About the geometry groupoid of a balanced equational variety

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Balanced equations

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Counter-example : left (or right) inverse $x_0^{-1} * x_0 = 1$.

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For instance, the varieties of semigroups, of monoids or also of inverse semigroups (in commutative or non-commutative version) are balanced, while that of groups is not.

Substitution operations

Given a signature Σ, a set X and a term $t \in \Sigma[\mathbb{N}]$, P. Dehornoy defined

$$\mathbf{Subst}_X(t) := \{ \hat{\sigma}(t) \in \Sigma[X] \colon \sigma \colon \mathbb{N} \to \Sigma[X] \} .$$

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Let us consider the sub-monoid $\mathbf{G}_R(\mathbf{V})$ (also denoted by $\mathbf{G}(\mathbf{V})$) of partial bijections of $\Sigma[X]$ generated by $\rho_p^{(s,t)} \colon \Sigma[X] \to \Sigma[X]$, $(s,t) \in R$ or $(t,s) \in R$, and the positions p.

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This object $\mathbf{G}(\mathbf{V})$, introduced by P. Dehornoy, was called the monoid of geometry of the variety $\mathbf{V}$.

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It is relevant to study rewrite term system.

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Let recall that an inverse monoid is a monoid with a unary operation $(-)^\star \colon M \to M$ that satisfies the relations $(xy)^\star = y^\star x^\star$, $1^\star = 1$, $xx^\star x = x$, $(x^\star)^\star = x$, $x^\star x y^\star y = y^\star y x^\star x$, et $(xy)^\star = y^\star x^\star$.

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Geometry ?

Geometric property (Dehornoy)

The monoid $\mathbf{G}(\mathbf{V})$ acts on $\Sigma[X]$ and the homogeneous space $\Sigma[X]/\mathbf{G}(\mathbf{V})$ (set of orbits) associated with this action is the free algebra $\mathbf{V}[X]$ in $\mathbf{V}$ on X.

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Theorem (Dehornoy)

For every (non void) partial bijection θ in $\mathbf{G}(\mathbf{V})$, there is a pair of balanced terms $(s_\theta, t_\theta) \in \Sigma[\mathbb{N}]^2$, unique up to renaming of variables, such that $s_\theta \cong t_\theta$ and

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Moreover for each set of relations R and R' that generate $\cong$, $\mathbf{G}_R(\mathbf{V}) \cong \mathbf{G}_{R'}(\mathbf{V})$.

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Hence the geometry monoid $\mathbf{G}(\mathbf{V})$ is essentially the fully invariant congruence $\cong$ on $\Sigma[\mathbb{N}]$ generated by the relations R, and thus is largely independent of the choice of R and the set X, and, by HSP Theorem of G. Birkhoff, is intrinsically related to the variety $\mathbf{V}$ itself.

Groupoid structure on $\mathbf{G}(\mathbf{V})$

For $\theta_1, \theta_2 \in \mathbf{G}(\mathbf{V})$, one defines (classical construction) the restricted product $\theta_2 \cdot \theta_1 := \theta_2 \circ \theta_1$ if, and only if, $\mathrm{dom}(\theta_2) = \mathrm{im}(\theta_1)$.

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Remark

There is an isomorphism of categories between inverse semigroups and inductive ordered groupoids (C. Ehresmann, M. Lawson).

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One defines the orbit space $\Sigma_F/\mathbf{G}$ as $\Sigma_F/\sim_F$.

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A (left) action $\mathbf{G}$ on (E, π) is a map from the fibered product $\mathbf{Arr}(G) \; {}_{d_0}{\times}_{\pi} \; E$ to E, denoted by $(f, x) \mapsto f \cdot x$, that satisfies a certain number of axioms.

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Theorem (PL, 2014)

Both versions of the definition of a groupoid action are equivalent.

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But actually it is not important that π is a permutation on the whole X. So one can restrict to endo-functions of X which are bijective only on a finite set, and consider two such functions as equal as soon as they coincide on the finite set: germs of local bijections.

Groupoid of germs of local bijections

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The quotient category $\mathbf{Germ}_\infty(X)$ is actually a groupoid, with $[\sigma, A]^{-1} = [(\sigma_{|_A})^{-1}, \sigma(A)]$, called the groupoid of germs of bijections of X.

Action of $\mathbf{Germ}_{\infty}(X)$ on $\Sigma[X]$

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The orbit $\mathcal{O}(t)$ is just the set of terms obtained from t by renaming of variables, and so $s \sim t$ if, and only if, s and t are equal up to renaming of variables.

Action on a balanced congruence

If $\cong$ is a balanced (i.e., $u \cong v \Rightarrow \mathbf{var}(s) = \mathbf{var}(t)$) and fully invariant (i.e., invariant under all endomorphisms) congruence of $\Sigma[X]$,

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$\mathcal{O}(t)$ and $\mathcal{O}$ denote respectively the orbit of $t \in \Sigma[X]$ and any orbit under the action of the groupoid of germs of bijections, $\mathcal{O}^{(2)}(s,t)$ and $\mathcal{O}^{(2)}$ denote respectively the orbit of (s,t) with $s \cong t$, and any orbit under the (diagonal) action of $\mathbf{Germ}_\infty(X)$ on $\cong$.

Reflexive directed graph

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$$d_i(\mathcal{O}^{(2)}(s,t)) = \mathcal{O}_{\partial_i(s,t)}$$

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- The loops: the map $i \colon \Sigma[X] \to \cong/\mathbf{Germ}_\infty(X)$, defined by $i(t) := \mathcal{O}^{(2)}(t,t)$, passes to the quotient to provide a map that satisfies $\iota(\mathcal{O}(t)) = \mathcal{O}^{(2)}(t,t)$ for every $t \in \Sigma[X]$.

Structure of multiplicative graph

Deformation of the groupoid structure of an equivalence relation

- Let us define $m\colon \{\, ((s,t),(r,s')) \in \cong^2 \colon s' \in \mathcal{O}(s) \,\} \to \Sigma[X] \times \Sigma[X]$ by

$$m((s,t),(r,s')) := (r, [\sigma, \mathbf{var}(s)] \cdot t)$$

where $[\sigma, \mathbf{var}(s)] \cdot s = s'$.

- One observes that $\mathrm{im}(m) \subseteq \cong$.

- It can be shown that there is a unique well-defined map $\gamma\colon (\cong / \mathbf{Germ}_\infty(X)) \;{}_{d_0}\times_{d_1} (\cong / \mathbf{Germ}_\infty(X)) \to (\cong / \mathbf{Germ}_\infty(X))$ such that

$$\gamma(\mathcal{O}^{(2)}(s,t), \mathcal{O}^{(2)}(r,s')) = \mathcal{O}^{(2)}_{m((s,t),(r,s'))} \; .$$

Groupoid structure

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This defines the geometry groupoid $\mathbf{Geom(V)}$ of the balanced variety $\mathbf{V}$ determined by $\cong$.

Refinement of G. Birkhoff's HSP Theorem

Theorem (PL, 2014)

The map $\mathbf{V} \mapsto \mathbf{Geom}(\mathbf{V})$ is a Galois connection between the lattice of balanced sub-varieties of Σ-algebras and a sub-poset of small groupoids.

Groupoid action on $\Sigma[X]$

Let X be a set.

Lemma (Dehornoy)

For all terms $s, t \in \Sigma[\mathbb{N}]$, $\mathbf{Subst}_X(s) = \mathbf{Subst}_X(t)$ if, and only if, $\mathcal{O}(s) = \mathcal{O}(t)$.

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$F_X(\mathcal{O}^{(2)}(s,t))\colon \mathbf{Subst}_X(\mathcal{O}(s)) \to \mathbf{Subst}_X(\mathcal{O}(t))$, $F_X(\mathcal{O}^{(2)}(s,t)) := \rho^{(s,t)}$
(does not depend on the choice of (s,t)).

Recover $\mathbf{G(V)}$ from $\mathbf{Geom(V)}$

When X is not void F_X is faithful and injective on objects.

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Remark

The equivalence relation $\sim$ induced by the action of $\mathbf{Germ}_\infty(X)$ on $\Sigma[X]$ (i.e., $u \sim v$ if, and only if, $\mathcal{O}(s) = \mathcal{O}(t)$) is not in general a congruence,

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For instance, let $x, y \in X$, $x \neq y$. Then $x \sim x$ and $x \sim y$ but $x * x \not\sim x * y$.

A Σ-algebra structure on the orbit space

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$$f'(u_1, \cdots, u_n) := \mathcal{O}(f(v_1, \cdots, v_n))$$

for $v_i \in \mathcal{O}(u_i)$, $i = 1, \cdots, n$ such that $\mathbf{var}(v_i) \cap \mathbf{var}(v_j) = \emptyset$, $i \neq j$ (it is possible since the set of variables is infinite).

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One shows that there exists one, and only one, well-defined map such that $\bar{f}(\mathcal{O}(u_1), \cdots, \mathcal{O}(u_n)) = \mathcal{O}(f(v_1, \cdots, v_n))$ with $v_i \in \mathcal{O}(u_i)$, $i = 1, \cdots, n$ such that $\textbf{var}(v_i) \cap \textbf{var}(v_j) = \emptyset$, $i \neq j$.

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Proposition (PL, 2014)

$\mathbf{Geom(V)}$ is a Σ-algebra in the category of (small) groupoids.

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The groupoid $\mathbf{Germ}_\infty(X)$ of germs of bijections acts also on $\mathbf{V}[X]$ by a quotient action

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The groupoid of geometry of a balanced sub-variety

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Then one can define a (small) groupoid $\mathbf{Geom_V(W)}$ whose set of objects is $\mathbf{V}[\mathbb{N}]/\mathbf{Germ}_\infty(\mathbb{N})$ and that of arrows is $\cong /\mathbf{Germ}_\infty(\mathbb{N})$, which is called the relative geometry groupoid of $\mathbf{W}$ (with respect to $\mathbf{V}$).

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Remark

Of course one recovers $\mathbf{Geom}(\mathbf{W})$ by considering $\mathbf{Geom}_{\mathbf{V}}(\mathbf{W})$ with $\mathbf{V}$ the variety of all Σ-algebras (which is balanced).

One can also equips $\mathbf{V}[\mathbb{N}]/\mathbf{Germ}_\infty(\mathbb{N})$ and $\cong /\mathbf{Germ}_\infty(\mathbb{N})$ with structures of Σ-algebras (as already done in case of $\Sigma[\mathbb{N}]$).

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Nevertheless one cannot go further in general: $\mathbf{V}[\mathbb{N}]/\mathbf{Germ}_{\infty}(\mathbb{N})$ is generally not an algebra of the variety $\mathbf{V}$. (A different number of occurrences of a same variable in two equivalent terms is an obstruction to this.)

Definition

A balanced congruence on $\Sigma[X]$ is said to be linearly generated (or simply linear) if it admits a set of generators $R \subseteq \Sigma[X]^2$ such that for each $(s, t) \in R$, every variable in s (hence in t) occurs one and only one time in s and in t.

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Definition

A balanced variety determined by a linear congruence is called a linear variety.

Algebra in $\mathbf{V}$

Theorem (PL, 2014)

Let $\mathbf{V}$ be a linear variety of Σ-algebras and let $\cong$ be a balanced and fully invariant congruence on $\mathbf{V}[\mathbb{N}]$ that (uniquely) determines a balanced sub-variety $\mathbf{W}$ of $\mathbf{V}$.

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Then $\mathbf{Geom}_{\mathbf{V}}(\mathbf{W})$ is an algebra in the variety $\mathbf{V}$ in the category of (small) groupoids.

Some examples

- Let $\cong$ be a balanced and fully invariant congruence on $\mathbb{N}^\star = \mathbf{Mon}[\mathbb{N}]$, and let $\mathbf{W}$ be the associated sub-variety of monoids.

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- In particular if $\cong$ is the commutativity, then $\mathbf{Geom_{Mon}}(\mathbf{ComMon})$ is a symmetric monoidal groupoid.

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- In particular if $\cong$ is the commutativity, then $\mathbf{Geom}_{\mathbf{Mon}}(\mathbf{ComMon})$ is a symmetric monoidal groupoid.

- Similarly $\mathbf{Geom}_{\star\mathbf{Mon}}(\mathbf{InvMon})$ is an "involutive" monoidal groupoid (where $\star\mathbf{Mon}$ is the variety of monoids with an involution).

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- Links with monads, Lawvere theories and clones.

- Of course, links with Lawson's work.