Jacobi algebras, in-between Poisson, differential, and Lie algebras

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Lie algebras

Definition

Let R be a commutative ring with a unit.

A Lie algebra $(\mathfrak{g}, [-,-])$ is the data of a R-module $\mathfrak{g}$ and a bilinear map $[-,-]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the Lie bracket, such that

- It is alternating: $[x, x] = 0$ for every $x \in \mathfrak{g}$.
- It satisfies the Jacobi identity

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

for each $x, y, z \in \mathfrak{g}$.

A Lie algebra is said to be commutative whenever its bracket is the zero map.

Universal enveloping algebra

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One has

$$\mathcal{U}(\mathfrak{g}) \simeq \mathsf{T}(\mathfrak{g})/\langle xy - yx - [x, y] \colon x, y \in \mathfrak{g} \rangle$$

where $\mathsf{T}(M)$ is the tensor algebra of a R-module M.

Poincaré-Birkhoff-Witt theorem

Let $\mathfrak{g}$ be a Lie algebra (over R).

Let $j\colon \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ be the Lie map defined as the composition $\mathfrak{g} \hookrightarrow \mathrm{T}(\mathfrak{g}) \xrightarrow{\pi} \mathcal{U}(\mathfrak{g})$ (where π is the canonical projection, and $\mathcal{U}(\mathfrak{g})$ is seen as a Lie algebra under its commutator bracket).

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PBW Theorem

If R is a field, then j is one-to-one.

More generally, P.M. Cohn proved in 1963 that if the underlying abelian group of $\mathfrak{g}$ is torsion-free, then j is one-to-one.

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Remark

Actually, PBW theorem states that the associated graded algebra of $\mathcal{U}(\mathfrak{g})$ and the symmetric algebra of $\mathfrak{g}$ are isomorphic.

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The first one is a somewhat "trivial" extension. Indeed, a derivation on an algebra is also a derivation for its commutator bracket. Moreover the universal enveloping algebra may be equipped with a (universal) derivation that extends the derivation of the Lie algebra, and the Poincaré-Birkhoff-Witt theorem remains unchanged.

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The other one is rather different (since it is not based on the commutator) and is sketched hereafter.

Wronskian bracket

Now, let us assume that $(A, \cdot, d)$ is a differential commutative algebra.

There is another Lie bracket given by the Wronskian

$$W(x, y) = x \cdot d(y) - d(x) \cdot y$$

which turns A into a (differential) Lie algebra.

The above correspondence is actually functorial.

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1. Does it admit a left adjoint ? In other terms, is there a universal enveloping differential (commutative) algebra ? (Call it the Wronskian enveloping algebra.)

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In this talk I will also provide some examples of embedding / non-embedding of Lie algebras into their differential associative envelope.

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A Σ-algebra is a pair (A, F), where A is a set, and F is a family of set-theoretic maps $(F(n)\colon \Sigma(n) \to A^{A^n})_n$ that makes possible to interpret the symbols of functions (resp., constants) by n-ary functions on (resp., members of) A.

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- There are signatures for (associative) R-algebras, Lie R-algebras, and their differential counterparts.

Equational varieties

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Some (counter-)examples

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Some (counter-)examples

- Semigroups, inverse semigroups, monoids, commutative monoids, groups, abelian groups, rings, R-algebras for a unital commutative ring R, Lie R-algebras, Jordan R-algebras, etc.
- Fields (inversion is only partially defined), small categories, and the category of monoids with invertible elements (groups!), because it is not closed under sub-algebras (e.g., the sub-monoid $\mathbb{N}$ of $\mathbb{Z}$).

Algebraic functors

One of the key features of equational varieties is the fact that they come equipped with a forgetful functor $U_{\mathbf{V}} \colon \mathbf{V} \to \mathbf{Set}$ (it maps an algebra to its carrier set). So they are concrete categories over **Set** (and even monadic).

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Let **V** and **W** be two equational varieties of algebras (not necessarily over the same signature). A functor $F \colon \mathbf{V} \to \mathbf{W}$ is said to be an algebraic functor if it preserves the forgetful functors, i.e., $U_{\mathbf{W}} \circ F = U_{\mathbf{V}}$.

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Generalities about differential algebras

Let R be a commutative ring with a unit.

Let **V** be a variety of (not necessarily associative nor unital) R-algebras (i.e., R-modules M with a R-bilinear operation $\cdot\colon M \times M \to M$ subject to some additional axioms).

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By considering algebras $(M, \cdot)$ of **V** with a derivation d and homomorphisms of algebras commuting with derivations, one gets a variety, say **DiffV**, of differential algebras (in **V**).

Differential ideals

A two-sided (differential) ideal I of a differential algebra $(M, \cdot, d)$ is just a two-sided ideal of $(M, \cdot)$ (i.e., a sub-module such that $M \cdot I \subseteq I \supseteq I \cdot M$) such that $d(I) \subseteq I$.

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Because an intersection of any family of differential ideals also is a differential ideal, it makes also sense to talk about the least differential ideal generated by a set.

A forgetful functor (1/2)

The free differential algebra generated by an algebra

There is an obvious forgetful functor **DiffV** $\to$ **V** which admits a left adjoint (since it is an algebraic functor).

Hence any algebra in **V** "freely generates" a differential algebra (in **V**).

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The construction: let $(M, \cdot)$ be an algebra in $\mathbf{V}$. Let $FDiffV(|M|)$ be the free differential algebra in $\mathbf{V}$ generated by the set $|M|$ (carrier set of $(M, \cdot)$), and let $j\colon |M| \to |FDiffV(|M|)|$ be the canonical map.

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Then, $FDiffV(|M|)/I$ is the free differential algebra generated by $(M, \cdot)$.

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Hence there is a unique differential algebra map $\tilde{\phi}\colon \mathit{FDiffV}(|M|)/I \to (N, \cdot, e)$ such that $\tilde{\phi} \circ \pi \circ j = \phi$.

Example

The free differential Lie algebra generated by a Lie algebra / by a set

One may apply the results from the previous slide with $\mathbf{V} = \mathbf{Lie}$ in order to obtain the free differential Lie algebra $\mathcal{DL}(\mathfrak{g}) := \mathit{FDiffLie}(|\mathfrak{g}|)/I$ generated by a Lie algebra $\mathfrak{g}$.

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One may apply the results from the previous slide with $\mathbf{V} = \mathbf{Lie}$ in order to obtain the free differential Lie algebra $\mathcal{DL}(\mathfrak{g}) := \mathit{FDiffLie}(|\mathfrak{g}|)/I$ generated by a Lie algebra $\mathfrak{g}$.

It is easily seen that any algebra $\mathfrak{g}$ canonically embeds into its differential envelope $\mathcal{DL}(\mathfrak{g})$ (because $(\mathfrak{g}, 0)$ is itself a differential Lie algebra).

One can even describe $\mathit{FDiffLie}(X)$ for a set X: Let M_X be the free magma on the set $X \times \mathbb{N}$, and let $A_X := RM_X$ be the free R-module generated by M_X.

A_X is a (non associative) algebra with bilinear multiplication extending the product in M_X. It is even the free (non associative) differential algebra with derivation d given on generators (x, i) by $d(x, i) := (x, i + 1)$, $x \in X$, $i \in \mathbb{N}$.

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$\mathit{FDiffLie}(X) = A_X/J$, with the quotient derivation, where J is the two-sided differential ideal of A_X generated by tt, $(rs)t + (st)r + (tr)s$, $r, s, t \in M_X$.

Another example

The free commutative differential algebra generated by an algebra / a set

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Let A be a commutative (associative) algebra with a unit. Then, $R\{|A|\}/I$, where I is the two-sided differential ideal generated by the relations that would turn the canonical map $j\colon |A| \to R\{|A|\}$ into an algebra map, is the free commutative differential algebra generated by A.

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- A embeds, as sub-algebra, into $R\{|A|\}/I$ by j.
- The above construction may be adapted for not necessarily commutative algebras.

Reflective sub-category (1/2)

$\mathbf{V} \hookrightarrow \mathbf{DiffV}$

The variety **V** embeds into the variety **DiffV** since any algebra in **V** may be seen as a differential algebra with the zero (or trivial) derivation.

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The construction: let $(M, \cdot, d)$ be a member of **DiffV**. Let I_d be the (algebraic) ideal generated $im(d)$. Thus, M/I_d is a member of **V**, and the natural projection $\pi\colon M \to M/I_d$ is a homomorphism of algebras.

Reflective sub-category (2/2)

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Reflective sub-category (2/2)

Universal property

Given an algebra $(N, \cdot)$ and a homomorphism of differential algebras $\phi\colon (M, \cdot, d) \to (N, \cdot, 0)$, because $\phi \circ d = 0$, it passes to the quotient and gives rise to a unique homomorphism of algebras $\hat{\phi}\colon (M/I_d, \cdot) \to (N, \cdot)$ such that $\hat{\phi} \circ \pi = \phi$.

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3. Differential Lie algebras and their enveloping differential algebras
4. The embedding problem
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Extension of the usual universal enveloping algebra to the differential setting

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One has $d([x, y]) = d(xy - yx) = d(x)y + xd(y) - d(y)x - yd(x) = [d(x), y] + [x, d(y)]$. Hence, $(A, [-, -], d)$ is a differential Lie algebra.

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This gives rise to a functor **DiffAss** $\to$ **DiffLie** which makes commute the following diagram (of forgetful functors).

$$\begin{array}{ccc}
\mathbf{DiffAss} & \xrightarrow{\textit{Comm. bracket}} & \mathbf{DiffLie} \\
{\scriptstyle \textit{forgets der.}}\downarrow & & \downarrow{\scriptstyle \textit{forgets der.}} \\
\mathbf{Ass} & \xrightarrow[\textit{Comm. bracket}]{} & \mathbf{Lie}
\end{array}$$

All functors in this diagram admit a left adjoint.

A construction

Let $(\mathfrak{g}, [-,-], d)$ be a differential Lie algebra.

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Let (A, D) be a differential algebra, and let $\phi\colon (\mathfrak{g}, [-,-], d) \to (A, [-,-], D)$ be a homomorphism of differential Lie algebras.

Then, there is a unique homomorphism of differential algebras $\hat{\phi}\colon (\mathcal{U}(\mathfrak{g}), \tilde{\partial}) \to (A, D)$ such that $\hat{\phi} \circ j = \phi$, where $j\colon \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ is the canonical differential Lie map.

Conclusion for the first approach

The universal enveloping algebra lifts to the realm of differential algebras. Hence symbolically one has

$$\begin{array}{ccc} (\mathfrak{g}, d) & \xrightarrow{j} & (\mathcal{U}(\mathfrak{g}), \tilde{\partial}) \\ \downarrow & & \downarrow \\ \mathfrak{g} & \xrightarrow[j]{} & \mathcal{U}(\mathfrak{g}) \end{array}$$

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PBW theorem remains unchanged.

The Wronskian bracket

The second approach

Let (A, d) be a commutative differential (associative and unital) R-algebra.

Let us define the Wronskian bracket

$$W(x, y) := xd(y) - d(x)y \ .$$

Of course it is alternating $W(x, x) = xd(x) - d(x)x = 0$ (since A is commutative).

Moreover it satisfies Jacobi identity.

Hence (A, W) turns to be a Lie algebra.

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While
$W(d(x), y) + W(x, d(y)) = d(x)d(y) - d^2(x)y + xd^2(y) - d(x)d(y)$.

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Hence (A, W, d) is a differential Lie algebra.

This defines a functor, say the Wronskian, $(A,d) \mapsto (A,W,d)$ from **DiffComAss** to **DiffLie**.

Remark

Composing with the obvious forgetful functor **DiffLie** $\to$ **Lie**, the above construction provides a functor $(A,d) \mapsto (A,W)$ from **DiffComAss** to **Lie**.

Wronskian enveloping algebra

One observes that the Wronskian functor preserves the obvious forgetful functors,

so it is an algebraic functor,

and it admits a left adjoint $\mathcal{W}$, the Wronskian enveloping algebra.

Construction of the differential enveloping algebra (1/2)

1st step: universal extension of the derivation on the symmetric algebra

Let $(\mathfrak{g}, [-,-], d)$ be a differential Lie algebra.

Let $\mathrm{S}(\mathfrak{g})$ be the symmetric algebra of the module $\mathfrak{g}$ which becomes a commutative differential algebra with the unique derivation ∂ that extends the map $\partial(x) = d(x)$ on the generators $x \in \mathfrak{g}$.

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Remark

Actually, one defines the derivation ∂ on the tensor algebra $\mathrm{T}(\mathfrak{g})$, and since it commutes to the permutation of variables, it factors through $\mathrm{S}(\mathfrak{g})$.

Construction of the Wronskian enveloping algebra (2/2)

2nd step: identify on generators the Wronskian and the original Lie bracket

Let us consider the (algebraic) ideal I generated by $d(x)y - xd(y) - [x, y]$, $x, y \in \mathfrak{g}$.

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One observes that $\partial(I) \subseteq I$. Hence I is actually a differential ideal.

Then, the Wronskian enveloping algebra $\mathcal{W}(\mathfrak{g}, [-,-], d)$ is $(\mathsf{S}(\mathfrak{g})/I, \tilde{\partial})$.

Universal property of the Wronskian enveloping algebra

Let (A, δ) be any commutative differential algebra, and let $\phi\colon (g, [-,-], d) \mapsto (A, W, \delta)$ be a homomorphism of differential Lie algebras.

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Then, there exists a unique differential algebra map $\tilde{\phi}\colon (\mathsf{S}(\mathfrak{g})/I, \tilde{\partial}) \to (A, \delta)$ such that $\tilde{\phi}(x + I) = \phi(x)$ for each $x \in \mathfrak{g}$.

Proof

Let $\hat{\phi}\colon \mathsf{S}(\mathfrak{g}) \to A$ be the unique algebra map that extends ϕ.

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Hence it factors through I and provides a unique homomorphism of differential algebras $\tilde{\phi}$ from $(\mathrm{S}(\mathfrak{g})/I, \tilde{\partial})$ to (A, δ) such that
$\tilde{\phi}(x + I) = \phi(x)$, $x \in \mathfrak{g}$. □

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Statement of the problem

Given a differential Lie R-algebra $(\mathfrak{g}, d)$, and its Wronskian enveloping algebra $(\mathcal{W}(\mathfrak{g}, d), \tilde{\partial})$, the (differential) Lie map $can\colon \mathfrak{g} \to \mathsf{S}(\mathfrak{g})/I$, $x \mapsto x + I$, is referred to as the canonical map.

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Under which conditions on $(\mathfrak{g}, d)$ and on R is the canonical map one-to-one?

Remark

can is one-to-one if, and only if, there are a differential commutative algebra (A, δ), and a one-to-one differential Lie map $\phi\colon (\mathfrak{g}, d) \to ((A, W), \delta)$.

Example: $\mathfrak{sl}_2(\mathbb{K})$

Let $\mathbb{K}$ be a field of characteristic zero.

The Lie algebra $\mathfrak{sl}_2(\mathbb{K})$ embeds into the algebra of vector fields of $\mathbb{K}[x]$ by the identification of the elements of its Chevalley basis $e = -1$, $h = -2x$, and $f = x^2$ (the familiar commutation rules are satisfied $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$).

It is a differential Lie algebra when equipped with the usual derivation of polynomials.

Hence it embeds into the commutative differential algebra $(\mathbb{K}[x], \frac{d}{dx})$ as a sub-Lie algebra under the Wronskian bracket, therefore it embeds into its Wronskian enveloping algebra.

Warning: The case of a non-differential Lie algebra (1/3)

For Lie algebras without derivation, there are two different notions for the Wronskian envelope, depending on whether or not one identifies **Lie** with a sub-category of **DiffLie** via the embedding functor $\mathfrak{g} \mapsto (\mathfrak{g}, 0)$.

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Therefore, there are two formulations for the embedding problem.

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As a Lie algebra with the zero derivation

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The differential ideal I is equal to the (algebraic) ideal generated by $[x, y]$, $x, y \in \mathfrak{g}$.

Hence it follows that in case $\mathfrak{g}$ is not commutative (i.e., $[-,-]$ does not vanish identically), $\mathfrak{g}$ does not embed into its universal enveloping differential (commutative) algebra $\mathcal{W}(\mathfrak{g})$ even if R is a field!

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In this case, the embedding problem is rather obvious (of course, any commutative Lie algebra embeds into its Wronskian envelope, which reduced to the symmetric algebra).

Warning: The case of a non-differential Lie algebra (3/3)

Composite of left adjoints

The composite forgetful functor **DiffComAss** $\xrightarrow{\textit{Wronskian bracket}}$ **DiffLie** $\xrightarrow{\textit{forgets der.}}$ **Lie** is an algebraic functor, hence admits a left adjoint.

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Thus, by composition of left adjoints, the Wronskian envelope of a Lie algebra $\mathfrak{g}$ may be defined as the the Wronskian envelope $\mathcal{W}(\mathcal{DL}(\mathfrak{g}))$ of the free differential Lie algebra $\mathcal{DL}(\mathfrak{g})$ generated by the Lie algebra $\mathfrak{g}$.

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Embedding problem

Under which conditions on $\mathfrak{g}$ and on R is the canonical map from $\mathfrak{g}$ to $\mathcal{W}(\mathcal{DL}(\mathfrak{g}))$ one-to-one?

Remark

The canonical map $\mathfrak{g} \to \mathcal{W}(\mathcal{DL}(\mathfrak{g}))$ is one-to-one if, and only if, there are a differential commutative algebra (A, δ), and a one-to-one Lie map $\phi \colon \mathfrak{g} \to (A, W)$.

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Indeed, in this case there is a unique differential Lie algebra map $\hat{\phi}\colon (\mathcal{DL}(\mathfrak{g}), d) \to ((A, W), \delta)$ such that $\hat{\phi} \circ can_{\mathfrak{g}} = \phi$, where $can_{\mathfrak{g}}\colon \mathfrak{g} \to \mathcal{DL}(\mathfrak{g})$ is the canonical map (a Lie algebra map).

Remark

The canonical map $\mathfrak{g} \to \mathcal{W}(\mathcal{DL}(\mathfrak{g}))$ is one-to-one if, and only if, there are a differential commutative algebra (A, δ), and a one-to-one Lie map $\phi\colon \mathfrak{g} \to (A, W)$.

Indeed, in this case there is a unique differential Lie algebra map $\hat{\phi}\colon (\mathcal{DL}(\mathfrak{g}), d) \to ((A, W), \delta)$ such that $\hat{\phi} \circ can_{\mathfrak{g}} = \phi$, where $can_{\mathfrak{g}}\colon \mathfrak{g} \to \mathcal{DL}(\mathfrak{g})$ is the canonical map (a Lie algebra map).

Then, there is a unique differential algebra map $\hat{\hat{\phi}}\colon (\mathcal{W}(\mathcal{DL}(\mathfrak{g})), d) \to (A, \delta)$ such that $\hat{\hat{\phi}} \circ can = \hat{\phi}$, hence $\hat{\hat{\phi}} \circ can \circ can_{\mathfrak{g}} = \phi$ which implies that $can \circ can_{\mathfrak{g}} = \mathfrak{g} \xrightarrow{can_{\mathfrak{g}}} \mathcal{DL}(\mathfrak{g}) \xrightarrow{can} \mathcal{W}(\mathcal{DL}(\mathfrak{g}))$ is one-to-one.

Augmented modules

Let (M, ϵ) be an augmented R-module, i.e., a R-module together with a linear map $\epsilon\colon M \to R$, called its augmentation map.

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Proposition

The associated Lie algebra of an augmented module embeds into its Wronskian envelope.

Sketch of the proof

Given an augmented module (M, ϵ), it can be shown that there is a unique derivation d_ϵ on the symmetric algebra $\mathsf{S}(M)$ of M that extends ϵ.

Let $u, v \in M$. Then, $W(u, v) = ud_\epsilon(v) - d_\epsilon(u)v = u\epsilon(v) - \epsilon(u)v = [u, v]_\epsilon$. Hence the canonical embedding $M \hookrightarrow \mathsf{S}(M)$ is a Lie map. □

Modules with a "rank one" projection

Let M be a R-module. Let $P \colon M \to M$ be a rank one (linear) projection, i.e., $P^2 = P$ and $im(P) \simeq R$ (as modules).

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Remark

It is essentially the same object as an augmented module (M, ϵ) with a surjective augmentation map ϵ, because in this case, since R is free on $\{ 1 \}$, the short exact sequence $0 \to \ker \epsilon \hookrightarrow M \xrightarrow{\epsilon} R \to 0$ splits, so $M \simeq \ker \epsilon \oplus Re$ (with $\epsilon(e) = 1$), and one has a rank one projection $P(x) := \epsilon(x)e$.

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Conversely, if P is a rank one projection on M, then for each $x \in M$ there is a unique scalar $\langle P(x) \mid e\rangle \in R$ such that $P(x) = \langle P(x) \mid e\rangle e$, where e a generator of $im(P) \simeq R$. Then, $\langle P(\cdot) \mid e\rangle\colon M \to R$ is a surjective augmentation map.

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Once chosen a generator e of $im(P)$, one has a Lie algebra structure on M given by $[u, v] = \langle P(v) \mid e\rangle u - \langle P(u) \mid e\rangle v$.

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Let (A, d) be a commutative R-algebra with a unit.

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Moreover, the restriction of d to $A^d \oplus Fix(A, d)$ is a linear projection with $im(d) = Fix(A, d)$.

An application (2/2)

Assuming that the ring of constants A^d is $R1_A \simeq R$, one gets a rank one projection $id - d$ on $R1_A \oplus Fix(A, d)$ onto $R1_A$.

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1. Let $R[x]$ with its usual derivation $d(x) = 1$. Then, $R[x]^d = R$ and $Fix(R[x], d) = (0)$. Then, $[r, s] = 0$ for all $r, s \in R$.

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2. Let d be the unique derivation of $R[x]$ such that $d(x) = x$. Then, $d(x^n) = nx^n$. It follows that $Fix(R[x], d) = Rx$ and $R[x]^d = R$. Hence $[rx + s, tx + v] = (rx + s)\langle tx + v - d(tx + v) \mid 1\rangle - (tx + v)\langle rx + s - d(rx + s) \mid 1\rangle = (rx + s)v - (tx + v)s = x(rv - st)$.

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Moreover, $(A, d)_{A^d}^d = A^d$, and $Fix((A, d)_{A^d}) = Fix(A, d)$ (as A^d-modules).

Therefore, the previous construction applies, and $A^d \oplus (A, d)$ turns to be a Lie A^d-algebra that embeds into its Wronskian envelope (via the canonical Lie map).

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Lie-Rinehart algebras

Let $(M, \cdot)$ be a (not necessarily associative) R-algebra. Let $\mathfrak{Der}_R(M, \cdot)$ be its Lie R-algebra of R-linear derivations (under the usual commutator bracket). When $(M, \cdot)$ is commutative, $\mathfrak{Der}_R(M, \cdot)$ becomes a A-module in an obvious way.

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Definition

A Lie-Rinehart algebra over R is a triple $(A, \mathfrak{g}, \mathfrak{d})$, where

- A is a commutative R-algebra with a unit,
- $\mathfrak{g}$ is a Lie R-algebra which is also a left A-module (with A-action denoted by $a \cdot x$),
- $\mathfrak{d} \colon \mathfrak{g} \to \mathfrak{Der}_R(A)$ is both a Lie R-algebra map, and a A-linear map $(\mathfrak{d}(a \cdot x)(b) = a(\mathfrak{d}(x)(b)))$ which turns A into a $\mathfrak{g}$-module,
- $[x, a \cdot y] = a \cdot [x, y] + \mathfrak{d}(x)(a) \cdot y$, $a \in A$, $x, y \in \mathfrak{g}$.

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- $[x, a \cdot y] = a \cdot [x, y] + \mathfrak{d}(x)(a) \cdot y$, $a \in A,\ x, y \in \mathfrak{g}$.

By abuse, $\mathfrak{d}$ is referred to as the anchor map of the Lie-Rinehart algebra $(A, \mathfrak{g})$.

Remark and example

The structure of a Lie-Rinehart algebra is modeled on the properties of the pair $(C^\infty(V), \mathfrak{X}(V))$, where V is a finite-dimensional smooth manifold, $C^\infty(V)$ is the ring of smooth functions on V, and $\mathfrak{X}(V)$ is the Lie algebra of smooth vector fields on V.

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Example

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Given a Lie-Rinehart algebra $(A, \mathfrak{g})$, the Lie algebra $\mathfrak{g}$, together with the anchor, is also referred to as a Lie (R, A)-pseudoalgebra.

Some functors

Any commutative differential R-algebra (A, d) may be turned into a Lie-Rinehart algebra $(A, (A, W))$ with anchor map $a \mapsto \mathfrak{d}(a) := ad$, and this is functorial. This allows to view **DiffComAss** as sub-category of **LieRin**.

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There is also a forgetful functor **LieRin** $\rightarrow$ **Lie**, and it admits a left adjoint given on objects by $\mathfrak{g} \mapsto (R, \mathfrak{g})$. (This may also be interpreted as an embedding of **Lie** into the category of Lie (R, R)-pseudoalgebras.)

Wronskian envelope of a Lie-Rinehart algebra (sketch)

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Let $(A, \mathfrak{g})$ be a Lie-Rinehart algebra with anchor map $\mathfrak{d}$. Let $\mathcal{D}(A, \mathfrak{g})$ be the free commutative differential R-algebra generated by the set $|A| \sqcup |\mathfrak{g}|$. Hence it is the commutative algebra of differential polynomials $R\{|A| \sqcup |\mathfrak{g}|\}$ with variables in $|A| \sqcup |\mathfrak{g}|$.

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Then, let $I(A, \mathfrak{g})$ be the differential ideal of $\mathcal{D}(A, \mathfrak{g})$ generated by the relations that turn the canonical map $(A, \mathfrak{g}) \to (\mathcal{D}(A, \mathfrak{g}), (\mathcal{D}(A, \mathfrak{g}), W))$ into a Lie-Rinehart map. Then, $\mathcal{D}(A, \mathfrak{g})/I(A, \mathfrak{g})$ is the free commutative differential algebra generated by $(A, \mathfrak{g})$.

Jacobi algebra

A Jacobi algebra is a commutative R-algebra with a unit, together with a Lie bracket (called a Jacobi bracket) over R which satisfies Jacobi-Leibniz rule:

$$[ab, c] = a[b, c] + b[a, c] - ab[1_A, c]$$

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It follows that $ad_{1_A} = [1_A, \cdot]\colon A \to A$ is a R-derivation of the associative algebra A, and that $[-, -] - W_{[1_A, -]}$ is an alternating biderivation.

Poisson and differential commutative algebras

Remark

Actually each triple (A, D, d) where A is a commutative algebra, D is an alternating biderivation, and d is a derivation such that $D + W_d$ is a Lie bracket provides a Jacobi algebra $(A, D + W_d)$.

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$$\mathbf{PoissCom} \hookrightarrow \mathbf{Jac} \hookleftarrow \mathbf{DiffComAss}.$$

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This provides two embedding functors

$$\mathbf{PoissCom} \hookrightarrow \mathbf{Jac} \hookleftarrow \mathbf{DiffComAss}.$$

Moreover, there is also a forgetful functor $\mathbf{Jac} \to \mathbf{DiffComAss}$, $(A, [-, -]) \mapsto (A, [1_A, -])$.

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PoissCom is reflective in **Jac**: given a Jacobi algebra $(A, [-,-])$, let us consider its Jacobi ideal I_{poiss} generated by $[1_A, x]$, $x \in A$, then A/I_{poiss} is the free commutative Poisson algebra generated by $(A, [-,-])$.

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One finally mentions the composite forgetful functor
Jac $\to$ **DiffComAss** $\to$ **LieRin**, $(A, [-,-]) \mapsto (A, (A, W_{ad_{1_A}}))$, which makes it possible to consider the Jacobi envelope of a Lie-Rinehart algebra as the Jacobi envelope of the free commutative differential algebra generated by a Lie-Rinehart algebra.

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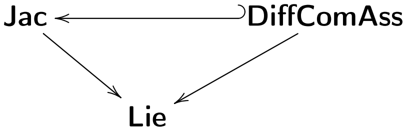
Then, $Jac(|\mathfrak{g}|)/J$ is the universal Jacobi envelope of $\mathfrak{g}$.

Relations between some envelopes

Because the following diagram of forgetful functors commutes, the Wronskian envelope of a Lie algebra $\mathfrak{g}$ may be described as the free differential commutative algebra generated by the Jacobi envelope of $\mathfrak{g}$.



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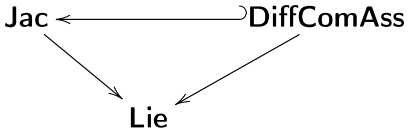
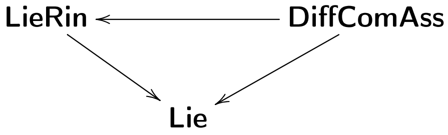


Moreover the following diagram of functors also commutes, implying that the Wronskian envelope of a Lie algebra $\mathfrak{g}$ is also the differential envelope of the Lie-Rinehart algebra $(R, \mathfrak{g})$.



Local Lie algebras

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Let $Sec(E)$ its space of global sections. E is said to be trivial whenever $E = V \times \mathbb{R}$ in which case $C^\infty(V) = Sec(E)$.

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Let $Sec(E)$ its space of global sections. E is said to be trivial whenever $E = V \times \mathbb{R}$ in which case $C^\infty(V) = Sec(E)$.

Following A. A. Kirillov (1976), a local Lie algebra is a structure of a Lie algebra on $Sec(E)$ which is local, i.e., the support of $[s_1, s_2]$ is contained in the intersection of the supports of s_1 and s_2 (recall that the support of a section is the closure of the set of points at which the section does not vanish).

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When E is a trivial line bundle, then the local Lie bracket is of the form

$$[s_1, s_2] = \Lambda(ds_1, ds_2) + s_1\Gamma(s_2) - \Gamma(s_1)s_2$$

where Λ is a bivector field, and Γ is a vector field.

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This implies that such a local Lie algebra $(C^\infty(V), [-,-])$ is precisely a Jacobi algebra.

Lie algebroids (1/2)

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The anchor map of the corresponding Lie-Rinehart algebra $(C^\infty(V), Sec(E))$ is described by a vector bundle morphism $d\colon E \to TV$ which induces the Lie map from $(Sec(E), [-,-])$ to the Lie algebra $(\mathfrak{X}(V), [-,-]_{vf})$ of vector fields on V.

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Example

1. A Lie algebroid on the tangent bundle TV is given by the canonical bracket $[-,-]_{vf}$ on $\mathfrak{X}(V) = Sec(TM)$.
2. Every Lie algebra is a Lie algebroid over the one point manifold.

Lie algebroids (2/2)

Lie algebroids on the trivial line bundle, hence Lie algebroids brackets on $C^\infty(V)$, are particular local Lie algebras of the form

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It follows that the underlying Lie algebra of the Lie pseudoalgebra $(C^\infty(V), [-,-])$ embeds into its Wronskian envelope.

Remark

Other examples of embedding of a Lie pseudoalgebra into its Wronskian envelope are given by Lie algebras of vector fields tangent to a given foliation with one-dimensional leaves.

Conclusion

The embedding problem of a (differential) Lie algebra into its Wronskian enveloping algebra seems to be quite harder than the classical situation and related to Lie algebras of (one-dimensional) vector field. But Lie algebras of vector fields satisfy some non-trivial identities.

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It might be useful to tackle this problem by dividing it into two parts: first the embedding problem of a Lie algebra into its Jacobi envelope, and secondly the embedding problem of a Jacobi algebra into its differential envelope.

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