Generalized ladder operators and a « normal form » for endomorphisms

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Weyl algebra: definition

Let $\mathbb{K}$ be any field.

The (first) Weyl algebra $A(\mathbb{K})$ is defined as the quotient algebra of the algebra of polynomials $\mathbb{K}\langle x, y\rangle$ in non-commuting variables by the two-sided ideal generated by the relation $[x, y] = 1$.

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Let $a = \pi(x)$ and $a^\dagger = \pi(y)$ where $\pi\colon \mathbb{K}\langle x, y\rangle \twoheadrightarrow A(\mathbb{K})$ is the canonical epimorphism.

Support of a polynomial

Definition: Support of a polynomial

The support $\mathrm{Supp}(P)$ of a polynomial $P \in \mathbb{K}\langle X \rangle$ is the (finite) set of words $w \in X^*$ such that $\langle P \mid w \rangle \neq 0$.

Weyl algebra: normal ordering basis

As a $\mathbb{K}$-vector space, $A(\mathbb{K})$ is free with basis $\{ (a^\dagger)^i a^j \}_{i,j\in\mathbb{N}}$ (this is a general fact from the theory of Ore extensions).

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This means that for every $\Omega \in A(\mathbb{K})$ there is a unique polynomial, call it

$$\mathcal{Pol}(\Omega) \in \mathbb{K}\langle x, y\rangle$$

with support $\mathrm{Supp}(\mathcal{Pol}(\Omega)) \subseteq \{\, y^{i}x^{j} : i, j \in \mathbb{N}\,\}$ such that $\pi(\mathcal{Pol}(\Omega)) = \Omega$ (in other terms, $\mathcal{Pol} : A(\mathbb{K}) \hookrightarrow \mathbb{K}\langle x, y\rangle$ is a section of π).

Weyl algebra: normal ordering - formal definition

We call normal ordering of a polynomial $P \in \mathbb{K}\langle x, y\rangle$, the polynomial

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Remark

Note that P and $\mathcal{N}(P)$ define the same element of A since $\pi(\mathcal{N}(P)) = \pi(\mathcal{Pol}(\pi(P))) = \pi(P)$.

Weyl algebra: normal ordering - an example

Let $P = y^2xy + x^3yx \in \mathbb{Q}\langle x, y\rangle$, then $\mathcal{N}(P) = y^2 + y^3x + 3x^3 + yx^4$.

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in such a way that

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Weyl algebra as an algebra of differential operators

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- Since $\rho([x, y]) = [\rho(x), \rho(y)] = Id_{\mathbb{K}[z]}$, it follows that there is a unique algebra map $\tilde{\rho}\colon A(\mathbb{K}) \to \mathrm{End}_{\mathbb{K}\text{-}\mathcal{V}ect}(\mathbb{K}[z])$ such that

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This representation is faithful, *i.e.*, $\ker \tilde{\rho} = (0)$ in such a way that $A(\mathbb{K})$ may be identified with the sub-algebra of $\mathrm{End}_{\mathbb{K}\text{-}\mathcal{V}ect}(\mathbb{K}[z])$ generated by the multiplication by z and the formal derivation $\frac{d}{dz}$.

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Then for every $p \in \mathbb{K}[z]$,

$$\tilde{\rho}(\Omega)(p) = z^2 p + z^3 p' + 3p''' + zp'''' \ .$$

Ladder operators

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Thus, a is a lowering operator, while $a^\dagger$ is a raising operator.

Both of them are ladder operators.

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Algebraic part of Jacobson's theorem

Let R be a (unitary) ring (commutative or not). If M is a left R-module, then we denote by $\nu\colon R \to \mathrm{End}_{\mathcal{Ab}}(M)$ the associated (module) structure map. (This is a ring map since it is a linear representation of R.)

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The ring R is said to be (left-)primitive if it has a faithful simple left-module.

Topological part: Compact-open topology

Let X and Y be two topological spaces, and let $\mathcal{C}(X, Y)$ be the set of all continuous mappings from X to Y.

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Let K be a compact subset of X and U be an open set in Y, then we define

$$V(K, U) = \{ f \in \mathcal{C}(X, Y) \colon f(K) \subseteq U \} .$$

Then the collection of all such sets $V(K, U)$ is a subbasis for the compact-open topology on $\mathcal{C}(X, Y)$.

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Then the collection of all such sets $V(K, U)$ is a subbasis for the compact-open topology on $\mathcal{C}(X, Y)$.

This means that for every non-void open set V in the compact-open topology, and every $f \in V$, there exists a finite number $K_1, \cdots, K_n$ of compact sets in X and a finite number $U_1, \cdots, U_n$ of open sets in Y such that

$$f \in \bigcap_{i=1}^{n} V(K_i, U_i) \subseteq V .$$

Compact-open topology: a remark

Let $\mathbb{D}$ be a skew-field (also called a division ring), and let V be a left vector space over $\mathbb{D}$.

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Then, the compact-open topology on $\mathrm{End}_{\mathbb{D}\text{-}\mathcal{V}ect}(V) \subseteq \mathcal{C}(V, V) = V^V$ is the same as the topology of simple convergence, *i.e.*, for every topological space X, a map $\phi\colon X \to \mathrm{End}_{\mathbb{D}\text{-}\mathcal{V}ect}(V)$ is continuous if, and only if, for every $v \in V$, the map

$$\phi_v \colon x \in X \to \phi(x)(v) \in V$$

is continuous.

Jacobson's density theorem

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The ring R is primitive if, and only if, it is a dense subring (in the compact-open topology) of a ring $\mathrm{End}_{\mathbb{D}\text{-}\mathcal{Vect}}(V)$ of linear operators of some (left) vector space V over a skew-field $\mathbb{D}$ (where V is assumed to be discrete).

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A much stronger result actually holds [Kurbanov and Maksimov, '86]:

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A much stronger result actually holds [Kurbanov and Maksimov, '86]:

For every linear operator ϕ on $\mathbb{K}[z]$, there is a summable family $(\Omega_n)_{n\in\mathbb{N}}$ of elements of $A(\mathbb{K})$ such that

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(Sum of a summable family.)

Moreover, the family is uniquely determined by ϕ (*i.e.*, $(\Omega_n)_n$ is a function of ϕ) and may be even explicitly computed.

An example: the integration operator

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According to Jacobson's density theorem, I may be seen as a differential operator of infinite degree (!):

$$I = \sum_{n \geq 0} (-1)^n \frac{z^{n+1}}{(n+1)!} \frac{d^n}{dz} \ .$$

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while a is the lowering operator associated to $(\frac{z^n}{n!})_{n \geq 0}$.

Decomposition of endomorphisms

Theorem [2010]

Let $E = (e_n)_n$ and $F = (f_n)_n$ be two bases of V over the field $\mathbb{K}$ such that $\operatorname{span}_{\mathbb{K}}\{ f_0 \} = \operatorname{span}_{\mathbb{K}}\{ e_0 \}$ (the two bases agree on degree zero).

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Then, there is a family $(P_n)_{n \geq 0}$ of polynomials in one variable such that

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and the map $\phi \in \mathrm{End}_{\mathbb{K}\text{-}\mathcal{V}ect}(V) \mapsto (P_n)_n \in \mathbb{K}[z]^{\mathbb{N}}$ is a linear isomorphism.

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- $\lambda P_0(E) = \phi(f_0)$.
- $\lambda P_{n+1}(E) = \phi(f_{n+1}) - \sum_{k=0}^{n} P_k(R_E) f_{n+1-k}$.

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Properties

- $\mathbb{K}\langle x, y\rangle\rangle$ is a sub $\mathbb{K}$-vector space of $\mathbb{K}\langle\langle x, y\rangle\rangle$.

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$$\mathbb{K}\langle x, y\rangle\rangle = \{ \sum_{n\geq 0} P_n(x)y^n : \forall n,\ P_n(x) \in \mathbb{K}[x] \}$$

where the (non-commutative) concatenation is denoted by a simple juxtaposition.Call it the space of non-commutative generating functions of polynomials.

Properties

- $\mathbb{K}\langle x, y\rangle\rangle$ is a sub $\mathbb{K}$-vector space of $\mathbb{K}\langle\langle x, y\rangle\rangle$.
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- Actually it is the completion (for the product topology with $\mathbb{K}[x]$ discrete) of the $\mathbb{K}[x]$-module of all $\sum_{n\geq 0} P_n(x)y^n \in \mathbb{K}\langle\langle x, y\rangle\rangle$ where only finitely many $P_n(x) \neq 0$.

A normal form for operators

Remarks

- We note that $xy = y \cdot x$ but yx does not belong to $\mathbb{K}\langle x, y\rangle\rangle$.

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- Actually, $\mathbb{K}\langle x, y\rangle\rangle$ is the completion of the free $\mathbb{K}[x]$-module on basis $\{\, y^n \colon n \geq 0 \,\}$, namely

$$\mathbb{K}[x] \otimes_{\mathbb{K}} \mathrm{span}_{\mathbb{K}}\{\, y^n \colon n \geq 0 \,\}$$

(with the obvious $\mathbb{K}[x]$-action), with respect to the coarsest topology that makes continuous the maps $x^i \otimes y^j \mapsto x^i$ for $\mathbb{K}[x]$ discrete.

A normal form for operators

According to the previous theorem, there exists a $\mathbb{K}$-linear isomorphism

$$\pi_{E,F} \colon \mathbb{K}\langle x, y\rangle\rangle \to \mathrm{End}_{\mathbb{K}\text{-}\mathcal{V}ect}(V)$$

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Let $\phi \in \mathrm{End}_{\mathbb{K}\text{-}\mathcal{V}ect}(V)$. The unique element $S \in \mathbb{K}\langle x, y\rangle\rangle$ such that $\pi_{E,F}(S) = \phi$ may be called the normal form of ϕ with respect to the bases E, F of V.

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in the following way: define a valuation $\nu\colon V \to \mathbb{N} \sqcup \{\,\infty\,\}$ such that $\nu(v) = \inf\{\, n \geq 0 \colon \langle v \mid e_n\rangle \neq 0\,\}$ for $v \neq 0$ and $\nu(0) = \infty$.

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With respect to the topology induced by this valuation, we can describe the completion $\widehat{V}$ of V as the infinite direct product $\prod_{n\geq 0} \mathrm{span}_{\mathbb{K}}(e_n)$.

Its elements are infinite linear combinations:

$$\sum_{n\geq 0} \alpha_n e_n$$

where all coefficients α_n are allowed to be different from zero.

Duality

Actually V and $\widehat{V}$ may be paired by

$$\langle S \mid P \rangle = \sum_{n \geq 0} \langle S \mid e_n \rangle \langle P \mid e_n \rangle$$

where $S \in \widehat{V}$ and $P \in V$ (similarly to $\mathbb{K}\langle\langle X \rangle\rangle$ and $\mathbb{K}\langle X \rangle$).

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Using this (non-degenerate) pairing,

$$V^* \cong \widehat{V}$$

and

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Transpose

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Let $\phi \in \mathrm{End}_{\mathbb{K}\text{-}\mathcal{Vect}}(V)$ and $\psi \in \mathrm{End}_{\mathbb{K}\text{-}\mathcal{TopVect}}(\widehat{V})$.

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Then we define ${}^{\dagger}\phi \in \mathrm{End}_{\mathbb{K}\text{-}\mathcal{VectTop}}(\widehat{V})$ by

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and $\psi^{\dagger} \in \mathrm{End}_{\mathbb{K}\text{-}\mathcal{Vect}}(V)$ by

$$\langle S \mid \psi^{\dagger}(P) \rangle = \langle \psi(S) \mid P \rangle \ .$$

Decomposition of continuous endomorphisms

Using this duality and transpose, we can prove that any continuous operator ψ on $\widehat{V}$ admits a decomposition as the sum of a summable family

$$\psi = \sum_{n \geq 0} R^n P_n(L) \ .$$

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$$\psi = \sum_{n \geq 0} R^n P_n(L) \ .$$

If follows in particular that we have a linear isomorphism

$$\mathrm{End}_{\mathbb{K}\text{-}\mathcal{Vect}}(V) \cong \mathrm{End}_{\mathbb{K}\text{-}\mathcal{TopVect}}(\widehat{V}) \ .$$

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Links with other well-known combinatorial structures

Any sequence of polynomials

$$(P_n(x))_{n\in\mathbb{N}} \in \mathbb{K}[x]^{\mathbb{N}}$$

is bi-univocally transformed into a doubly-infinite matrix with coefficients in $\mathbb{K}$

$$(\langle P_i(x) \mid x^j \rangle)_{i,j\geq 0}$$

where $\langle P \mid x^i \rangle$ is the coefficient of the monomial x^i in the polynomial P

(the so-called Dirac-Schützenberger bracket) in such a way that

$$P = \sum_{i\geq 0} \langle P \mid x^i \rangle x^i$$

(sum with only finitely many non-zero terms).

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So it follows that the set $\mathbb{K}[x]^{\mathbb{N}}$ of sequences of polynomials and the set $\mathbb{K}^{\mathbb{N}\times(\mathbb{N})}$ are equipotent by $(P_n)_n \mapsto (\langle P_i(x) \mid x^j \rangle)_{i,j}$.

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Actually there are isomorphic as $\mathbb{K}$-vector spaces.

Links with other well-known combinatorial structures

We have

$$\mathrm{End}_{\mathbb{K}\text{-}\mathcal{Vect}}(V) \cong_{\mathbb{K}\text{-}\mathit{Vect}} \mathbb{K}\langle x, y\rangle\rangle \cong_{\mathbb{K}\text{-}\mathit{Vect}} \mathbb{K}[x]^{\mathbb{N}} \cong_{\mathbb{K}\text{-}\mathit{Vect}} \mathbb{K}^{\mathbb{N}\times(\mathbb{N})} \ .$$

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$$\left(\sum_{i\geq 0} P_i(R_E)L_F^i\right) \# \left(\sum_{i\geq 0} Q_i(R_E)L_F^i\right) = \sum_{i\geq 0}\left(\sum_{j\geq 0}\langle P_i(x) \mid x^j\rangle Q_j(R_E)\right) L_F^i .$$

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This « new » product $\#$ on $\mathrm{End}_{\mathbb{K}\text{-}\mathcal{V}ect}(V)$ is a generalization of the umbral composition of polynomial sequences (*i.e.*, sequences of polynomials $(P_n(x))_n$ such that for all n, $\deg P_n = n$, or, equivalently, the associated matrix is lower triangular):

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$$(p_n(x))_n \# (q_n(x))_n = \left(\sum_{k \geq 0} \langle p_n(x) \mid x^k \rangle q_k(x) \right)_n .$$

Links with other well-known combinatorial structures

A polynomial sequence $(p_n(x))_n$ (thus $\deg p_n = n$) is said to be a Sheffer sequence if there are two formal power series g and ϕ such that $g(0) \neq 0$ and $\phi(0) = 0$, $\phi'(0) \neq 0$ such that

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Sheffer sequences form a group under umbral composition which is isomorphic to the Riordan group (following Shapiro's terminology) $\mathbb{K}[x]^* \rtimes x\mathbb{K}[x]$, also called the group of substitutions with prefunction.

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Need to understand the relations between these combinatorial objects in the setting of decomposition of operators.

Infinite commutation formula

As any operator, the commutator $[L_F, R_E]$ admits a decomposition in the form $\sum_{n\geq 0} P_n(R_E)L_F^n$.

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We obtain an infinite commutation formula !

Dziękuję za uwagę.