Hilbertian Frobenius algebras

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Main result (B. Coecke et al., *arXiv*ed in 2008, published in 2013)

Every commutative †-Frobenius monoid in *FdHilb* determines an orthogonal basis (consisting of its group-like elements) and every orthogonal basis determines a commutative †-Frobenius monoid in *FdHilb*. These correspondences are inverse to each other.

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What about the infinite-dimensional situation? In a joint paper (*arXiv*ed 2010, published 2012), S. Abramsky and C. Heunen explore the notion of commutative †-Frobenius semigroups in *Hilb* to "*provide a categorical way to speak about orthonormal bases and quantum observables in arbitrary dimension*".

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$\Leftarrow$ Conditions under which a commutative †-Frobenius semigroup in *Hilb* is semisimple.

$\Leftarrow$ Structure Theorem for Hilbertian Frobenius algebras.

Semisimple commutative Banach algebras

A (not necessarily unital) commutative algebra A is

1. *semisimple* when $J(A) = (0)$,
2. *radical* when $A = J(A)$.

Jacobson radical $J(A)$
$=$ intersection of all maximal modular ideals of A,
$=$ intersection of the kernels of the characters onto $\mathbb{C}$ (A Banach),
$=$ kernel of the Gelfand representation.

Gelfand representation $G\colon A \to C_0(\hat{A})$, $G(a)(\chi) := \chi(a)$
with $\hat{A}$ = set of all non-trivial characters of A.

Hilbertian algebras

= (commutative) semigroups in *Hilb*

Hilbertian algebras

are (commutative) semigroups in *Hilb*,

"are" Banach algebras (forgetful functor),

aren't Hilbert algebras (no involutions).

Hilbertian algebras

Group-like elements

By Riesz Representation Theorem,

$$\hat{A} \simeq G(A)$$

where $G(A)$ = set of non-zero group-like elements of $A = (H, \mu)$,

that is, those $0 \neq x \in H$ s.t. $\mu^\dagger(x) = x \otimes x$.

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Consequently,

$$J(A) = G(A)^\perp$$

and

$$H = J(A) \oplus_2 J(A)^\perp = G(A)^\perp \oplus_2 \overline{\langle G(A) \rangle} \text{ (qua Hilbert spaces).}$$

Hilbertian algebras

Semisimplicity

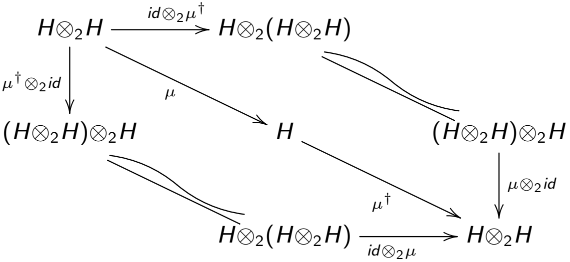
$A = (H, \mu)$ is

1. *semisimple* when $H = \overline{\langle G(A) \rangle} = J(A)^{\perp}$,
2. *radical* when $G(A) = \emptyset$.

Hilbertian Frobenius algebras

a.k.a. commutative †-Frobenius algebras

A commutative semigroup $H \otimes_2 H \xrightarrow{\mu} H$ in *Hilb* s.t.



(1)

Call it *special* when furthermore μ is a coisometry, that is, $\mu \circ \mu^\dagger = id$.

Structure Theorem for Hilbertian Frobenius Algebras

The main result

Theorem

Let $A = (H, \mu)$ be a Hilbertian Frobenius algebra.

Then, $H = J(A) \oplus_2 J(A)^\perp$, where $J(A)$ and $J(A)^\perp$ are both ideals ($J(A)$ is the annihilator of A) and subcoalgebras. $J(A)$ and $J(A)^\perp$ are Frobenius, radical and semisimple respectively.

$\overline{\langle G(A)\rangle} = J(A)^{\perp}$ is a closed subalgebra and subcoalgebra of A for a Hilbertian Frobenius algebra $A = (H, \mu)$

Proof:

Is $\overline{\langle G(A)\rangle}$ a subcoalgebra?

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Proof:

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By general properties of the Hilbert adjoint, for a Hilbertian algebra $A = (H, \mu)$,

I closed ideal of $A \Leftrightarrow I^{\perp}$ is a subcoalgebra of $(H, \mu^{\dagger})$.

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$0 = \langle x \otimes x^2, u \otimes x \rangle$

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Quasi-nilpotent elements

Let A be a (not necessarily commutative) Banach algebra and let $u \in A$.

u is quasi-nilpotent when its spectral radius is equal to zero, that is, $\|u^n\|^{\frac{1}{n}} \to 0$.

When A is commutative, $J(A)$ coincides with the set of all quasi-nilpotent elements of A.

For a Hilbert space H, call quasi-nilpotent operator a bounded linear map $H \xrightarrow{f} H$ quasi-nilpotent as a member of $\mathcal{B}(H)$.

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Frobenius $\Rightarrow$ multiplication operators are normal

The statement

Let $A = (H, \mu)$ be a Hilbertian algebra. For $u \in H$, define $M_u \colon H \to H$, $M_u(v) := uv$.

Proposition

Let us assume that $A = (H, \mu)$ is Frobenius. Then, for each $u \in H$, M_u is normal, that is, $M_u \circ M_u^\dagger = M_u^\dagger \circ M_u$.

Consequently, $J(A)$ is equal to the annihilator of A.

Frobenius $\Rightarrow$ multiplication operators are normal

The proof

In any Hilbertian algebra $A = (H, \mu)$, $u \in J(A) \Rightarrow M_u$ is a quasi-nilpotent operator.

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If furthermore M_u is normal, then u belongs to the annihilator of A, that is, $M_u(v) = 0$ for all v, because the spectral radius of a normal operator coincides with its operator norm.

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The first assertion is obtained by direct inspection (using the Frobenius conditions). □

Structure Theorem for Hilbertian Frobenius Algebras

Main result

Let $A = (H, \mu)$ be a Hilbertian Frobenius algebra.

Then, $H = J(A) \oplus_2 J(A)^\perp$, where $J(A)$ and $J(A)^\perp$ are both ideals ($J(A)$ is the annihilator of A) and subcoalgebras and subcoalgebras. $J(A)$ and $J(A)^\perp$ are Frobenius, radical and semisimple respectively.

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Proof:

That $J(A)^\perp$ is an ideal follows from the facts that $J(A)^\perp$ is a subalgebra and $J(A)$ is the annihilator of A. Consequently, $J(A)^{\perp\perp} = J(A)$ is a subcoalgebra. □

The conditions for semisimplicity

In arbitrary dimension

Corollary

Let $A = (H, \mu)$ be a Hilbertian Frobenius algebra. The following assertions are equivalent.

1. A is semisimple.
2. μ has a dense range.
3. $\mu^\dagger$ is one-to-one.
4. $\mu \circ \mu^\dagger$ is one-to-one.

In particular, if A is special, then A is semisimple.

Remark: Commutativity is necessary

Let X be a non void set and $x_0 \in X$.

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Then, $(\ell^2(X), \mu_{x_0})$ is a non commutative (whenever $|X| > 1$) special Hilbertian Frobenius algebra, with $\mu_{x_0}(u \otimes v) := v(x_0)u$.

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Then, $(\ell^2(X), \mu_{x_0})$ is a non commutative (whenever $|X| > 1$) special Hilbertian Frobenius algebra, with $\mu_{x_0}(u \otimes v) := v(x_0)u$.

However it is not (Jacobson) semisimple. E.g., $\{ \delta_{x_0} \}^\perp$ consists entirely of nilpotent elements! Its Jacobson radical is precisely $\{ \delta_{x_0} \}^\perp$, while its annihilator is (0). (The left annihilator is (0) and the right annihilator is $\{ \delta_{x_0} \}^\perp$.)

The conditions for semisimplicity

In finite dimension

Corollary

Let $A = (H, \mu)$ be a finite-dimensional Hilbertian Frobenius algebra. The following assertions are equivalent.

1. A has a unit.
2. μ is onto.
3. $\mu^\dagger$ is one-to-one.
4. A is semisimple.

In particular, if A is special, then A is unital.

In the finite-dimensional situation, structures of (resp. special) Frobenius algebras on H correspond one-one to orthogonal (resp. orthonormal) bases of H.

This is almost the same when is dropped the finite-dimensional requirement.

Call *bounded above* (resp. *bounded below*) a set X of a Hilbert space such that there exists $C > 0$ with $\|x\| \leq C$ (resp. $C \leq \|x\|$) for each $x \in X$.

Bounded below or bounded above: a matter of taste!

$x \mapsto \frac{x}{\|x\|^2}$ transforms bijectively a bounded above orthogonal set into a bounded below orthogonal set (and vice versa). Moreover it is a bijection between the sets of all bounded below orthogonal sets and of all bounded above orthogonal sets.

Relations with Hilbertian bases

Theorem

Let H be a Hilbert space. There are (among others) one-one correspondences between

1. Bounded above orthogonal bases of H and bounded linear maps $\mu\colon H \otimes_2 H \to H$ with a dense range such that (H, μ) is a Frobenius algebra.
2. Orthonormal bases of H and bounded linear coisometries $\mu\colon H \otimes_2 H \to H$ such that (H, μ) is a Frobenius algebra.

Relations with Hilbertian bases

Elements of the proof

Let X be a bounded above orthogonal basis of H. Define $\mu_X(u \otimes v) := \sum_{x \in X} \frac{1}{\|x\|} \langle u, x \rangle \langle v, x \rangle \frac{x}{\|x\|}$. Then, (H, μ_X) is a semisimple Hilbertian algebra.

Let (H, μ) be a semisimple Frobenius algebra. Then, $G(H, \mu)$ is a bounded above orthogonal basis of H (because for each $x \in G(H, \mu)$, $\|x\| \leq \|\mu\|_{op}$).

The operations are invertible and inverse one from the other.

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The main theorem as an equivalence of categories

Frob (resp. ${}_{semisimple}Frob$) = full subcategory of ${}_{c}Sem(Hilb)$ spanned by the (resp. semisimple) Frobenius algebras.

Proposition

The categories *Frob* and ${}_{semisimple}Frob \times Hilb$ are equivalent.

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Proof: By the main theorem, each radical Frobenius algebra is trivial, that is, it has the zero multiplication. Consequently it is essentially a Hilbert space.

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There is a functor $Frob \times Frob \to Frob$, $((H, \mu), (K, \gamma)) \mapsto (H \oplus_2 K, \rho)$, where ρ acts like μ on $H \otimes_2 H$, like γ on $K \otimes_2 K$, and is zero everywhere else (here is used additivity of $\otimes_2$). Moreover $J(H \oplus_2 K, \rho) = J(H, \mu) \oplus_2 J(K, \gamma)$.

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Weighted pointed sets

A weighted pointed set is a triple (X, x_0, α), where (X, x_0) is a pointed set and $\alpha\colon X \setminus \{ x_0 \} \to [C, +\infty[$ is a map, where $C > 0$.

A morphism $(X, x_0, \alpha) \xrightarrow{f} (Y, y_0, \beta)$ is a base-point preserving map $(X, x_0) \xrightarrow{f} (Y, y_0)$ such that

1. for each $y \neq y_0$, $|f^{-1}(\{ y \})| < +\infty$,
2. there exists a real number $M_f \geq 0$ such that for all $y \neq y_0$, $\sum_{x \in f^{-1}(\{ y \})} \alpha(x) \leq M_f \beta(y)$.

This provides the category $\mathit{WSet}_\bullet$.

The $\ell^2_\bullet$-functor

Let (X, x_0, α) be a weighted pointed set.

Define the following subspace of $\ell^2(X)$
$\ell^2_\bullet(X, x_0, \alpha) := \{ f \in \mathbb{C}^X : f(x_0) = 0, \sum_{x \in X \setminus \{x_0\}} \alpha(x)|f(x)|^2 < +\infty \}$.
Under pointwise product of maps, it is a semisimple Frobenius algebra.

This construction extends to a functor $\ell^2_\bullet : \mathit{WSet}_\bullet^{op} \to \mathit{Frob}$.

The set of minimal ideals functor

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Let $Min(A) := \{ \mathbb{C}x \colon x \in G(A) \}$ and $w_A \colon Min(A) \to [C, +\infty[$, $w_A(I) := \frac{1}{\|g_I\|^2}$, where g_I is the group-like generator of I.

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$(Min_\bullet(A), 0, w_A)$, with $Min_\bullet(A) = Min(A) \cup \{ 0 \}$, is a weighted pointed set, and this construction extends to a functor $Min_\bullet \colon Frob \to WSet_\bullet^{op}$.

An adjunction and its restricted equivalence

One has an adjunction $Min_\bullet \dashv \ell^2_\bullet \colon Frob \to WSet_\bullet^{op}$

which restricts to an equivalence of categories

$$_{semisimple}Frob \simeq WSet_\bullet^{op}.$$

Consequently, $Frob \simeq WSet_\bullet^{op} \times Hilb$ (equivalence).

Some derived equivalences

From $\mathit{Frob} \simeq \mathit{WSet}_{\bullet}^{op} \times \mathit{Hilb}$, by restriction:

$$\mathit{FdFrob} \simeq \mathit{FinSet}_{\bullet}^{op} \times \mathit{FdHilb}$$

Some derived equivalences

From *Frob* $\simeq$ *WSet*$_\bullet^{op} \times$ *Hilb*, by restriction:

$$FdFrob \simeq FinSet_\bullet^{op} \times FdHilb$$

and

$$_1FdFrob \simeq FdFrob_{proper} \simeq FinSet^{op} \times FdHilb.$$

A morphism $(H,\mu) \xrightarrow{f} (K,\gamma)$ of Hilbertian algebras is called *proper* when $ran(f)$ is included in no maximal modular ideals of (K,γ). Alternatively, for each group-like element y of (K,γ), there exists $u \in H$ such that $\langle f(u), y\rangle \neq 0$.