Topological Duality and Row-finite Matrices

Laurent Poinsot

LIPN - UMR CNRS 7030
Université Paris-Nord XIII - Institut Galilée

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Goal of this talk

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Let X, Y be any sets.
To each R-linear map $\phi \colon R^X \to R^Y$ is associated a « matrix » $\mathcal{M}_\phi$ with entries in $Y \times X$ and coefficients in R whose (y, x)-entry is given by

$$\mathcal{M}_\phi(y, x) = (\phi(\delta_x))(y)$$

where $\delta_x \in R^X$ the Dirac mass at x.

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where $\delta_x \in R^X$ the Dirac mass at x.

It is similar to the decomposition of a linear map in some bases. Note however that when X is infinite, then $(\delta_x)_{x \in X}$ is not an algebraic basis for R^X.

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When $X = Y = \mathbb{N}$, a $\mathbb{N} \times \mathbb{N}$-matrix M is row-finite if for every $i \in \mathbb{N}$, the ith row $(M(i, j))_{j \in \mathbb{N}}$ is finite.

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The main goal of this talk is to prove the following result:

Theorem

If a linear map $\phi\colon R^X \to R^Y$ is continuous, then its matrix $\mathcal{M}_\phi$ is row-finite.

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Therefore, if $\mathcal{M}_\phi$ is row-finite, then ϕ is continuous (with respect to the product topologies) for all Hausdorff field topologies on R.

In other terms, if ϕ is continuous for the product topologies relative to one given Hausdorff field topology on R, then ϕ is continuous for all Hausdorff field topologies on R.

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In other terms, if ϕ is continuous for the product topologies relative to one given Hausdorff field topology on R, then ϕ is continuous for all Hausdorff field topologies on R.

Actually, this follows from a deeper result: for all Hausdorff field topologies on R, the topological duals of R^X (R^X has the product topology) are the same.

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The modules are also assumed to be unitary.

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It is characterized by the following property:

Let (X, τ) be a topological space, and $f \colon X \to \prod_{i \in I} E_i$. Then, f is continuous if, and only if, $\pi_j \circ f \colon X \to E_j$ is continuous for each $j \in I$.

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This topology is Hausdorff if, and only if, each space (E, τ_i) is separated.

Example

Let (E, τ) be a topological space, and X be a set.

Then $E^X \cong \prod_{x \in X} E_x$ where $E_x = E$ for every $x \in X$.

The product topology on E^X is the coarset topology that makes continuous the projections $f \mapsto f(x)$, $x \in X$.

We recover the topology of simple convergence on E^X.

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For instance any ring (or field) is a topological ring (or field) with either the trivial or the discrete topologies.

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When R is a topological field, then (M, τ) is said to be a topological R-vector space.

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Let R be a topological ring, and X be a set.

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The R-module R^X of all maps from X to R with the product topology is a topological R-module.

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Let $\mathbb{K}$ be a Hausdorff topological field and let us assume that $\mathbb{K}^X$ has the product topology.

In this part we prove that the topological dual of $\mathbb{K}^X$ is isomorphic (as a $\mathbb{K}$-vector space) to $\mathbb{K}^{(X)}$.

Algebraic and topological duals

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$$\mathcal{H}om_{R\text{-}\mathcal{T}op\mathcal{M}od}(M, N) \; .$$

In particular, $M' = \mathcal{H}om_{R\text{-}\mathcal{T}op\mathcal{M}od}(M, R)$ is the topological dual of M.

Duality bracket

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The map

$$\begin{array}{cccc} \langle \cdot \mid \cdot \rangle & R^X \times R^{(X)} & \to & R \\ & (f, p) & \mapsto & \langle f \mid p \rangle = \sum_{x \in X} f(x)p(x) \end{array}$$

is a duality bracket (it means that $\langle \cdot \mid \cdot \rangle$ is R-bilinear and $\langle f \mid \cdot \rangle$ and $\langle \cdot \mid p \rangle$ have a null kernel for every f, p that is $\langle \cdot \mid \cdot \rangle$ is said to be non-degenerated).

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Let $\mathbb{K}$ be a separated topological field, and X be a set.

Let us assume that $\mathbb{K}^X$ has the product topology.

Then the topological dual $(\mathbb{K}^X)'$ of $\mathbb{K}^X$ is isomorphic to $\mathbb{K}^{(X)}$.

Lemma 1

Let R be a ring, and X be a set.

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The map

$$\begin{array}{rccc} \Phi\colon & R^{(X)} & \to & (R^X)^* \\ & p & \mapsto & \left(\begin{array}{rccc} \Phi(p)\colon & R^X & \to & R \\ & f & \mapsto & \langle f \mid p \rangle \end{array} \right) \end{array}$$

is R-linear and one-to-one.

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So $p(x) = \Phi(p)(\delta_x) = 0$ for every $x \in X$.

Lemma 2

Let R be a topological ring, and X be a set. We assume that R^X has the product topology.

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For all $p \in R^{(X)}$, $\Phi(p)$ is continuous, *i.e.*, $\Phi(p) \in (R^X)'$.

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(Proof: Obvious.)

Lemma 3

Recall: summability

Let (G, τ) be a separated topological Abelian group. A family $(g_i)_{i \in I}$ of members of G is summable with sum $g \in G$, which is denoted by $\sum_{i \in I} g_i = g$, if for every open neighbourhood U of zero, there exists a finite subset $J \subseteq I$ such that $\sum_{j \in J} g_j - g \in U$.

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Let R be a separated topological ring, and let us assume that R^X has the product topology.

For each $f \in R^X$, the family $(f(x)\delta_x)_{x \in X} = (\langle f \mid \delta_x \rangle \delta_x)_{x \in X}$ is summable with sum f,

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For each $f \in R^X$, the family $(f(x)\delta_x)_{x \in X} = (\langle f \mid \delta_x \rangle \delta_x)_{x \in X}$ is summable with sum f, that is

$$f = \sum_{x \in X} f(x)\delta_x \ .$$

Proof of Lemma 3

It is sufficient to prove that for each $y \in X$, the family $(\pi_y(f(x)\delta_x))_{x \in X}$ is summable in R, with sum $\pi_y(f)$, where $\pi_y \colon R^X \to R$ is the canonical projection onto R.

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But this projection is given by $f \mapsto \langle f \mid \delta_y \rangle$.

Therefore we need to prove that for each $y \in X$, the family $(\langle f(x)\delta_x \mid \delta_y \rangle)_{x \in X} = (f(x)\delta_y(x))_{x \in X}$ is summable with sum $f(y)$, which is obvious.

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If $\ell \in (R^X)'$,

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We have a direct consequence:

Lemma 5

Let $\mathbb{K}$ be a Hausdorff topological field.

If $\ell \in (\mathbb{K}^X)'$, then $\ell(\delta_x) = 0$ for all $x \in X$, except a finite number.

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From general properties of summability we know that for every open neighbourhood U of 0 in R, $f_\ell(x)\ell(\delta_x) \in U$ for all, except finitely many, $x \in X$.

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From general properties of summability we know that for every open neighbourhood U of 0 in R, $f_\ell(x)\ell(\delta_x) \in U$ for all, except finitely many, $x \in X$.

Since R is assumed Hausdorff, there is an open neighbourhood U of zero such that $1 \notin U$.

Because $1 = f_\ell(x)\ell(\delta_x) \notin U$ for all $x \in Y_\ell$, if Y_ℓ is not finite, this leads to a contradiction.

Lemma 6

Under the same assumptions as Lemma 5,

$$\Phi\colon \mathbb{K}^{(X)} \to (\mathbb{K}^X)'$$

is onto.

Proof of Lemma 6

Let $\ell \in (\mathbb{K}^X)'$ be fixed, and let us define $p_\ell \colon X \to \mathbb{K}$ by $p_\ell(x) = \ell(\delta_x)$.

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According to Lemma 5, $p_\ell \in \mathbb{K}^{(X)}$.

Let $f \in \mathbb{K}^X$. We have

$$\Phi(p_\ell)(f) = \langle f \mid p_\ell \rangle = \sum_{x \in X} f(x) p_\ell(x) = \sum_{x \in X} f(x) \ell(\delta_x) = \ell(f) \ .$$

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Lemma 6 (Φ is onto).

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Remark

For infinite-dimensional spaces, the $\mathbb{K}$-linear map $\phi \mapsto \mathcal{M}_\phi$ is not one-to-one.

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Since $\mathbb{K}^{(X)} = \ker \pi_V$, for every $x, y \in X$, $\langle \pi_V(\delta_y) \mid \delta_X \rangle = 0$ so that $\mathcal{M}_{\pi_V}$ is the null matrix, while $\pi_V \neq 0$.

Some definitions

A matrix $M \in R^{Y \times X}$ is said to be row-finite if for every $y \in Y$, the map $M(y, \cdot)\colon x \in X \to M(y, x) \in R$ is finitely supported, that is $M(y, \cdot) \in R^{(X)}$.

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Convention

In what follows, $\mathbb{K}$ denotes a Hausdorff topological field and $\mathbb{K}^Z$ has the product topology for every set Z.

A first result

Lemma 7

For every $\phi \in \mathcal{Hom}_{\mathbb{K}\text{-}\mathcal{TopVect}}(\mathbb{K}^X, \mathbb{K}^Y)$, $\mathcal{M}_\phi$ is row-finite, that is $\mathrm{Im}(\mathcal{M}) \subseteq \mathbb{K}^{Y \times (X)}$.

Proof of Lemma 7

Let $\phi \in \mathcal{Hom}_{\mathbb{K}\text{-}\mathcal{TopVect}}(\mathbb{K}^X, \mathbb{K}^Y)$ be given.

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For every $y \in Y$, the map $f \in \mathbb{K}^X \mapsto \langle \phi(f) \mid \delta_y \rangle \in \mathbb{K}$ belongs to $(\mathbb{K}^X)'$ by composition of continuous maps.

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According to the previous theorem, there exists a unique $p_{\phi,y} \in \mathbb{K}^{(X)}$ such that $\langle f \mid p_{\phi,y} \rangle = \langle \phi(f) \mid \delta_y \rangle$.

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Therefore $\mathrm{Supp}(p_{\phi,y}) = \{\, x \in X \colon \mathcal{M}_\phi(y,x) \neq 0 \,\}$ so that $\mathcal{M}_\phi \in \mathbb{K}^{Y \times (X)}$.

Lemma 8

The map $\mathcal{M} : \phi \in \mathcal{Hom}_{\mathbb{K}\text{-}\mathcal{TopVect}}(\mathbb{K}^X, \mathbb{K}^Y) \mapsto \mathcal{M}_\phi \in \mathbb{K}^{Y \times (X)}$ is one-to-one.

Proof of Lemma 8

Let $\phi \in \mathcal{Hom}_{\mathbb{K}\text{-}\mathcal{TopVect}}(\mathbb{K}^X, \mathbb{K}^Y)$ such that $\mathcal{M}_\phi = 0$.

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Therefore ϕ is the null linear map on $\mathbb{K}^{(X)}$.

Since ϕ is assumed to be continuous, and $\mathbb{K}^{(X)}$ is dense in $\mathbb{K}^X$, $\phi = 0$.

Lemma 9

The map $\mathcal{M} : \phi \in \mathcal{H}om_{\mathbb{K}\text{-}\mathcal{T}op\mathcal{V}ect}(\mathbb{K}^X, \mathbb{K}^Y) \mapsto \mathcal{M}_\phi \in \mathbb{K}^{Y \times (X)}$ is onto.

Lemma 9

The map $\mathcal{M} : \phi \in \mathcal{H}om_{\mathbb{K}\text{-}\mathcal{T}op\mathcal{V}ect}(\mathbb{K}^X, \mathbb{K}^Y) \mapsto \mathcal{M}_\phi \in \mathbb{K}^{Y \times (X)}$ is onto.

(Proof: It is an easy exercice.)

From lemmas 7, 8 and 9, we easily obtain the following result:

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Proposition

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and for every $\phi \in \mathcal{H}om_{\mathbb{K}\text{-}\mathcal{T}op\mathcal{V}ect}(\mathbb{K}^X, \mathbb{K}^Y)$, and $\psi \in \mathcal{H}om_{\mathbb{K}\text{-}\mathcal{T}op\mathcal{V}ect}(\mathbb{K}^Y, \mathbb{K}^Z)$, we have

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In particular, $\mathcal{M}$ is an isomorphism of algebras from $\mathcal{E}nd_{\mathbb{K}\text{-}\mathcal{T}op\mathcal{V}ect}(\mathbb{K}^X)$ into $\mathbb{K}^{X\times(X)}$.

Conclusion

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Note however that it is not true that $\mathcal{M}_\phi$ is row-finite implies that the linear map ϕ is continuous. Because it is not always the case that $\phi = \psi_{\mathcal{M}_\phi}$. For instance $\mathcal{M}_{\pi_V} = 0$ is row-finite but π_V is not continuous (if it was the case, by injectivity of $\mathcal{M}$, $\pi_V = 0$).