M, I(M)\}$. We define the exponential generating function of N by

$$\mathsf{EGF}(N; z) := \sum_{n=0}^{\infty} \mu(N[n]) \frac{z^n}{n!} . \qquad (6)$$

(Recall that $\mu(N[n])$ is the common value of $\mu(N_Y)$ for every finite subset Y of X of cardinality n.)

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Exponential formula for M

We have

$$\mathsf{EGF}(M;\mathrm{z}) = \mu(0) - 1 + e^{\mathsf{EGF}(I(M);\mathrm{z})}\,. \tag{7}$$

In particular, if $\mu(0) = 1$,

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Example

Let $\mathfrak{E}$ be the set of all equivalence relations on finite subsets of $\mathbb{N}$:
$E \in \mathfrak{E}$ means that there is $X \subseteq \mathbb{N}$, X finite, such that E is an equivalence relation on X. Every element of $\mathfrak{E}$ may be identified with its graph. Let $S_2(n,k)$ be the Stirling number of second kind, *i.e.*, the number of equivalence relations on a set of cardinality n with exactly k connected components. We choose as statistic

$$\mu(\mathfrak{E}[n]) := \sum_{k\geq 0} S_2(n,k)\mathrm{x}^k \ . \qquad (9)$$

Then, we can prove that

$$\mathsf{EGF}(I(\mathfrak{E}); \mathrm{z}) = \mathrm{x}(e^{\mathrm{z}} - 1) \qquad (10)$$

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Let $\mathfrak{E}$ be the set of all equivalence relations on finite subsets of $\mathbb{N}$: $E \in \mathfrak{E}$ means that there is $X \subseteq \mathbb{N}$, X finite, such that E is an equivalence relation on X. Every element of $\mathfrak{E}$ may be identified with its graph. Let $S_2(n,k)$ be the Stirling number of second kind, *i.e.*, the number of equivalence relations on a set of cardinality n with exactly k connected components. We choose as statistic

$$\mu(\mathfrak{E}[n]) := \sum_{k \geq 0} S_2(n,k)\mathrm{x}^k \,. \tag{9}$$

Then, we can prove that

$$\mathsf{EGF}(I(\mathfrak{E}); \mathrm{z}) = \mathrm{x}(e^{\mathrm{z}} - 1) \tag{10}$$

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