Jacobi algebras, in-between Poisson, differential, and Lie algebras

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Monday, July 20, 2015

AADIOS

Algebraic and Algorithmic Differential and Integral Operator Session

ACA 2015

Kalamata, Greece, 20-23 July 2015

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Lie algebras

Definition

Let R be a commutative ring with a unit.

A Lie algebra $(\mathfrak{g}, [-,-])$ is the data of a R-module $\mathfrak{g}$ and a bilinear map $[-,-]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the Lie bracket, such that

- It is alternating: $[x, x] = 0$ for every $x \in \mathfrak{g}$.
- It satisfies the Jacobi identity

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

for each $x, y, z \in \mathfrak{g}$.

A Lie algebra is said to be commutative whenever its bracket is the zero map.

Universal enveloping algebra

Any (say unital and associative) algebra $(A, \cdot)$ may be turned into a Lie algebra when equipped with the commutator bracket

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Actually this defines a functor from the category **Ass** to the category **Lie**.

This functor admits a left adjoint namely the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$.

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PBW Theorem

If R is a field, then j is one-to-one.

In other words, $\mathfrak{g}$ canonically embeds into its universal enveloping algebra as a sub-Lie algebra.

Wronskian bracket functor

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By general abstract nonsense, this functor admits a left adjoint that makes possible the definition of the Wronskian enveloping algebra of a Lie algebra.

Wronskian enveloping algebra

Universal property

Let $(\mathfrak{g}, [-,-])$ be a Lie algebra (over a commutative ring R).

Its Wronskian enveloping algebra is a differential commutative algebra $(\mathcal{W}, D)$ together with a homomorphism *can* of Lie algebras from $(\mathfrak{g}, [-,-])$ to $(\mathcal{W}, W_D)$ that satisfies the following universal property:

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Given another differential commutative algebra (A, d), and a homomorphism of Lie algebras $\phi\colon (\mathfrak{g}, [-,-]) \to (A, W_d)$, there is a unique homomorphism of differential algebras $\hat{\phi}\colon (\mathcal{W}, D) \to (A, d)$ such that

$$\hat{\phi} \circ can = \phi.$$

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Remark

The Wronskian enveloping algebra of a Lie algebra is unique up to isomorphism.

Embedding problem

Under which conditions on the base ring R and the Lie algebra $\mathfrak{g}$, is the canonical map *can* one-to-one?

Remark

If there is any differential commutative algebra (A, d) and a one-to-one Lie map $\phi\colon \mathfrak{g} \to (A, W_d)$, then *can* automatically is also one-to-one.

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Indeed, under these assumptions there is a unique differential algebra map $\hat{\phi}\colon (\mathcal{W}, D) \to (A, d)$ such that $\hat{\phi} \circ can = \phi$, whence *can* is one-to-one.

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But before, some examples.

Example 1

The Lie algebra $\mathfrak{sl}_2(\mathbb{K})$, where $\mathbb{K}$ is a field of characteristic zero, embeds into $(\mathbb{K}[x], \frac{d}{dx})$, hence it embeds into its Wronskian envelope.

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It can be shown that this Lie algebra $A \cdot d$ of "vector fields on the line A" embeds into its Wronskian envelope.

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Let $u, v \in M$. Then,
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Therefore, $(M, [-, -]_\epsilon)$ embeds into its Wronskian envelope.

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Lie-Rinehart algebras

Let $(A, \cdot)$ be a (not necessarily associative) R-algebra. Let $\mathfrak{Der}_R(A, \cdot)$ be its Lie R-algebra of R-linear derivations (under the usual commutator bracket). When $(A, \cdot)$ is commutative, $\mathfrak{Der}_R(A, \cdot)$ becomes a A-module in an obvious way.

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Definition

A Lie-Rinehart algebra over R is a triple $(A, \mathfrak{g}, \mathfrak{d})$, where

- A is a commutative R-algebra with a unit,
- $\mathfrak{g}$ is a Lie R-algebra which is also a left A-module (with A-action denoted by $a \cdot x$),
- $\mathfrak{d} \colon \mathfrak{g} \to \mathfrak{Der}_R(A)$ is both a Lie R-algebra map, and a A-linear map ($\mathfrak{d}(a \cdot x)(b) = a(\mathfrak{d}(x)(b))$) which turns A into a $\mathfrak{g}$-module,
- $[x, a \cdot y] = a \cdot [x, y] + \mathfrak{d}(x)(a) \cdot y$, $a \in A$, $x, y \in \mathfrak{g}$.

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- $[x, a \cdot y] = a \cdot [x, y] + \mathfrak{d}(x)(a) \cdot y$, $a \in A,\ x, y \in \mathfrak{g}$.

By abuse, $\mathfrak{d}$ is referred to as the anchor map of the Lie-Rinehart algebra $(A, \mathfrak{g})$.

Remark and example

The structure of a Lie-Rinehart algebra is modeled on the properties of the pair $(C^\infty(M), \mathfrak{X}(M))$, where M is a finite-dimensional smooth manifold, $C^\infty(M)$ is the ring of smooth functions on M, and $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on M.

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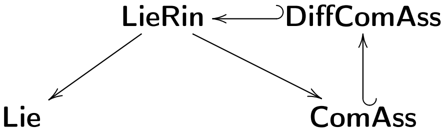
Example

Let A be a commutative R-algebra with a unit. Then, $(A, \mathfrak{Der}_R(A))$ is a Lie-Rinehart algebra.

Given a Lie-Rinehart algebra $(A, \mathfrak{g})$, the Lie algebra $\mathfrak{g}$, together with the anchor, is also referred to as a Lie (R, A)-pseudoalgebra.

Some (forgetful) functors (and their adjoints)



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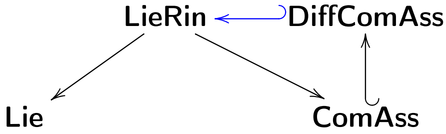
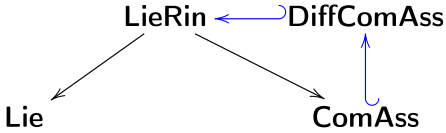


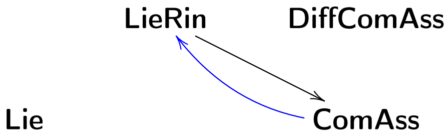
Any commutative differential R-algebra (A, d) may be turned into a Lie-Rinehart algebra $(A, (A, W_d))$ with anchor map $a \mapsto \mathfrak{d}(a) := ad$, and this is functorial. This allows to view **DiffComAss** as a sub-category of **LieRin**.

Some (forgetful) functors (and their adjoints)



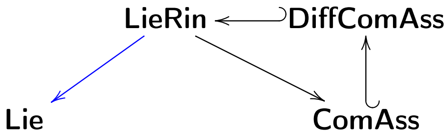
In particular, any commutative R-algebra A, viewed as a differential algebra with the zero derivation, provides a Lie-Rinehart algebra $(A, (A, 0))$.

Some (forgetful) functors (and their adjoints)



A commutative R-algebra A also provides another Lie-Rinehart algebra, namely $(A, (0))$, which is even the free Lie-Rinehart algebra generated by A.

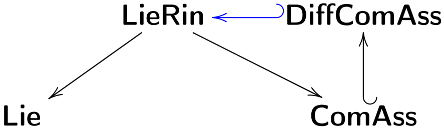
Some (forgetful) functors (and their adjoints)



There is also a forgetful functor **LieRin** $\to$ **Lie**, and it admits a left adjoint given on objects by $\mathfrak{g} \mapsto (R, \mathfrak{g})$. (This may also be interpreted as an embedding of **Lie** into the category of Lie (R, R)-pseudoalgebras.)

Wronskian envelope of a Lie-Rinehart algebra (sketch)

DiffComAss is a reflective sub-category of **LieRin**, i.e., the inclusion functor below admits a left adjoint.



Wronskian envelope of a Lie-Rinehart algebra (sketch)

DiffComAss is a reflective sub-category of **LieRin**.

Let $(A, \mathfrak{g})$ be a Lie-Rinehart algebra with anchor map $\mathfrak{d}$. Let $\mathcal{D}(A, \mathfrak{g})$ be the free commutative differential R-algebra generated by the set $|A| \sqcup |\mathfrak{g}|$. Hence it is the commutative algebra of differential polynomials $R\{|A| \sqcup |\mathfrak{g}|\}$ with variables in $|A| \sqcup |\mathfrak{g}|$.

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Then, let $I(A, \mathfrak{g})$ be the differential ideal of $\mathcal{D}(A, \mathfrak{g})$ generated by the relations that turn the canonical map $(A, \mathfrak{g}) \to (\mathcal{D}(A, \mathfrak{g}), (\mathcal{D}(A, \mathfrak{g}), W))$ into a Lie-Rinehart map. Then, $\mathcal{D}(A, \mathfrak{g})/I(A, \mathfrak{g})$ is the free commutative differential algebra generated by $(A, \mathfrak{g})$.

Jacobi algebra

A Jacobi algebra is a commutative R-algebra A with a unit, together with a Lie bracket (called a Jacobi bracket) over R which satisfies Jacobi-Leibniz rule:

$$[ab, c] = a[b, c] + b[a, c] - ab[1_A, c]$$

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It follows that $ad_{1_A} = [1_A, \cdot] \colon A \to A$ is a R-derivation of the associative algebra A, and that $[-, -] - W_{ad_{1_A}}$ is an alternating biderivation.

Poisson and differential commutative algebras

Remark

Actually each triple (A, D, d) where A is a commutative algebra, D is an alternating biderivation, and d is a derivation such that $D + W_d$ is a Lie bracket provides a Jacobi algebra $(A, D + W_d)$.

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A commutative Poisson algebra thus is a Jacobi algebra whose associated derivation ad_{1_A} is zero.

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A commutative differential algebra, with its Wronskian bracket, is a Jacobi algebra whose associated biderivation $[-,-] - W_{ad_{1_A}}$ is zero.

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This provides two embedding functors

$$\mathbf{PoissCom} \hookrightarrow \mathbf{Jac} \hookleftarrow \mathbf{DiffComAss}.$$

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This provides two embedding functors

$$\mathbf{PoissCom} \hookrightarrow \mathbf{Jac} \hookleftarrow \mathbf{DiffComAss}.$$

Moreover, there is also a forgetful functor $\mathbf{Jac} \to \mathbf{DiffComAss}$, $(A, [-,-]) \mapsto (A, [1_A, -])$.

Various envelopes

PoissCom is reflective in **Jac**.

Various envelopes

PoissCom is reflective in **Jac**: given a Jacobi algebra $(A, [-,-])$, let us consider its Jacobi ideal I_{poiss} generated by $[1_A, x]$, $x \in A$, then A/I_{poiss} is the free commutative Poisson algebra generated by $(A, [-,-])$.

Various envelopes

DiffComAss is reflective in **Jac**, since the embedding functor is an algebraic functor between (equational) varieties (Bill Lawvere).

Various envelopes

There is also a notion of a Jacobi envelope of a differential commutative algebra since the functor $\mathbf{Jac} \to \mathbf{DiffComAss}$ is an algebraic functor. One observes that any differential commutative algebra embeds into its Jacobi envelope.

Various envelopes

One finally mentions the composite forgetful functor **Jac** $\to$ **DiffComAss** $\to$ **LieRin**, $(A, [-,-]) \mapsto (A, (A, W_{ad_{1_A}}))$, which makes it possible to consider the Jacobi envelope of a Lie-Rinehart algebra as the Jacobi envelope of the free commutative differential algebra generated by a Lie-Rinehart algebra.

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The Lie side

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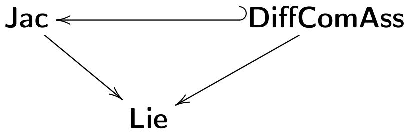
Let J be its Jacobi ideal generated by the relations that make the canonical image of $\mathfrak{g}$ in $Jac(|\mathfrak{g}|)$ a Lie algebra.

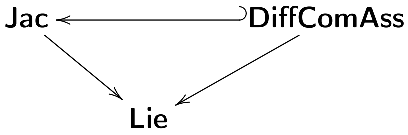
Then, $Jac(|\mathfrak{g}|)/J$ is the universal Jacobi envelope of $\mathfrak{g}$.

Relations between some envelopes

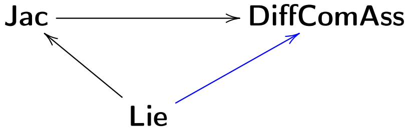


The above diagram of forgetful functors commutes.

Relations between some envelopes

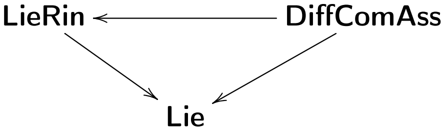


Hence the Wronskian envelope of a Lie algebra $\mathfrak{g}$ may be described as the free differential commutative algebra generated by the Jacobi envelope of $\mathfrak{g}$ as illustrated below.



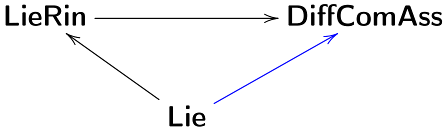
Relations between some envelopes

Moreover the following diagram of functors also commutes.



Relations between some envelopes

This implies that the Wronskian envelope of a Lie algebra $\mathfrak{g}$ is also the differential envelope of the Lie-Rinehart algebra $(R, \mathfrak{g})$.



Local Lie algebras

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Following A. A. Kirillov (1976), a local Lie algebra is a structure of a Lie algebra on $Sec(E)$ which is local, i.e., the support of $[s_1, s_2]$ is contained in the intersection of the supports of s_1 and s_2 (recall that the support of a section is the closure of the set of points at which the section does not vanish).

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When E is a trivial line bundle, then the local Lie bracket is of the form

$$[s_1, s_2] = \Lambda(ds_1, ds_2) + s_1\Gamma(s_2) - \Gamma(s_1)s_2$$

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This implies that such a local Lie algebra $(C^\infty(M), [-,-])$ is precisely a Jacobi algebra.

Lie algebroids (1/2)

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Example

1. A Lie algebroid on the tangent bundle TM is given by the canonical bracket $[-,-]_{vf}$ on $\mathfrak{X}(M) = Sec(TM)$.
2. Every Lie algebra is a Lie algebroid over the one point manifold.

Lie algebroids (2/2)

Lie algebroids on the trivial line bundle, hence Lie algebroid brackets on $C^\infty(M)$, are particular local Lie algebras of the form

$$[f, g] = f\Gamma(g) - \Gamma(f)g$$

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Remark

Other examples of embedding of a Lie pseudoalgebra into its Wronskian envelope are given by Lie algebras of vector fields tangent to a given foliation with one-dimensional leaves.

Conclusion

The embedding problem of a (differential) Lie algebra into its Wronskian enveloping algebra seems to be difficult, and related to Lie algebras of (one-dimensional) vector fields. But Lie algebras of vector fields satisfy some non-trivial identities.

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It might be useful to tackle this problem by dividing it into two parts: first the embedding problem of a Lie algebra into its Jacobi envelope, and secondly the embedding problem of a Jacobi algebra into its differential envelope.

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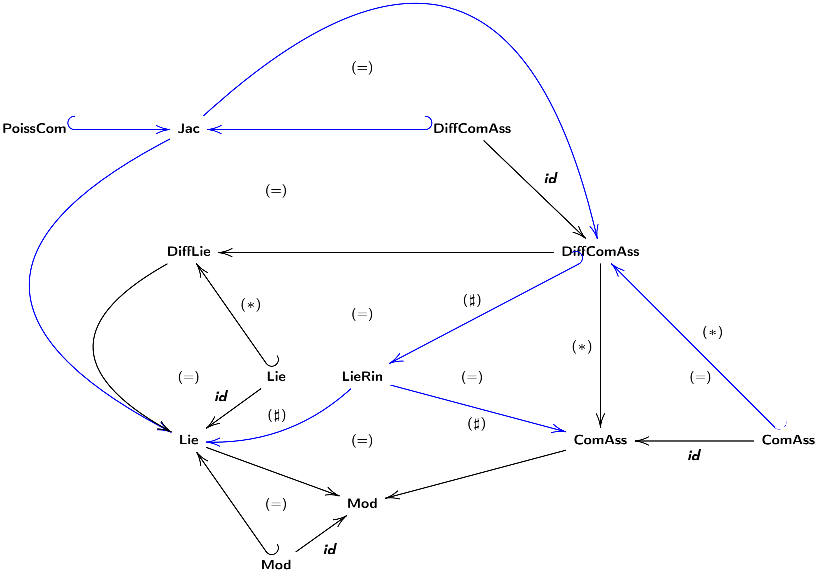

The mark $(*)$ on arrows means functors with both left and right adjoints, while $(\sharp)$ means "non-algebraic functors".