From "combinatorial" monoids to bialgebras and Hopf algebras, functorially

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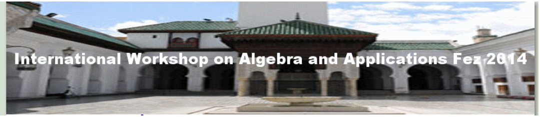

June, 20th 2014

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• Finite decomposition monoid: For each $x \in M$, there are only finitely many $y, z \in M$ such that $x = y * z$.

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- Locally finite monoid: For each $x \in M$, there are only finitely many $x_1, \cdots, x_n \in M \setminus \{ 1 \}$ such that $x = x_1 * \cdots * x_n$.

Motivations

Large algebra

The class of finite decomposition monoids is the larger class for which convolution of functions is possible.

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Let R be a commutative ring with a unit. Let M be a finite decomposition monoid. Then one can define the R-coalgebra $R^{(M)}$ (free module with basis M)

$$\Delta(x) = \sum_{x=y*z} y \otimes z$$

and

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It follows that one can consider its dual R-algebra $R[[M]]$, called the large algebra of M, of all functions from M to R. Its multiplication is given by convolution

$$(f * g)(x) = \sum_{x=y*z} f(y)g(z) \ .$$

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Möbius inversion formula: let $\zeta = \sum_{x \in M} x$ (called the zêta function of M), and let $\mu = \zeta^{-1}$ (called the Möbius function of M).

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Möbius inversion formula: let $\zeta = \sum_{x \in M} x$ (called the zêta function of M), and let $\mu = \zeta^{-1}$ (called the Möbius function of M). Then for all $f, g \in R[[M]]$,

$$g(x) = \sum_{x = y * z} f(y) \Leftrightarrow f(x) = \sum_{x = y * z} g(y)\mu(z) \ .$$

Purpose of the talk

• Give a category-theoretic interpretation of these combinatorial monoids as monoid objects in a monoidal category.

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• Explain some known and new results using monoidal functors.

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In details this means that for each $x \in M$, there are only finitely many $y, z \in M$ such that $x = y * z$.

Category-theoretic interpretation

Let us consider the category **FinFibSet** of all sets with finite-fiber maps. It admits a structure of a symmetric monoidal category inherited from the set-theoretic product.

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The category of monoid objects in **FinFibSet** in then the category of finite decomposition monoids (homomorphisms of monoids with finite fibers).

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Clearly $\varprojlim_{A \in \mathfrak{P}_{\mathrm{fin}}(X)} R^X / R^{(X \setminus A)} \cong \varprojlim_{A \in \mathfrak{P}_{\mathrm{fin}}(X)} R^A \cong R^X$, hence R^X is complete in the inverse limit topology (where all R^A are discrete), this topology is equivalent to the product topology (with R discrete).

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Let us denote by ${}_R\mathbf{TopFreeMod}$ the category of all topologically free modules with continuous linear maps.

Completed tensor product

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And

$$R^{X \times Y} \cong \varprojlim_{A,B} R^{A \times B} \cong \varprojlim_{A,B} (R^X \otimes_R R^Y)/K_{A,B} \ .$$

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One thus defines $R^X \hat{\otimes}_R R^Y = R^{X \times Y}$ ($\hat{\otimes}$ is a bifunctor), so that $R^X \hat{\otimes}_R R^Y$ is the completion of $R^X \otimes_R R^Y$ (in the linear topology).

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Theorem (Universal property of $\hat{\otimes}$)

Let $\phi \colon R^X \times R^Y \to R^Z$ be a continuous R-bilinear map. Then, there exists a unique continuous R-linear map $\phi_0 \colon R^X \hat{\otimes}_R R^Y \to R^Z$ such that $\phi_0 \circ can = \phi$.

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and for $\phi\colon X \to Y$, let $R^{\phi}\colon R^X \to R^Y$ be given by

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R^{-} is a monoidal functor, hence it lifts to a functor between categories of monoid objects (it is a property of monoidal functors). One recovers $M \mapsto R[[M]]$, where M is a finite decomposition monoid, and this corrects the lack of functoriality of the large algebra as defined usually.

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Filtered sets

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The category of all filtered sets admits a monoidal tensor $(X, (X_n)_n) \otimes (Y, (Y_n)_n) = (X \times Y, (T^n(X, Y))_n)$ with

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The unit is the one-point set $*$ with filtration $*_n = \emptyset$ for all $n > 0$ and $*_0 = *$.

Sub-monoidal categories

A filtered set $(X, (X_n)_n)$ is

- Exhausted if $X = X_0$;
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A set X with an exhausted and separated filtration is equivalent to a set X with a length function $\ell \colon X \to \mathbb{N}$. ($X_n := \{ x \in X \colon \ell(x) \geq n \}$ and $\ell(x) := \sup\{ n \in \mathbb{N} : x \in X_n \}$.)

A filtered set is connected if, and only if, there is a unique element of length zero.

Locally finite monoid

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Hence a locally finite monoid is also a monoid object in the monoidal category of connected filtered sets.

Monoid objects

One now considers the category cSet of all connected filtered sets with finite-fiber and filtration-preserving maps. It is a monoidal category.

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Proof: A monoid object in **cSet** is thus a usual monoid M with a connected filtration $(M_n)_n$ of (two-sided) ideals of M. Let ℓ be its associated length function. It thus satisfies $\ell(x * y) \geq \ell(x) + \ell(y)$. Since it is connected, $\ell^{-1}(\{ 0 \}) = \{ 1 \}$. Let us assume that there exists some $x \in M$ with arbitrary long non-trivial decompositions. Then, for every n, $\ell(x) \geq n$ (since $x = x_1 * \cdots * x_m$, $m \geq n$, $x_i \neq 1$) which is impossible since the filtration is separated. □

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Monoid objects: Filtered (complete) R-algebras.

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$R[[M]]$ is complete in this topology but is not necessarily the completion of $R[M]$ with the induced topology.

Remark

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Remark

$R[[M]]$ is an augmented algebra with augmentation ideal $\mathfrak{I}_1$ (this is due to the fact that M is connected as a filtered set).

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From large algebra to representable functor

Let M be a finite decomposition monoid.

Let us define a functor $(-)^M \colon {}_c\mathbf{Alg}_R \to \mathbf{Set}$ by $A \mapsto A^M$.

It is representable with coordinate ring $R[x_a \colon a \in M]$ (polynomial ring in the indeterminates x_a, $a \in M$).

Ring scheme (or Hopf ring)

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Of course both structures are related so that ring axioms hold.

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Proof: This comes from ${}_c\mathbf{Alg}_R(R[x_a \colon a \in M], R) \cong R[[M]]$ (of course as sets but also as rings). $\square$

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The antipode S is given by $S(x_a) = \mu(a)$ for each $a \in M \setminus \{ 1 \}$, where μ is the Möbius function of M.