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sur le sujet :

Variations of the graph coloring problem : theoretical aspects and algorithms

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Contents

Introduction

Ce document présente mes activités de recherche et les principaux résultats obtenus aprés l'obtention du diplôme de doctorat en Informatique à l'Université Paris-Sud en décembre 2000. J'ai travaillé pendant quatre ans (janvier 2001 - décembre 2004) comme enseignant assistant au département des mathématiques de l'Université des Andes (Colombie). Pendant cette période, j'ai enseigné et travaillé en théorie des graphes, théorie algébrique des graphes, théorie des groupes et algorithmique. De janvier 2005 à juin 2006, j'ai enseigné algorithmique au département des mathématiques appliquées de l'Université de Padoue (Italie). C'est en 2006 que j'ai été recruté comme Maître de Conférences à l'Université Paris-Nord, où je travaille dans l'équipe *Optimisation Combinatoire et Algorithmique Distribuée* (OCAD) au sein du *Laboratoire d'Informatique de l'Université Paris-Nord* UMR CNRS 7030.

Mes travaux de recherche font partie des *Mathématiques Discrètes*, plus précisément, de la *Théorie des Graphes*. Ils portent sur l'étude de certaines variantes du problème de coloration des graphes. Je m'intéresse à l'étude de nouvelles propriétés pour l'obtention des algorithmes polynomiaux ainsi comme à l'analyse de la complexité algorithmique de certaines variantes de la coloration dans certaines familles des graphes. Ma recherche porte aussi sur l'étude de problèmes algèbriques concernant la coloration des graphes. Ce document est composé principalement des trois chapitres suivants.

Chapitre 2 : *b-coloration des graphes*

Dans ce chapitre, on montre d'abord qu'il n'existe pas une constante $\varepsilon > 0$ pour laquelle le problème de la détermination du nombre b-chromatique d'un graphe peut être approché avec un facteur de $120/113-\varepsilon$ en temps polynomial, sauf si P $=$ NP. Ce résultat est jusqu'à présent, l'unique résultat concernant la difficulté d'approximer tel paramètre dans les graphes. Ce travail à été réalisé avec la collaboration de Sylvie Corteel (CNRS, France) et Juan Vera (University of Waterloo, Canada) (voir référence [19]).

Ensuite, on montre que les graphes P_4-sparse (et en particulier, les cographes) sont b-continus et b-monotones. En plus, il est donné un algorithme de programmation dynamique pour déterminer en temps polynomial le nombre b-chromatique dans cette famille des graphes. Ce travail à été réalisé avec la collaboration de Flavia Bonomo, Guillermo Dúran (Universidad de Buenos Aires, Argentina), Frédéric Maffray (CNRS, France), et Javier Marenco (Universidad Nacional de General Sarmiento, Argentina) (voir référence [8]).

Chapitre 3 : *Produit direct de certains graphes sommet-transitifs*

Dans ce chapitre, on étudie d'abord le nombre d'indépendence et le nombre chromatique du produit direct de graphes circulaires, graphes de Kneser et de graphes puissance des cycles via homomorphismes. Ce travail à été réalisé avec la collaboration de Juan Vera (University of Waterloo, Canada) (voir référence [80]).

Ensuite, en utilisant des méthodes algébriques classiques, on obtient une charactérisation complète des partitions idomatiques du produit direct de trois graphes complets ainsi que la détermination du nombre idomatique de ces graphes (voir référence [78]).

Chapitre 4 : *Somme-coloration de graphes*

Dans ce chapitre, on étudie d'abord le probléme de la somme-coloration minimum (MSC) dans les graphes P_4-sparse. Dans ce cas, le but consiste en colorier les sommets du graphe avec des entiers positifs, tout en minimisant la somme des couleurs affectées aux sommets. On montre qu'il y a une grande sous-famille des graphes P_4-sparse pour laquelle ce probléme peut être résolu en temps polynomial. Ce travail à été réalisé avec la collaboration de Flavia Bonomo (Universidad de Buenos Aires, Argentina) (voir référence [10]).

Le probléme de la somme-arête-coloration minimum d'un graphe (MSEC) peut se définir de manière analogue. Dans ce chapitre, on étudie le problème MSEC dans le cas des multicycles (cycles avec des arêtes multiples). On propose des algorithmes simples (pseudo) polynomiaux pour résoudre ce problème dans cette famille de multigraphes. Ce travail à été réalisé avec la collaboration de Jean Cardinal (Université Libre de Bruxelles, Belgique) et Vlady Ravelomanana (Université Paris-Nord, France) (voir références [13, 14]).

Finalement, on a défini le problème de la somme-ensemble-coloration minimum (MSSC) d'un graphe, lequel consiste à affecter un ensemble d'entiers positifs de taille $\omega(v)$ à chaque sommet v du graphe, de telle sorte que l'intersection des ensembles affectés à des sommets adjacents soit vide et que la somme totale des ensembles des entiers affectés aux sommets soit minimum. Il est clair que si $\omega(v) = 1$ pour tout sommet v du graphe, alors le problème MSSC deviens le problème MSC. On montre que, dans les arbres, le problème MSSC peut être résolu en temps polynomial (resp. est un problème NP-difficile) dans le cas où l'ensemble des entiers affectés à chaque sommet est un intervalle consécutif (resp. non-consécutif). On montre aussi que, dans le cas des graphes représentatifs des arêtes des arbres, le problème MSSC est NP-difficile quand l'ensemble des entiers affectés à chaque sommet est un intervalle consécutif. Ce travail à été réalisé avec la collaboration de Flavia Bonomo, Guillermo Dúran (Universidad de Buenos Aires, Argentina), et Javier Marenco (Universidad Nacional de General Sarmiento, Argentina) (voir référence [9]).

Conclusions et Perspectives

Le dernier chapitre conclut ce document et s'ouvre sur les différentes perspectives de recherche que pourront être dans le futur traitées par ultérieures collaborations nationales et internationales ainsi qu'être à la base de nouvelles thématiques de thèse en Informatique Théorique ou en Mathématiques Discrètes.

Chapter 1

Introduction

This manuscript presents the principal results that I have obtained after my Ph.D. degree in Computer Science at Paris-Sud University in December 2000. After my Ph.D. degree I have worked four years (from January 2001 to December 2004) as Assistant Professor at the Mathematical Department of ”Los Andes” University in Bogotá (Colombia). During this period, I have had the opportunity of teaching and working on Algebraic Graph Theory, Graph Theory, Group Theory and Algorithmic. From January 2005 till June 2006, I have taught some Algorithmic courses at the Mathematical Department of the University of Padova (Italy). From September 2006, I have obtained a tenure position as ”Maître de Conférences” in Computer Science at Paris-Nord University. Actually, I'm working in the Computer Science Department of the Paris-Nord University in the OCAD team (*”Optimisation Combinatoire et Algorithmique Distribuée”*).

My research domain is *Discrete Mathematics*. Precisely, the main goal of my research is the study of several problems on variations of the famous classical problem of graph coloring. I have studied new properties to develop algorithms for solving some of the proposed problems and to address the computational complexity of some of these variations of graph coloring in different graph classes, not overlooking the practical applications of these models. This manuscript is composed of three principal chapters as in the following.

Chapter 2 : *b-coloring of graphs*

In this chapter, it is proved that there is no constant $\varepsilon > 0$ for which the problem of determining the b-chromatic number of a graph can be approximated within a factor of $120/113 - \varepsilon$ in polynomial time, unless P $=$ NP. This result is until now the only hardness approximation result known for this parameter. This work has been done with the collaboration of Sylvie Corteel (CNRS, France) and Juan Vera (University of Waterloo, Canada) (see reference [19]).

Next, it is proved that P_4-sparse graphs (and, in particular, cographs) are b-continuous and b-monotonic. Besides, it is given a dynamic programming algorithm to compute the b-chromatic number in polynomial time within these graph classes. These algorithms rely on the structural properties of the corresponding classes and are based on the notion of dominance vector. This work has been done with the collaboration of Flavia Bonomo and

Guillermo Dúran (Universidad de Buenos Aires, Argentina), Frédéric Maffray (CNRS, France), and Javier Marenco (Universidad Nacional de General Sarmiento, Argentina) (see reference [8]).

Chapter 3 : *Direct products of some vertex-transitive graphs*

In this chapter, the independence and chromatic numbers of finite direct products graphs of circular graphs, Kneser graphs and powers of cycles is studied. In the case of circular and Kneser graphs, this is done via classical homomorphisms. For the direct product graph of powers of cycles, first it is analyzed its independence number and then such a parameter is used to compute its chromatic number. This work has been done with the collaboration of Juan Vera (University of Waterloo, Canada) (see reference [80]).

Next, by using an standard algebraic approach, a full characterization of the idomatic partitions of the direct product of three complete graphs is given, and it is shown how to use such a characterization in order to construct idomatic partitions of the direct product of four or more complete graphs (see reference [78]).

Chapter 4 : *Sum-coloring of graphs*

In this chapter, the Minimum Sum Coloring (MSC) problem on P_4-sparse graphs is studied. In the MSC problem, the goal is to assign natural numbers to vertices of a graph such that adjacent vertices get different numbers, and the sum of the numbers assigned to the vertices is minimum. It is introduced the concept of maximal sequence associated with an optimal solution of the MSC problem of any graph. Based in such maximal sequences, it is shown that there is a large sub-family of P_4-sparse graphs for which the MSC problem can be solved in polynomial-time. This work has been done with the collaboration of Flavia Bonomo (Universidad de Buenos Aires, Argentina) (see reference [10]).

In an analogous way, it has been defined the edge coloring version of the MSC problem : the *Minimum Sum Edge Coloring* (MSEC) problem. The *chromatic edge strength* of a graph is the minimum number of colors required in a minimum sum edge coloring of this graph. In this part, it is studied the case of multicycles, defined as cycles with parallel edges, and it is given a closed-form expression for the chromatic edge strength of a multicycle, thereby extending a theorem due to Berge. It is shown that the minimum sum can be achieved with a number of colors equal to the chromatic index. It is also proposed simple algorithms for finding a minimum sum edge coloring of a multicycle. These results are generalized to a large family of minimum cost coloring problems. This work has been done with the collaboration of Jean Cardinal (Université Libre de Bruxelles, Belgique) and Vlady Ravelomanana (Université Paris-Nord, France) (see references [13, 14]).

Finally, it is defined the Minimum Sum Set Coloring (MSSC) problem which consists in assign a set of $\omega(v)$ positive integers to each vertex v of a graph so that the intersection of sets assigned to adjacent vertices be empty and the sum of the assigned set of numbers to each vertex of the graph is minimum. Clearly, when $\omega(v) = 1$ for each vertex v of the graph, the MSSC problem becomes the MSC problem. It is shown that the MSSC problem on trees is polynomial-time solvable in the *non-preemptive* case (i.e. the set of integers assigned to each vertex is a consecutive interval) but NP-hard in the *preemptive* case. Finally, it is shown that the *non-preemptive* case of the MSSC problem is NP-hard for line graphs of trees. This work has been done with the collaboration of Flavia Bonomo and Guillermo Dúran (Universidad de Buenos Aires, Argentina), and Javier Marenco (Universidad Nacional de General Sarmiento, Argentina) (see reference [9]).

Conclusions and Perspectives

This final chapter contains the conclusions of this manuscript and gives some perspectives for a future work at the basis of the formulation of new Ph.D. subjects in Theoretical Computer Science and Discrete Mathematics.

Chapter 2

b-coloring of graphs

We consider finite undirected graphs without loops or multiple edges. A *vertex coloring* (i.e. *proper coloring*) of a graph $G = (V, E)$ is an assignment of colors to the vertices in V such that adjacent vertices receive different colors. We assume that the colors are positive integers. A vertex t-coloring of a graph G is a coloring where the color of each vertex in V is taken from the set $\{1, 2, \ldots, t\}$. The smallest number t such that G admits a t-coloring is called the *chromatic number* of G and is denoted by $\chi(G)$.

Given a coloring of a graph G with t colors, a vertex v is said to be *dominant* or t*-dominant* if v is adjacent to at least one vertex receiving each of the $t - 1$ colors not assigned to v. A *b-coloring* of a graph is a coloring such that every color class admits a dominant vertex. Note that every coloring of G with $\chi(G)$ colors is a b-coloring. The *b-chromatic number* of a graph G, denoted by $\chi_b(G)$, is the maximum number t such that G admits a b-coloring with t colors. This parameter has been introduced by R. W. Irving and D. F. Manlove [45], by considering proper colorings that are minimal with respect to a partial order defined on the set of all the partitions of the vertex set of G. They proved that determining $\chi_b(G)$ is NP-hard for general graphs, but polynomial-time solvable for trees. In [57], Kratochvil, Tuza and Voigt show that determining $\chi_b(G)$ is NP-hard even if G is a connected bipartite graph.

Several related concepts concerning b-colorings of graphs have been studied in [25, 42, 43, 53, 54, 63]. A graph G is defined to be *b-continuous* [25] if it admits a b-coloring with t colors, for every $t = \chi(G), \ldots, \chi_b(G)$. For example, the graph in Figure 2.1 is not b-continuous since it admits b-colorings with 2 colors and 4 colors, but no b-coloring with 3 colors. In [53] (see also [25]) it is proved that chordal graphs and some planar graphs are b-continuous.

Hoàng and Kouider [42] defined the concept of *b-perfectness* of a graph. A graph G is *b-perfect* if $\chi_b(H) = \chi(H)$ for every induced subgraph H of G. The property $\chi_b(G) = \chi(G)$ is not hereditary: the graph in Figure 2.2 has $\chi_b(G) = \chi(G) = 3$ but it contains an induced subgraph H with $\chi_b(H) = 4$ and $\chi(H) = 3$. Also a graph G is *b-imperfect* if it is not b-perfect, and *minimally b-imperfect* if it is b-imperfect and every proper induced subgraph of G is b-perfect (see [54, 43, 63] and references therein).

We define a graph G to be *b-monotonic* if $\chi_b(H_1) \geq \chi_b(H_2)$ for every induced subgraph H_1 of G, and every induced subgraph H_2 of H_1. This property does not hold in general, see Figure 2.2. Notice that, by the monotonicity of the chromatic number, both

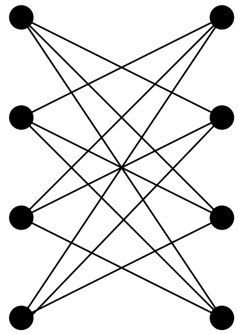


Figure 2.1: A non-b-continuous graph, admitting b-colorings with 2 and 4 colors but no b-coloring with 3 colors.

b-perfect and minimally non b-perfect graphs are b-monotonic.

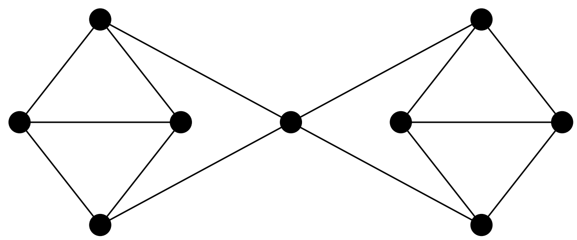


Figure 2.2: A non-b-monotonic graph G. We have $\chi_b(G) = 3$, but the subgraph H obtained from G by deleting the central vertex has $\chi_b(H) = 4$.

Recently, the theory of b-colorings of graphs has been applied in [21] to some clustering problems in data mining processes. In fact, clustering is generally defined as an unsupervised data mining process which aims to divide a set of data into groups, or clusters, such that the data within the same group are similar to each other while data from different groups are dissimilar. However, additional background information (namely constraints) is available in some domains and must be considered in the clustering solutions. A b-coloring-based approach exhibits more important clustering features an enables to build a fine partition of the data set in clusters when the number of clusters is not predefined.

In this chapter, we first prove in Section 2.1 that, there is no constant $\varepsilon > 0$ for which the problem of determining the b-chromatic number of a graph can be approximated within a factor of $120/113 - \varepsilon$ in polynomial time, unless P $=$ NP. This result is until now, the only hardness approximation result known for this parameter. This work has been done with the collaboration of Sylvie Corteel (CNRS, France) and Juan Vera (University of Waterloo, Canada) (see reference [19]). In Section 2.2, we prove that P_4-sparse graphs (and, in particular, cographs) are b-continuous and b-monotonic. Besides, we describe a dynamic programming algorithm to compute the b-chromatic number in polynomial time within these graph classes. These algorithms rely on the structural properties of the corresponding classes, and are based on the notion of dominance vector that we will introduce in Section 2.2. This work has been done with the collaboration of Flavia Bonomo and Guillermo Dúran (Universidad de Buenos Aires, Argentina), Frédéric Maffray (CNRS, France), and Javier Marenco (Universidad Nacional de General Sarmiento, Argentina) (see reference [8]).

2.1 On approximating the b-chromatic number

2.1.1 Preliminaries

Let $\mathcal{P}$ be a maximization problem and let $\alpha \geq 1$. For an instance x of $\mathcal{P}$ let $OPT(x)$ be the optimal value. An *α-approximation algorithm* for $\mathcal{P}$ is a polynomial time algorithm $\mathcal{A}$ such that on each input instance x of $\mathcal{P}$ it outputs a number $\mathcal{A}(x)$ such that $OPT(x)/\alpha \leq \mathcal{A}(x) \leq OPT(x)$.

To show the hardness of approximating the b-chromatic number we relate it to the hardness of approximating the optimization version of the k-ESAT problem. Let k be an integer greater than 1.

k-ESAT problem.
Instance: A set $X = \{x_1, x_2, \ldots, x_n\}$ of boolean variables, a collection $C = \{c_1, c_2, \ldots, c_p\}$ of disjunctive clauses with exactly k different literals, where a literal is a variable or a negated variable in X.
Question: Does there exist a truth assignment for the variables in X such that each clause in C is satisfied?

The decision version of the k-ESAT problem is NP-complete for $k \geq 3$ [29]. Johnson showed in [52] the following result.

Theorem 1 *(Theorem 3 in [52]) Let (X, C) be an instance of the k-ESAT problem. Then, there is a deterministic polynomial time algorithm that finds a truth assignment for variables in X which satisfies at least $|C|(1 - 1/2^k)$ clauses in C.*

The **MAX k-ESAT** problem is the optimization version of the k-ESAT problem in which, given an instance of k-ESAT, the goal consists of finding the maximum number of clauses that can be satisfied simultaneously by any truth assignment of the boolean variables. The MAX k-ESAT problem is NP-hard [29].

Note that in the case $k = 3$, Theorem 1 gives an $8/7$-approximation algorithm for the MAX 3-ESAT problem. Moreover, Håstad showed in [37] the following inapproximability result for the MAX 3-ESAT problem.

Theorem 2 *(Theorem 6.1 in [37]) The MAX 3-ESAT problem is not approximable within $8/7 - \varepsilon$ for any $\varepsilon > 0$, unless P = NP.*

In the following section, we use Theorem 2 restricted to a special kind of instances in order to obtain an inapproximability result for the b-chromatic number problem of a graph.

Definition 1 *We say that an instance (X, C) of MAX 3-ESAT is non-trivial if $|C| > 4$, and for all $x \in X$*

- *There is no* $c \in C$ *such that* $x, \overline{x} \in c$*,*
- *There are* $c, d \in C$ *such that* $x \in c$ *and* $\overline{x} \in d$*.*

We now show that Theorem 2 holds when restricted to non-trivial instances of MAX 3-ESAT.

Corollary 1 *The MAX 3-ESAT problem is not approximable within* $8/7 - \varepsilon$ *for any* $\varepsilon > 0$*, even when restricted to non-trivial instances.*

Proof : We present a proof by contradiction. Assume that there is an $(8/7-\varepsilon)$-approximation algorithm running in polynomial time $p(|X|+|C|)$ for non-trivial instances (X, C) of the MAX 3-ESAT problem, for some $0 < \varepsilon \leq 1/7$. We prove that there is an $(8/7 - \varepsilon)$-approximation algorithm for the MAX 3-ESAT problem. This contradicts Theorem 2.

We prove this by induction on $|X| + |C|$. The base case is trivial. Now, let $k > 1$ and assume that the statement holds for all instances (X, C) such that $|X| + |C| < k$, and let (X, C) be an instance of MAX 3-ESAT such that $|X| + |C| = k$. If the instance is non-trivial, the statement follows from our initial assumption. If not we have three possible cases:

- There is $x \in X$ such that there is $c \in C$ with $x, \overline{x} \in c$. Let $C' = C \setminus \{c\}$. By induction hypothesis applied to (X, C'), we can get, in polynomial time, a truth assignment for the variables in X that satisfies at least $\frac{|C'|}{8/7-\varepsilon}$ clauses in C'. This assignment also satisfies c and therefore satisfies at least

$$\frac{|C'|}{8/7-\varepsilon} + 1 \geq \frac{|C|}{8/7-\varepsilon}$$

  clauses of C.

- There is $x \in X$ such that no clause $c \in C$ contains $\overline{x}$. Let $X' = X \setminus \{x\}$ and $C' = C \setminus \{c \in C : x \in c\}$. By induction hypothesis we can get, in polynomial time, a truth assignment for the variables in X' that satisfies at least $\frac{|C'|}{8/7-\varepsilon}$ clauses in C'. Now we assign the value True to x, and all clauses in C containing it are satisfied. Therefore we have a truth assignment satisfying at least

$$\frac{|C'|}{8/7-\varepsilon} + |C \setminus C'| \geq \frac{|C|}{8/7-\varepsilon}$$

  clauses.

- There is $x \in X$ such that no clause $c \in C$ contains x. This case is analogous to the previous one.

Therefore, there is a $(8/7 - \varepsilon)$-approximation algorithm for the MAX 3-ESAT problem running in polynomial-time $O(k^2)p(k)$, where the $O(k^2)$ term represents the time needed to find the desired x and construct X' and C' and is certainly not the best possible. □

2.1.2 Hardness of approximation

In this section we prove the hardness result for approximating the b-chromatic number problem of a graph.

Let (X, C) be an instance of the 3-ESAT problem. We define $\mathbf{G}(X, C) = (V, E)$ to be the graph constructed as follows:

Let $X = \{x_1, x_2, \ldots, x_n\}$ be the set of boolean variables, and let $C = \{c_1, c_2, \ldots, c_p\}$ be the collection of disjunctive clauses, with $c_i = \{l_{i,1}, l_{i,2}, l_{i,3}\}$ for $i = 1, 2, \ldots, p$, where $l_{i,j} = x_k$ or $l_{i,j} = \overline{x_k}$ for some $1 \leq k \leq n$.

Let

$$\begin{aligned} V =& \{v\} \cup \{z_i : 1 \leq i \leq p-1\} \cup \{w_j : 1 \leq j \leq 2p\} \\ & \cup \{y_i : 1 \leq i \leq p\} \cup \{x_{i,j}, \overline{x_{i,j}} : 1 \leq i \leq n, 1 \leq j \leq p\}, \end{aligned}$$

and let

$$\begin{aligned} E =& \{\{z_i, w_j\} : 1 \leq i \leq p-1, 1 \leq j \neq i \leq 2p\} \\ & \cup \{\{v, z_i\} : 1 \leq i \leq p-1\} \cup \{\{v, y_i\} : 1 \leq i \leq p\} \\ & \cup \{\{y_i, y_j\} : 1 \leq i < j \leq p\} \\ & \cup \{\{x_{i,j}, \overline{x_{i,k}}\} : 1 \leq i \leq n, 1 \leq j, k \leq p\} \\ & \cup \{\{y_i, x_{j,k}\} : 1 \leq i \leq p, 1 \leq j \leq n, 1 \leq k \leq p, x_j \in c_i\} \\ & \cup \{\{y_i, \overline{x_{j,k}}\} : 1 \leq i \leq p, 1 \leq j \leq n, 1 \leq k \leq p, \overline{x_j} \in c_i\}. \end{aligned}$$

Notice that $|V| = 2np + 4p$.

The resulting graph $\mathbf{G}(X, C) = (V, E)$ is shown in Figure 2.3.

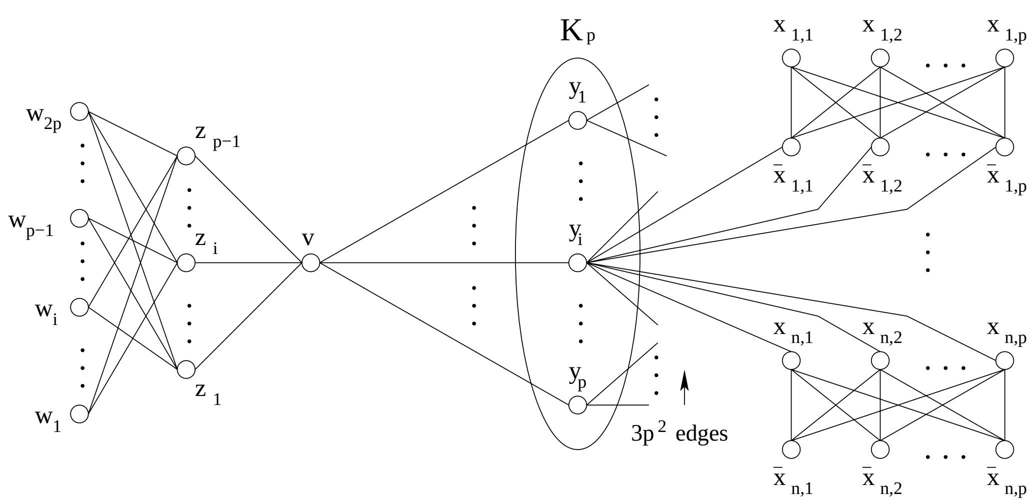


Figure 2.3: Partial construction of G from (X, C), where the clause $c_i \in C$ contains the literals $\overline{x_1}$ and x_n.

Theorem 3 *Let (X, C) be a non-trivial instance of the 3-ESAT problem, where $|X| = n$ and $|C| = p$. Then, $\chi_b(\mathbf{G}(X, C)) = p + t$ where t is the maximum number of clauses that can be satisfied in C.*

The proof of Theorem 3 requires Propositions 1 and 2 below.

Proposition 1 *Let (X, C) be a non-trivial instance of the 3-ESAT problem, where $|X| = n$ and $|C| = p$. Let t be the maximum number of clauses that can be satisfied in C. Then there is a b-coloring of $\mathbf{G}(X, C)$ with $p + t$ colors.*

Proof : Fix a truth assignment of the variables that satisfies exactly t clauses. W.l.o.g. assume that the clauses satisfied in C are $c_1, c_2, \ldots, c_t$.

Color the vertices of $\mathbf{G}(X, C)$ with $p + t$ colors as follows:

- for $1 \leq i \leq p - 1$, assign color i to vertex z_i,
- assign color p to vertex v,
- for $1 \leq i \leq t$, assign color $p + i$ to vertex y_i.

The previous vertices will be the dominating vertices of each one of the $p+t$ color classes.

For $1 \leq j \leq p + t$, assign color j to vertex w_j, and for $1 \leq j \leq p - t$, assign color $p + t$ to vertex w_{p+t+j}. In this way, the vertex z_i is dominating for the color class i.

Vertex v is already a dominating vertex for the color class p.

For $t + 1 \leq i \leq p$, assign to vertex y_i the color $i - t$.

For every $1 \leq i \leq n$, do the following. If x_i is true, choose $1 \leq s \leq p$ such that $x_i \in c_s$. Notice that c_s is satisfied and therefore $s \leq t$. Assign to each $x_{i,j}$ color j and to $\overline{x_{i,j}}$ color $p + s$, for $1 \leq j \leq p$. If x_i is false then $\overline{x}_i$ is true, and proceed in the analogous way.

Now, we just need to check that the coloring is proper and that for $1 \leq i \leq t$, y_i is a dominating vertex for its color class.

The coloring is not proper only if there are $1 \leq i \leq p$, $1 \leq j \leq n$ and $1 \leq k \leq p$ such that there is an edge between y_i and $l_{j,k}$, where $l_{j,k} = x_{j,k}$ or $l_{j,k} = \overline{x_{j,k}}$, with y_i and $l_{j,k}$ of the same color (all the other edges are taken care of directly by the construction). Without loss of generality we assume $l_{j,k} = x_{j,k}$, because the other case is analogous. By construction of $\mathbf{G}(X, C)$, we know that $x_j \in c_i$. There are two cases. If $1 \leq i \leq t$, as the color of $x_{j,k}$ is the same as the color of y_i, and this is $p + i > p$, then x_j is false, so $\overline{x_j}$ is true. Therefore by the construction of the coloring $\overline{x_j} \in c_i$, but then $x_j, \overline{x_j} \in c_i$ contradicting the non-triviality of the instance. If $t < i \leq p$, as the color of $x_{j,k}$ is the same as the color of y_i, and this is $i - t < p$, x_j is true. Therefore c_i is satisfied, but this contradicts our assumption that the truth assignment satisfies exactly the first t clauses.

Now, consider $1 \leq i \leq t$, and let l_i be a literal in clause c_i such that the truth assignment satisfies l_i. Notice that y_i is adjacent to the p vertices that correspond to this literal, and they received colors $1, \ldots, p$. Since vertex y_i is also adjacent to every other vertex y_j, for $1 \leq j \neq i \leq t$, vertex y_i is a dominating vertex. □

Proposition 2 *Let (X, C) be a non-trivial instance of MAX 3-ESAT and let $1 < t$. If there is a b-coloring of $\mathbf{G}(X, C)$ with $p + t$ colors, Then there exists a truth assignment for X such that at least t clauses are satisfied in C.*

Proof : Fix a b-coloring of $\mathbf{G}(X, C)$ with $p + t$ colors. There are three possible cases:

- There exist $1 \leq j \leq n$ and $1 \leq k \leq p$ such that $x_{j,k}$ is a dominating vertex. In this case, vertex $x_{j,k}$ is adjacent at least to $p + t - 1$ other vertices and therefore $x_{j,k}$ is adjacent to at least $t - 1$ of the vertices $y_i's$. This implies x_j belongs to at least $t - 1$ of the $c_i's$. If x_j belongs to at least t of the $c_i's$, any truth assignment where x_j is true will satisfy t clauses in C. If x_j belongs to exactly $t - 1$ $y_i's$, take $c \in C$ such that $x_j \notin c$, and let $j' \neq j$, $1 \leq j' \leq n$, be such that $x_{j'} \in c$ (or $\overline{x_{j'}} \in c$). Then any truth assignment where x_j is true and $x_{j'}$ is true (resp. $x_{j'}$ is false) will satisfy at least t clauses in C.

- There are $1 \leq j \leq n$ and $1 \leq k \leq p$ such that $\overline{x_{j,k}}$ is a dominating vertex. This case is completely analogous to the first one.

- For every $1 \leq j \leq n$ and $1 \leq k \leq p$ neither $x_{j,k}$ nor $\overline{x_{j,k}}$ is a dominating vertex. In this case the dominating vertices are among the set $\{v\} \cup \{z_i : 1 \leq i \leq p-1\} \cup \{y_i : 1 \leq i \leq p\}$. Now let B the set of dominating vertices belonging to $\{y_i : 1 \leq i \leq p\}$. Then $|B| \geq t$. Without loss of generality assume that for $1 \leq i \leq p$ the color of each y_i is i and that the color assigned to v is $p + 1$. Now define the following truth assignment for the boolean variables:

  $$\sigma(x_j) \text{ is True if and only if for all } 1 \leq k \leq p \text{ the color of } \overline{x_{j,k}} \text{ is not } p + 2.$$

  Now, let $1 \leq i \leq p$ be such that $y_i \in B$. As y_i is a dominating vertex, it has to be connected to some vertex of color $p + 2$, and this one has to be one of the $x_{j,k}$ or $\overline{x_{j,k}}$ for some $1 \leq j \leq n$ and $1 \leq k \leq p$. Notice that if $x_{j,k}$ has color $p + 2$ then for all $1 \leq l \leq p$, the color of $\overline{x_{j,l}}$ is not $p + 2$ and thus $\sigma(x_j)$ is True. On the other hand if $\overline{x_{j,k}}$ has color $p + 2$ then $\sigma(x_j)$ is False. In either case σ satisfies c_i. □

Proof of the Theorem 3. From Theorem 1, $t \geq 7p/8 > 1$, and the result follows from Propositions 1 and 2. □

By Corollary 1 and Theorem 3, the hardness approximation result for the b-chromatic number problem now follows.

Theorem 4 *The b-chromatic number problem is not approximable within $120/113 - \varepsilon$ for any $\varepsilon > 0$, unless P = NP.*

Proof : Suppose that the b-chromatic number problem can be approximated within a factor of $120/113 - \varepsilon$, for some $\varepsilon > 0$. Let (X, C) be a non-trivial instance of 3-ESAT, as defined in Section 2. Let p be the number of clauses in C, and let t be the maximum number of clauses of C that can be satisfied by a truth assignment to X. By Theorem 3, we can construct in polynomial time a graph G, namely $\mathbf{G}(X,C)$, such that $\chi_b(G) = p + t$. By the assumption, we can compute in polynomial time a b-coloring for G with l colors such that

$$\frac{\chi_b(G)}{120/113 - \varepsilon} \le l \le \chi_b(G),$$

and by Proposition 2, we can derive a truth assignment of (X,C) which satisfies at least $l - p$ clauses. Then

$$\frac{p+t}{120/113-\varepsilon} - p \le l - p \le t.$$
$$\frac{113t - 7p + 113p\varepsilon}{120 - 113\varepsilon} \le l - p \le t$$

But, from Theorem 1, $p \le 8t/7$, therefore

$$\frac{t}{8/7-\varepsilon} = \frac{105t}{120-105\varepsilon} \le \frac{105t + 113p\varepsilon}{120-113\varepsilon} \le \frac{113t - 7p + 113p\varepsilon}{120-113\varepsilon} \le l - p \le t$$

Thus, we can get a $8/7 - \varepsilon$ approximation to t which contradicts Corollary 1. □

2.2 On the b-coloring of cographs and P_4-sparse graphs

2.2.1 Definitions and preliminary results

Given a graph G, we define the *dominance sequence* to be $\text{dom}_G \in \mathbb{Z}^{\mathbb{N}_{\ge \chi(G)}}$, such that $\text{dom}_G[t]$ is the maximum number of distinct color classes admitting dominant vertices in any coloring of G with t colors, for every $t \ge \chi(G)$. Note that it suffices to consider this sequence until $t = |V(G)|$, since $\text{dom}_G[t] = 0$ for $t > |V(G)|$. The algorithmic treatment of this sequence will be based on this observation, i.e., we shall consider the *dominance vector* $(\text{dom}_G[\chi(G)], \ldots, \text{dom}_G[|V(G)|])$ instead of the whole sequence. For example, the dominance vector of the graph in Figure 2.4 is $(3, 3, 2, 0)$.

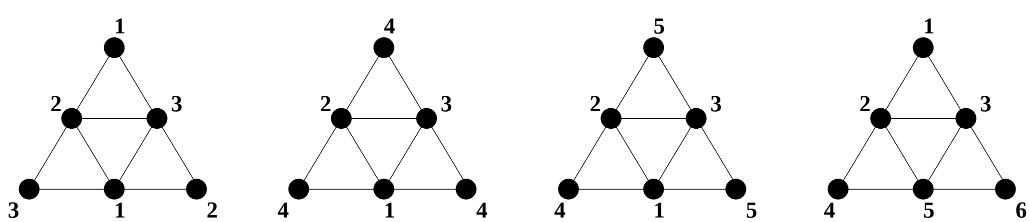


Figure 2.4: A pyramid and its coloring with 3, 4, 5 and 6 colors admitting 3, 3, 2 and 0 distinct color classes with dominant vertices, respectively.

Notice that a graph G admits a b-coloring with t colors if and only if $\text{dom}_G[t] = t$. Moreover, it is clear that $\text{dom}_G[\chi(G)] = \chi(G)$.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with $V_1 \cap V_2 = \emptyset$. The *union* of G_1 and G_2 is the graph $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$, and the *join* of G_1 and G_2 is the graph $G_1 \vee G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup V_1 \times V_2)$ (i.e., $\overline{G_1 \vee G_2} = \overline{G_1} \cup \overline{G_2}$).

A *cograph* is a P_4-free graph [17], i.e. a graph that does not contain a path with four vertices P_4 as an induced subgraph. Many NP-complete problems are polynomial time solvable on cographs; but there are some expections, e.g. achromatic number [6], list coloring [50], etc. The b-coloring problem on cographs was studied in [54], where b-perfect cographs have been characterized. Nevertheless, the complexity of computing the b-chromatic number of a cograph was not known. Cographs have a really nice structure, since they admit a fully decomposition theorem.

Proposition 3 [17] *Every non-trivial cograph is either union or join of two smaller cographs.*

To each cograph G one can associate a corresponding decomposition rooted tree T, called the *cotree* of G, in the following way. Each non-leaf node in the tree is labeled with either "$\cup$" (union-nodes) or "$\vee$" (join-nodes) and each leaf is labeled with a vertex of G. Each non-leaf node has two or more children. Let T_x be the subtree of T rooted at node x and let V_x be the set of vertices corresponding to the leaves in T_x. Then, each node x of the cotree corresponds to the graph $G_x = (V_x, E_x)$. An union-node (join-node) corresponds to the disjoint union (join) of the cographs associated with the children of the node. Finally, the cograph that is associated with the root of the cotree is just G, the cograph represented by this cotree. In the sequel, we assume that the union and join nodes in the cotree alternate on each path from a leaf to the root. The cotree associated to a cograph can be computed in linear time [18].

The chromatic number of a cograph can be recursively calculated from its cotree by applying the following result.

Theorem 5 [18] *If G is the trivial graph, then $\chi(G) = 1$. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs such that $V_1 \cap V_2 = \emptyset$. Then,*

(i) $\chi(G_1 \cup G_2) = \max\{\chi(G_1), \chi(G_2)\}$.

(ii) $\chi(G_1 \vee G_2) = \chi(G_1) + \chi(G_2)$.

For b-coloring there is a similar result, but the relation between the b-chromatic number of two graphs and the b-chromatic number of their union is weaker.

Theorem 6 [54] *If G is the trivial graph, then $\chi_b(G) = 1$. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs such that $V_1 \cap V_2 = \emptyset$. Then,*

(i) $\chi_b(G_1 \cup G_2) \geq \max\{\chi_b(G_1), \chi_b(G_2)\}$.

(ii) $\chi_b(G_1 \vee G_2) = \chi_b(G_1) + \chi_b(G_2)$.

The graph H in Figure 2.2 is an example of a graph verifying the strict inequality in Theorem 6: $\chi_b(H_1) = \chi_b(H_2) = 3$, but $\chi_b(H) = 4$.

A *spider* is a graph whose vertex set can be partitioned into S, C and R, where $S = \{s_1, \ldots, s_k\}$ ($k \geq 2$) is a stable set; $C = \{c_1, \ldots, c_k\}$ is a complete set; s_i is adjacent to c_j if and only if $i = j$ (a *thin spider*), or s_i is adjacent to c_j if and only if $i \neq j$ (a *thick* *spider*); R is allowed to be empty and if it is not, then all the vertices in R are adjacent to all the vertices in C and non-adjacent to all the vertices in S. Clearly, the complement of a thin spider is a thick spider, and viceversa. The triple (S, C, R) is called the *spider partition*, and can be found in linear time [46].

A graph is P_4-*sparse* if every 5-vertex subset contains at most one P_4. P_4-sparse graphs were introduced in [41], they generalize cographs, can be recognized in linear time [46], and are a subclass of perfect graphs. Besides, b-perfect P_4-sparse graphs have been characterized in [42]. P_4-sparse graphs have also a nice decomposition theorem.

Theorem 7 [41, 47] *If G is a non-trivial P_4-sparse graph, then either G or $\overline{G}$ is not connected, or G is a spider.*

In fact, Theorem 7 say that if G is a non-trivial P_4-sparse graph, then either (i) $G_1, \ldots, G_p$ ($p > 1$) are the connected components of G ($\overline{G}$) and G is the disjoint union (join) of G_i's ($\overline{G_i}$'s), or (ii) G and $\overline{G}$ are connected and G is a spider.

Let G be a graph and $A \subset V(G)$. Denote by $G[A]$ the subgraph of G induced by A. In [47] is observed that if G is a spider with vertex partition (S, C, R), then G is P_4-sparse if and only if $G[R]$ is P_4-sparse.

In [48], it is implicitly stated the following lemma, which allows to compute the chromatic number of a P_4-sparse graph recursively in linear time.

Lemma 1 [48] *Let G be a spider with spider partition (S, C, R). If R is empty, then $\chi(G) = |C|$. Otherwise, $\chi(G) = |C| + \chi(G[R])$.*

The algorithm is based on the decomposition theorem and the recognition algorithm, which finds the decomposition tree in linear time.

2.2.2 b-continuity in cographs

Minimally b-imperfect cographs, i.e., graphs G such that $\chi_b(G) > \chi(G)$ but $\chi_b(H) = \chi(H)$ for every proper induced subgraph H of G, are characterized in [54]. Such graphs are the disjoint union of two diamonds and the disjoint union of three P_3. In both cases $\chi_b(G) = \chi(G) + 1$ holds. It is natural to ask whether there exist cographs with a bigger difference between their chromatic number and their b-chromatic number.

Let B_n be the graph composed by $n+1$ copies of the star $K_{1,n}$. We have that $\chi(B_n) = 2$ and $\chi_b(B_n) = n + 1$. A b-coloring with $n + 1$ colors is obtained by coloring each of the $n + 1$ central vertices with a different color and, for every star, coloring each of the n non-central vertices with a different color (such that this color does not coincide with the color assigned to the corresponding central vertex). In such a coloring, all the central vertices are $(n + 1)$-dominant, and each color class admits a dominant vertex.

Since there are cographs with arbitrarily large difference between their b-chromatic number and their chromatic number, it makes sense to analyze b-continuity in cographs.

Lemma 2 *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs such that $V_1 \cap V_2 = \emptyset$. If G_1 and G_2 are b-continuous and $G = G_1 \cup G_2$, then G is b-continuous.*

Proof. Assume G admits a b-coloring with $t+1$ colors such that $t+1 > \chi(G)$. We shall show that there exists a b-coloring of G with t colors. Since $\chi(G) = \max\{\chi(G_1), \chi(G_2)\}$ by Theorem 5, then $t+1 > \chi(G_i)$ for $i = 1, 2$. We are going to eliminate color $t+1$ and obtain a b-coloring of G with t colors. To this end, consider the following cases for $i = 1, 2$:

(a) If G_i does not admit dominant vertices assigned the color $t+1$, recolor every vertex $v \in G_i$ receiving color $t+1$ with some color between 1 and t not used by any neighbor of v.

(b) If G_i admits dominant vertices assigned the color $t+1$ but no dominant vertex assigned color j for some $j \neq t+1$, then swap the colors $t+1$ and j, and then resort to Case (a), since now there are no dominant vertices with color $t+1$ in G_i.

(c) If G_i admits dominant vertices for every color, then we have a b-coloring of G_i with $t+1$ colors, where $t+1 > \chi(G_i)$. Since G_i is b-continuous, there exists a b-coloring of G_i with t colors.

After these operations, it is clear that the resulting coloring is a b-coloring of G with t colors (since we have yet dominant vertices with every color from 1 to t) . □

Lemma 3 *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs such that $V_1 \cap V_2 = \emptyset$. If G_1 and G_2 are b-continuous and $G = G_1 \vee G_2$, then G is b-continuous.*

Proof. Assume G admits a b-coloring with $t+1$ colors such that $t+1 > \chi(G)$. We shall show that there exists a b-coloring of G with t colors. We have that $\chi(G) = \chi(G_1) + \chi(G_2)$ and $\chi_b(G) = \chi_b(G_1) + \chi_b(G_2)$ by Theorems 5 and 6. Furthermore, any b-coloring of G_1 and G_2 generates a b-coloring of G by renaming the colors assigned to G_2 starting by the largest color assigned to G_1 plus one, and any b-coloring of G restricted to G_1 (resp. G_2) is also a b-coloring. Therefore, in the b-coloring of G with $t+1$ colors, either G_1 or G_2 (perhaps both) is colored with more colors than its chromatic number. Suppose without loss of generality that this is the case for G_1. By restricting the coloring of G to G_1, we obtain a b-coloring of G_1 with $k+1$ colors such that $k+1 > \chi(G_1)$. Since G_1 is b-continuous, there exists a b-coloring of G_1 with k colors. Combine this coloring with the original b-coloring of G restricted to G_2, thus constructing a b-coloring of G with t colors. □

Theorem 8 *Cographs are b-continuous.*

Proof. We proceed by induction, using Proposition 3, Lemma 2, and Lemma 3, since the trivial graph is b-continuous. □

2.2.3 A polynomial time algorithm for b-coloring cographs

Theorem 6 does not lead to an algorithm to compute the b-chromatic number of a cograph. In fact, it is not difficult to build examples showing that the b-chromatic number of the graph $G_1 \cup G_2$ does not depend only on the b-chromatic numbers of G_1 and G_2. For this reason, we introduce the notion of dominance vector. Our goal is to recursively compute this vector using the decomposition theorem for cographs, hence obtaining the b-chromatic number of the graph as the maximum t such that $\text{dom}_G[t] = t$.

Theorem 9 *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs such that $V_1 \cap V_2 = \emptyset$. If $G = G_1 \cup G_2$ and $t \geq \chi(G)$, then*

$$\text{dom}_G[t] = \min\{t, \text{dom}_{G_1}[t] + \text{dom}_{G_2}[t]\}.$$

Proof. Let $t \geq \chi(G)$. If $t > |V(G)|$, then $t > |V(G_1)|$ and $t > |V(G_2)|$, hence $\text{dom}_G[t] = 0 = \min\{t, \text{dom}_{G_1}[t] + \text{dom}_{G_2}[t]\}$. If $t \leq |V(G)|$, we take a coloring of G with t colors and $\text{dom}_G[t]$ color classes with dominant vertices. Let a_1 be the number of color classes with dominant vertices in G_1, and let a_2 be the number of color classes with dominant vertices in G_2 not having dominant vertices in G_1. Then $\text{dom}_G[t] = a_1 + a_2$. Notice that, for $i = 1, 2$, if $a_i > 0$ then the t colors are used in G_i. Therefore, by restricting the coloring to G_1 (resp. G_2) we obtain $\text{dom}_{G_1}[t] \geq a_1$ and $\text{dom}_{G_2}[t] \geq a_2$, so $\text{dom}_G[t] \leq \text{dom}_{G_1}[t] + \text{dom}_{G_2}[t]$. Since clearly $\text{dom}_G[t] \leq t$, we conclude $\text{dom}_G[t] \leq \min\{t, \text{dom}_{G_1}[t] + \text{dom}_{G_2}[t]\}$.

On the other hand, since $t \geq \chi(G)$ then $t \geq \chi(G_1)$ and $t \geq \chi(G_2)$. If $t > |V(G_1)|$ and $t > |V(G_2)|$, then $\text{dom}_G[t] = 0 = \min\{t, \text{dom}_{G_1}[t] + \text{dom}_{G_2}[t]\}$. If $t > |V(G_1)|$ but $t \leq |V(G_2)|$, then $\text{dom}_{G_1}[t] = 0$ and $\text{dom}_G[t] = \text{dom}_{G_2}[t] = \min\{t, \text{dom}_{G_1}[t] + \text{dom}_{G_2}[t]\}$ holds. If $t \leq |V(G_1)|$ and $t \leq |V(G_2)|$, take a coloring of G_1 (resp. a coloring of G_2) with t colors and $\text{dom}_{G_1}[t]$ (resp. $\text{dom}_{G_2}[t]$) color classes with dominant vertices. If $\text{dom}_{G_1}[t] + \text{dom}_{G_2}[t] \leq t$, then we can rename the colors in G_2 in such a way that the dominant vertices use $\text{dom}_{G_2}[t]$ color classes differing from the $\text{dom}_{G_1}[t]$ color classes with dominant vertices in G_1. This implies $\text{dom}_G[t] \geq \text{dom}_{G_1}[t] + \text{dom}_{G_2}[t] = \min\{t, \text{dom}_{G_1}[t] + \text{dom}_{G_2}[t]\}$. If $\text{dom}_{G_1}[t] + \text{dom}_{G_2}[t] > t$, then we can rename the colors in G_2 in such a way that the dominant vertices use the $t - \text{dom}_{G_1}[t]$ color classes differing from the $\text{dom}_{G_1}[t]$ color classes with dominant vertices in G_1 plus some additional colors. We conclude, therefore, that $\text{dom}_G[t] \geq t = \min\{t, \text{dom}_{G_1}[t] + \text{dom}_{G_2}[t]\}$. □

Theorem 10 *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs such that $V_1 \cap V_2 = \emptyset$. Let $G = G_1 \vee G_2$ and $\chi(G) \leq t \leq |V(G)|$. Let $a = \max\{\chi(G_1), t - |V(G_2)|\}$ and $b = \min\{|V(G_1)|, t - \chi(G_2)\}$. Then $a \leq b$ and*

$$\text{dom}_G[t] = \max_{a \leq j \leq b}\{\text{dom}_{G_1}[j] + \text{dom}_{G_2}[t - j]\}.$$

Proof. We will show first four inequalities that imply $a \leq b$. By Theorem 5, $\chi(G_1) + \chi(G_2) = \chi(G) \leq t$, so $\chi(G_1) \leq t - \chi(G_2)$; on the other hand, $\chi(G_1) \leq |V(G_1)|$ and $\chi(G_2) \leq |V(G_2)|$, so $t - |V(G_2)| \leq t - \chi(G_2)$; finally, $t \leq |V(G)| = |V(G_1)| + |V(G_2)|$, so $t - |V(G_2)| \leq |V(G_1)|$.

Consider any t-coloring of G with $\text{dom}_G[t]$ color classes with dominant vertices. Let t_1 (resp. t_2) be the number of colors used by this coloring in G_1 (resp. G_2), and let a_1 (resp. a_2) be the number of color classes with dominant vertices in G_1 (resp. G_2). Notice that any coloring of G assigns disjoint color sets to G_1 and G_2, so $t = t_1 + t_2$ and $\text{dom}_G[t] = a_1 + a_2$. Since the t_1 colors from G_1 are not used in G_2, the t-dominant vertices in G which are in G_1 are t_1-dominant vertices in the coloring restricted to G_1, hence $a_1 \leq \text{dom}_{G_1}[t_1]$. Similarly, the t-dominant vertices in G which are in G_2 are t_2-dominant vertices in the coloring restricted to G_2. Since $t_2 = t - t_1$, we obtain $a_2 \leq \text{dom}_{G_2}[t-t_1]$, implying $\text{dom}_G[t] = a_1 + a_2 \leq \text{dom}_{G_1}[t_1] + \text{dom}_{G_2}[t-t_1]$. As $t_1 \geq \chi(G_1)$, $t_1 \geq t - |V(G_2)|$, $t_1 \leq |V(G_1)|$ and $t_1 \leq t - \chi(G_2)$, then $a \leq t_1 \leq b$. Therefore, we have $\text{dom}_G[t] \leq \max_{a \leq j \leq b}\{\text{dom}_{G_1}[j] + \text{dom}_{G_2}[t-j]\}$.

Consider any t_1 such that $a \leq t_1 \leq b$. Since $\chi(G_1) \leq t_1 \leq |V(G_1)|$, there exists some coloring of G_1 with exactly t_1 colors. Take any such coloring having $\text{dom}_{G_1}[t_1]$ color classes with dominant vertices. Let $t_2 = t - t_1$. Since $\chi(G_2) \leq t_2 \leq |V(G_2)|$, there exists some coloring of G_2 with exactly t_2 colors. Take any such coloring having $\text{dom}_{G_2}[t_2]$ color classes with dominant vertices, and rename these t_2 colors in such a way that only colors in $\{t_1 + 1, \ldots, t\}$ are used in the new coloring. By combining these two colorings for G_1 and G_2, we obtain a coloring of G with exactly t colors. Each t_1-dominant vertex in G_1 has in G all the vertices in G_2 as neighbors, hence it admits a neighbor with every color in $\{t_1 + 1, \ldots, t\}$ and, therefore, it is a t-dominant vertex in G. Similarly, each t_2-dominant vertex in G_2 can be shown to be t-dominant in G. Conversely, every t-dominant vertex in G is either t_1-dominant in G_1 or t_2-dominant in G_2. Moreover, since the color sets corresponding to G_1 and G_2 are disjoint, the number of t-dominant vertices in such a coloring of G is $\text{dom}_{G_1}[t_1] + \text{dom}_{G_2}[t_2]$. Therefore, $\text{dom}_G[t] \geq \max_{a \leq j \leq b}\{\text{dom}_{G_1}[j] + \text{dom}_{G_2}[t-j]\}$. □

Theorem 11 *The dominance vector and the b-chromatic number of a cograph can be computed in $O(n^3)$ time.*

Proof. The previous results give a dynamic programming algorithm to compute the dominance vector of a cograph from its cotree. If $G = G_1 \cup G_2$ (as in Theorem 9) the value of $\text{dom}_G[t]$ is obtained directly from $\text{dom}_{G_1}[t]$ and $\text{dom}_{G_2}[t]$. If $G = G_1 \vee G_2$ (as in Theorem 10), then at most n values of j must be examined. Moreover, each of these two theorems reduces the computation of $\text{dom}_G[t]$ to the computation on two disjoint subgraphs. Thus, there are at most n occurrences of such reduction steps. In total, the computation time is $O(n^2)$ for every value of t, and so $O(n^3)$ for all possible values of t. From the dominance vector of a graph G, the b-chromatic number can be computed easily as the maximum t such that $\text{dom}_G[t] = t$. □

2.2.4 b-monotonicity in cographs

The monotonicity on induced subgraphs is a desirable property that holds for many known optimization parameters of a graph, like chromatic number, maximum clique, maximum degree. This is not the case of the b-chromatic number in general, so it is interesting to analyze the monotonicity of the b-chromatic number within different classes of graphs. In this section we study the b-monotonicity of cographs. We first state some preliminary properties of the dominance vector of a graph, and then use the decomposition theorem to analyze the b-monotonicity in this class of graphs.

Lemma 4 *If G is a graph and $t \geq \chi(G)$, then either* $\mathrm{dom}_G[t+1] = t+1$ *or* $\mathrm{dom}_G[t+1] \leq \mathrm{dom}_G[t]$.

Proof. If $t+1 > |V(G)|$, then $0 = \mathrm{dom}_G[t+1] \leq \mathrm{dom}_G[t]$. Assume, therefore, $t+1 \leq |V(G)|$ and $\mathrm{dom}_G[t+1] < t+1$. Take any coloring of G with $t+1$ colors having $\mathrm{dom}_G[t+1]$ color classes with dominant vertices. Since $\mathrm{dom}_G[t+1] < t+1$, there exists some color class with no dominant vertices, say the color $t+1$. For every vertex v with color $t+1$, change the color of v to any color in $\{1,\ldots,t\}$ not used by any of the neighbors of v. The resulting coloring is a coloring of G with t colors. Note that every dominant vertex in the original coloring is dominant in the new coloring, and that the number of color classes with dominant vertices in the new coloring is at least the same. Therefore, $\mathrm{dom}_G[t] \geq \mathrm{dom}_G[t+1]$. □

A direct consequence of this lemma is the following.

Corollary 2 *Let G be a graph. The maximum value of* $\mathrm{dom}_G[t]$ *is attained in $t = \chi_b(G)$.*

Lemma 5 *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs such that $V_1 \cap V_2 = \emptyset$, and let $G = G_1 \cup G_2$. Assume that for every $t \geq \chi(G_i)$ and every induced subgraph H of G_i we have* $\mathrm{dom}_H[t] \leq \mathrm{dom}_{G_i}[t]$, *for $i = 1, 2$. Then, for every $t \geq \chi(G)$ and every induced subgraph H of G,* $\mathrm{dom}_H[t] \leq \mathrm{dom}_G[t]$ *holds.*

Proof. Let H be an induced subgraph of G and let $t \geq \chi(G)$. By Theorem 9, we have $\mathrm{dom}_G[t] = \min\{t, \mathrm{dom}_{G_1}[t] + \mathrm{dom}_{G_2}[t]\}$. If $\mathrm{dom}_G[t] = t$, then $\mathrm{dom}_H[t] \leq \mathrm{dom}_G[t]$ clearly holds. Assume, therefore, $\mathrm{dom}_G[t] = \mathrm{dom}_{G_1}[t] + \mathrm{dom}_{G_2}[t]$. If H is completely contained in G_i, for $i = 1$ or $i = 2$, then $\mathrm{dom}_H[t] \leq \mathrm{dom}_{G_i}[t] \leq \mathrm{dom}_G[t]$. Otherwise, $H = H_1 \cup H_2$, where H_i is an induced subgraph of G_i, for $i = 1, 2$. By the hypothesis, $\mathrm{dom}_{H_i}[t] \leq \mathrm{dom}_{G_i}[t]$, hence $\mathrm{dom}_{H_1}[t] + \mathrm{dom}_{H_2}[t] \leq \mathrm{dom}_{G_1}[t] + \mathrm{dom}_{G_2}[t]$. Therefore, we conclude $\mathrm{dom}_H[t] \leq \mathrm{dom}_G[t]$. □

Lemma 6 *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two b-continuous graphs such that $V_1 \cap V_2 = \emptyset$, and let $G = G_1 \vee G_2$. Assume that for every $t \geq \chi(G_i)$ and for every induced subgraph H of G_i we have* $\mathrm{dom}_H[t] \leq \mathrm{dom}_{G_i}[t]$, *for $i = 1, 2$. Then, for every $t \geq \chi(G)$ and for every induced subgraph H of G,* $\mathrm{dom}_H[t] \leq \mathrm{dom}_G[t]$ *holds.*

Proof. Let H be an induced subgraph of G, and let $t \geq \chi(G)$. By hypothesis, G_1 and G_2 are b-continuous. So, by Theorem 3, we have that G is b-continuous. Hence it suffices to consider $t > \chi_b(G)$, otherwise $t = \mathrm{dom}_G[t] \geq \mathrm{dom}_H[t]$. Recall that, by Theorem 6, $\chi_b(G) = \chi_b(G_1) + \chi_b(G_2)$, so $t > \chi_b(G_1) + \chi_b(G_2)$. By Theorem 10, we have

$\text{dom}_G[t] = \max_{a \le j \le b}\{\text{dom}_{G_1}[j] + \text{dom}_{G_2}[t-j]\}$, where $a = \max\{\chi(G_1), t - |V(G_2)|\}$ and $b = \min\{|V(G_1)|, t - \chi(G_2)\}$, and $a \le b$ holds.

If H is completely contained in G_1 or G_2, say G_1, by the hypothesis we have $\text{dom}_H[t] \le \text{dom}_{G_1}[t]$. Let $j' = \max\{a, \chi_b(G_1)\}$. We know that $a \le b$, and $\chi_b(G_1) \le |V(G_1)|$. Furthermore, $\chi_b(G_1) < t - \chi_b(G_2) \le t - \chi(G_2)$, hence $a \le j' \le b$. Finally, $t > \chi_b(G_1) \ge \chi(G_1)$ and clearly $t \ge t - |V(G_2)|$, hence $t \ge j'$. Since $t \ge j' \ge \chi_b(G_1)$ and $\text{dom}_{G_1}[t] < t$, by Lemma 4, $\text{dom}_{G_1}[t] \le \text{dom}_{G_1}[j']$, and since $\text{dom}_{G_2}[t-j'] \ge 0$, we have $\text{dom}_G[t] = \max_{a \le j \le b}\{\text{dom}_{G_1}[j] + \text{dom}_{G_2}[t-j]\} \ge \text{dom}_{G_1}[j'] + \text{dom}_{G_2}[t-j'] \ge \text{dom}_{G_1}[j'] \ge \text{dom}_{G_1}[t] \ge \text{dom}_H[t]$. Therefore, $\text{dom}_H[t] \le \text{dom}_G[t]$.

If H is not completely contained in G_1 or G_2, then $H = H_1 \vee H_2$, where H_i is an induced subgraph of G_i, for $i = 1, 2$. By the hypothesis, $\text{dom}_{H_i}[j] \le \text{dom}_{G_i}[j]$ for each $j \ge \chi(G_i)$. By Theorem 10, we have $\text{dom}_H[t] = \max_{a' \le j \le b'}\{\text{dom}_{H_1}[j] + \text{dom}_{H_2}[t-j]\}$, where $a' = \max\{\chi(H_1), t - |V(H_2)|\}$ and $b' = \min\{|V(H_1)|, t - \chi(H_2)\}$, and $a' \le b'$ holds. Let $j' \in \{a', \dots, b'\}$ be the color realizing such maximum, and consider the following three possible cases:

(a) If $a \le j' \le b$, then $\text{dom}_{H_1}[j'] \le \text{dom}_{G_1}[j']$ and $\text{dom}_{H_2}[t-j'] \le \text{dom}_{G_2}[t-j']$, hence $\text{dom}_G[t] = \max_{a \le j \le b}\{\text{dom}_{G_1}[j] + \text{dom}_{G_2}[t-j]\} \ge \text{dom}_{G_1}[j'] + \text{dom}_{G_2}[t-j'] \ge \text{dom}_{H_1}[j'] + \text{dom}_{H_2}[t-j'] = \text{dom}_H[t]$.

(b) If $j' < a$, then in particular $a' < a$ and, since $t - |V(H_2)| \ge t - |V(G_2)|$, we have $a = \chi(G_1)$. Therefore, $\text{dom}_{H_1}[j'] \le j' < a = \text{dom}_{G_1}[a]$. Since $t > \chi_b(G_1) + \chi_b(G_2)$ and $a \le \chi_b(G_1)$, it holds $t - a > \chi_b(G_2)$. Since $t - j' > t - a > \chi_b(G_2)$, then Lemma 4 implies $\text{dom}_{G_2}[t-j'] \le \text{dom}_{G_2}[t-a]$. Finally, as $\text{dom}_{H_2}[t-j'] \le \text{dom}_{G_2}[t-j']$, we obtain $\text{dom}_G[t] = \max_{a \le j \le b}\{\text{dom}_{G_1}[j] + \text{dom}_{G_2}[t-j]\} \ge \text{dom}_{G_1}[a] + \text{dom}_{G_2}[t-a] \ge \text{dom}_{H_1}[j'] + \text{dom}_{H_2}[t-j'] = \text{dom}_H[t]$.

(c) If $j' > b$ the argumentation is similar. We have $b' > b$ and, since $|V(H_1)| \le |V(G_1)|$, we have $b = t - \chi(G_2)$. Therefore, $\text{dom}_{H_2}[t-j'] \le t - j' < t - b = \chi(G_2) = \text{dom}_{G_2}[t-b]$. Following the same argument as in Case (b), we conclude that $\text{dom}_{H_1}[j'] \le \text{dom}_{G_1}[b]$, hence $\text{dom}_G[t] = \max_{a \le j \le b}\{\text{dom}_{G_1}[j] + \text{dom}_{G_2}[t-j]\} \ge \text{dom}_{G_1}[b] + \text{dom}_{G_2}[t-b] \ge \text{dom}_{H_1}[j'] + \text{dom}_{H_2}[t-j'] = \text{dom}_H[t]$.

In the three cases we obtain $\text{dom}_H[t] \le \text{dom}_G[t]$. □

Theorem 12 *Cographs are b-monotonic.*

Proof. As cographs are hereditary, it is enough to prove that given a cograph G, $\chi_b(G) \ge \chi_b(H)$, for every induced subgraph H of G. By applying Proposition 3, Theorem 8, Lemma 5, and Lemma 6, an induction argument shows that for every cograph G, every $t \ge \chi(G)$, and every induced subgraph H of G, $\text{dom}_H[t] \le \text{dom}_G[t]$ holds. Let G be a cograph, and let H be an induced subgraph of G. If $\chi_b(H) < \chi(G)$, then $\chi_b(H) < \chi_b(G)$. Otherwise, $\chi_b(H) = \text{dom}_H[\chi_b(H)] \le \text{dom}_G[\chi_b(H)]$, and by Corollary 2 $\text{dom}_G[\chi_b(H)] \le \text{dom}_G[\chi_b(G)] = \chi_b(G)$. Hence $\chi_b(G) \ge \chi_b(H)$. □

2.2.5 P_4-sparse graphs

In this section we extend the results about cographs to a superclass of them: P_4-sparse graphs. For the b-chromatic number of a spider, a result similar to Lemma 1 can be proved.

Lemma 7 *Let G be a spider with spider partition (S, C, R). If R is empty then $\chi_b(G) = |C|$. Otherwise, $\chi_b(G) = |C| + \chi_b(G[R])$.*

Proof. Let G be a spider with spider partition (S, C, R), where $|S| = |C| = k \geq 2$. If R is empty then $\chi(G) = k$, and the vertices in S have degree at most $k-1$, thus they cannot be dominant in a coloring with more than k colors. So, $\chi_b(G) = k = |C|$. Assume now that R is non-empty. Then, by Lemma 1, $\chi_b(G) \geq \chi(G) = k + \chi(G[R]) \geq k+1$. Any b-coloring of $G[R]$ with p colors generates a b-coloring of G with $p+k$ colors, by using k new colors on C and coloring each vertex in S with a color used by a non-neighbor of it in C, thus $\chi_b(G) \geq k + \chi_b(G[R])$; conversely, any b-coloring of G with t colors, when restricted to $G[R]$ is also a b-coloring with $t-k$ colors, since the color sets used in C and R are disjoint and vertices in S cannot be dominant in a coloring with more than k colors, so $\chi_b(G) \leq k + \chi_b(G[R])$. Hence, the lemma holds. □

Nevertheless, in order to compute recursively the b-chromatic number of a P_4-sparse graph, we will need to calculate the dominance vector of a spider instead.

Theorem 13 *Let G be a spider with spider partition (S, C, R), and $k = |S| = |C| \geq 2$.*

(a) *If R is empty and G is a thin spider, then* $\text{dom}_G[k] = \text{dom}_G[k+1] = k$*, and* $\text{dom}_G[j] = 0$ *for* $j > k+1$.

(b) *If R is non-empty and G is a thin spider, then* $\text{dom}_G[k+r] = k + dom_{G[R]}[r]$ *for* $\chi(G[R]) \leq r \leq |R|$, $\text{dom}_G[k+|R|+1] = k$*, and* $\text{dom}_G[j] = 0$ *for* $j > k+|R|+1$.

(c) *If R is empty and G is a thick spider, then* $\text{dom}_G[k+s] = \min\{k, 2k-2s\}$ *for* $0 \leq s \leq k$*, and* $\text{dom}_G[j] = 0$ *for* $j > 2k$.

(d) *If R is non-empty and G is a thick spider, then* $\text{dom}_G[k+r] = k + dom_{G[R]}[r]$ *for* $\chi(G[R]) \leq r \leq |R|$, $\text{dom}_G[k+|R|+s] = \min\{k, 2k-2s\}$ *for* $1 \leq s \leq k$*, and* $\text{dom}_G[j] = 0$ *for* $j > 2k+|R|$.

Proof. Let G be a spider with spider partition (S, C, R), and $k = |S| = |C| \geq 2$. Let $C = \{c_1, \ldots, c_k\}$ and $S = \{s_1, \ldots, s_k\}$.

(a) If R is empty and G is a thin spider, then $\chi(G) = k$, implying $\text{dom}_G[k] = k$. The k vertices in C have degree k and the vertices in S have degree 1, hence $\text{dom}_G[k+1] \leq k$ and $\text{dom}_G[j] = 0$ for $j > k+1$, since G does not admit any vertex with degree at least $k+1$. Finally, a coloring of G with $k+1$ colors and k colors with dominant vertices can be obtained by assigning colors 1 to k to the vertices in C, and color $k+1$ to the vertices in S.

(b) If R is non-empty and G is a thin spider, then, by Lemma 1, $\chi(G) = k + \chi(G[R])$, implying dom$_G[k + \chi(G[R])] = k + \chi(G[R])$. The k vertices $c_1, \ldots, c_k$ in C have degree $k + |R|$, the vertices in R have degree at most $k + |R| - 1$ and the vertices in S have degree 1, hence dom$_G[k + |R| + 1] \leq k$ and dom$_G[j] = 0$ for $j > k + |R| + 1$. On the other hand, a coloring of G with $k + |R| + 1$ colors and k color classes with dominant vertices can be obtained by assigning the colors $1, \ldots, k$ to the vertices in C, the colors $k+1, \ldots, k+|R|$ and the color $k+|R|+1$ to the vertices of S. For $\chi(G[R]) < r \leq |R|$, in a coloring with $k + r$ colors, the vertices of S cannot be dominant. Moreover, if they use a color non present in $C \cup R$, then at most the k vertices in C can be dominant. Suppose now that all the colors used in S are also present in $C \cup R$. Then all the vertices in C are dominant, and they have pairwise different colors. In fact, they are dominant also in the coloring restricted to $G[C \cup R]$ as well as the dominant vertices in R, since there are no edges between R and S. Besides, any coloring of $G[C \cup R]$ can be extended to G without introducing new colors. So, dom$_G[k + r] =$ dom$_{G[C \cup R]}[k + r]$, and, by Theorem 10, dom$_G[k + r] = k + dom_{G[R]}[r]$.

(c) If R is empty and G is a thick spider, then $\chi(G) = k$, implying dom$_G[k] = k$. Furthermore, the vertices in S have degree $k - 1$ and the vertices in C have degree $2k - 2$, hence dom$_G[j] = 0$ for $j \geq 2k$. Finally, for $s = 1, \ldots, k$, the vertices in S cannot be dominant in a coloring of G with $k + s$ colors, thus dom$_G[k + s] \leq k$. In any coloring of G the vertices in C are assigned pairwise different colors, say the colors $1, \ldots, k$. Moreover, the vertex c_i is dominant if and only if the color assigned to s_i is also assigned to some other vertex in G. By symmetry, at least s vertices from S must be assigned the s colors between $k + 1$ and $k + s$, say $s_1, \ldots, s_s$. If $k < 2s$ then at least $s - (k - s)$ of them get a color not used by any other vertex in G. This implies dom$_G[k + s] \leq k - (s - (k - s)) = 2k - 2s$. As in the case $k \geq 2s$ we have $2k - 2s \geq k$ and this already is an upper bound, we obtain dom$_G[k + s] \leq \min\{k, 2k - 2s\}$. A coloring attaining this bound is obtained by assigning the colors $1, \ldots, k$ to the vertices in C, and the colors $k + 1, \ldots, k + s$ to the vertices $s_1, \ldots, s_s$. If $k \geq 2s$, the vertices $s_{s+1}, \ldots, s_{2s}$ receive the colors $k + 1, \ldots, k + s$, and for $i > 2s$, s_i gets the same color as c_i. In this case, we have k color classes with dominant vertices, since every vertex from C is dominant. If $k < 2s$, the vertices $s_{s+1}, \ldots, s_k$ are assigned the colors $k + 1, \ldots, 2k - s$. Here, we get $2k - 2s$ color classes with dominant vertices. Therefore, dom$_G[k + s] = \min\{k, 2k - 2s\}$.

(d) If R is non-empty and G is a thick spider, then, by Lemma 1, $\chi(G) = k + \chi(G[R])$, implying dom$_G[k + \chi(G[R])] = k + \chi(G[R])$. Furthermore, the vertices in S have degree $k - 1$, the vertices in C have degree $2k + |R| - 2$, and the vertices in R have degree at most $k + |R| - 1$ hence dom$_G[j] = 0$ for $j \geq 2k + |R|$. For $s = 1, \ldots, k$, neither the vertices in S nor the vertices in R can be dominant in a coloring of G with $k + |R| + s$ colors, hence dom$_G[k + |R| + s] \leq k$. In any coloring of G, the vertices from C are assigned pairwise different colors, say the colors $1, \ldots, k$. Moreover, the vertex c_i is dominant if and only if the color assigned to s_i is also assigned to some other vertex in G. Since the vertices in R can use at most $|R|$

colors, say $k+1,\dots,k+|R|$, then by symmetry, at least s vertices from S must be assigned the s colors between $k+|R|+1$ and $k+|R|+s$, say $s_1,\dots,s_s$. If $k<2s$, at least $s-(k-s)$ of them are assigned a color not used by any other vertex in G. Therefore, $\text{dom}_G[k+|R|+s] \leq k-(s-(k-s)) = 2k-2s$. As in the case $k \geq 2s$ we have $2k-2s \geq k$ and this already is an upper bound, we obtain $\text{dom}_G[k+|R|+s] \leq \min\{k, 2k-2s\}$. A coloring attaining this bound can be constructed by assigning the colors $1,\dots,k$ to the vertices in C, the colors $k+1,\dots,k+|R|$ to the vertices in R, and the colors $k+|R|+1,\dots,k+|R|+s$ to the vertices $s_1,\dots,s_s$. If $k \geq 2s$, the vertices $s_{s+1},\dots,s_{2s}$ are assigned the colors $k+|R|+1,\dots,k+|R|+s$, and s_i gets the same color as c_i, for $i>2s$. In this case, we obtain k color classes with dominant vertices, since all the vertices in C are dominant. If $k<2s$, the vertices $s_{s+1},\dots,s_k$ are assigned the colors $k+|R|+1,\dots,2k+|R|-s$. In this case, we have $2k-2s$ color classes with dominant vertices. Therefore, $\text{dom}_G[k+1+s] = \min\{k, 2k-2s\}$. For $\text{dom}_G[k+r]$ with $\chi(G[R]) < r \leq |R|$, we can use the same argumentation as in case (b), so $\text{dom}_G[k+r] = \text{dom}_{G[C\cup R]}[k+r] = k + dom_{G[R]}[r]$.

□

Theorem 14 *The dominance vector and the b-chromatic number of a P_4-sparse graph can be computed in $O(n^3)$ time.*

Proof. By combining Theorem 9, Theorem 10, Theorem 7, and Theorem 13, and since P_4-sparse graphs are a hereditary class, we can recursively calculate the dominance vector and, consequently, the b-chromatic number of a P_4-sparse graph in $O(n^3)$ time. The complexity analysis is the same as for Theorem 11, noting that for the base cases (spiders) the computation of $\text{dom}_G[t]$ for each value of t is given directly in the proof of Theorem 13. □

Now, we study the b-continuity on P_4-sparse graphs.

Theorem 15 *P_4-sparse graphs are b-continuous.*

Proof. We proceed by induction, using Theorem 7 and Lemmas 2 and 3. So, it remains to analyze the case of spiders. Suppose that G is a spider P_4-sparse graph, with spider partition (S,C,R), where $|S|=|C|=k\geq 2$. Assume G admits a b-coloring with $t+1$ colors such that $t+1>\chi(G)$. We shall show that there exists a b-coloring of G with t colors. We have that $\chi(G)=k+\chi(G[R])$ and $\chi_b(G)=k+\chi_b(G[R])$ by Lemmas 1 and 7. So, R must be non-empty and $\chi_b(G[R])>\chi(G[R])$. As observed in the proof of Lemma 7, any b-coloring of $G[R]$ with p colors generates a b-coloring of G with $p+k$ colors and, conversely, any b-coloring of G restricted to $G[R]$ is also a b-coloring. Therefore, by restricting the b-coloring of G with $t+1$ colors to $G[R]$, we obtain a b-coloring of $G[R]$ with $t+1-k$ colors, and $t+1-k>\chi(G[R])$. Since P_4-sparse is a hereditary graph class, $G[R]$ is a P_4-sparse graph, and by inductive hypothesis, there exists a b-coloring of $G[R]$ with $t-k$ colors. As observed before, this b-coloring of $G[R]$ generates a b-coloring of G with t colors. □

Finally, we analyze the b-monotonicity on P_4-sparse graphs.

Lemma 8 *Let G be a spider with spider partition (S, C, R). Assume that for every $t \geq \chi(G[R])$ and for every induced subgraph H of $G[R]$ we have $\text{dom}_H[t] \leq \text{dom}_{G[R]}[t]$. Then, for every $t \geq \chi(G)$ and for every induced subgraph H of G, $\text{dom}_H[t] \leq \text{dom}_G[t]$ holds.*

Proof. Let G be a spider with spider partition (S, C, R), where $|S| = |C| = k \geq 2$, and let H be an induced subgraph of G. For convenience, if a graph is empty define its dominance sequence as the zero sequence, beginning at zero. Let R_H be $V(H) \cap R$. By hypothesis, $dom_{G[R_H]}[r] \leq dom_{G[R]}[r]$, for each $r \geq \chi(G[R])$. Following the arguments used in the proof of Theorem 13, it can be seen that:

- If G is a thin spider, then $\text{dom}_H[k+r] \leq k + dom_{G[R_H]}[r] \leq k + dom_{G[R]}[r] = \text{dom}_G[k+r]$ for $\chi(G[R]) \leq r \leq |R|$, $\text{dom}_H[k+|R|+1] \leq k = \text{dom}_G[k+|R|+1]$, and $\text{dom}_H[j] = 0 = \text{dom}_G[j]$ for $j > k+|R|+1$.

- If G is a thick spider, then $\text{dom}_H[k+r] \leq k + dom_{G[R_H]}[r] \leq k + dom_{G[R]}[r] = \text{dom}_G[k+r]$ for $\chi(G[R]) \leq r \leq |R|$, $\text{dom}_H[k+|R|+s] \leq \min\{k, 2k-2s\} = \text{dom}_G[k+|R|+s]$ for $1 \leq s \leq k$, and $\text{dom}_H[j] = 0 = \text{dom}_G[j]$ for $j > 2k+|R|$.

We conclude $\text{dom}_G[t] \geq \text{dom}_H[t]$, for every $t \geq \chi(G)$. □

Theorem 16 *P_4-sparse graphs are b-monotonic.*

Proof. As P_4-sparse graphs are hereditary, it is enough to prove that given a P_4-sparse graph G, $\chi_b(G) \geq \chi_b(H)$, for every induced subgraph H of G. By applying Lemma 5, Theorem 15, Lemma 6, Lemma 8, and Theorem 7, since P_4-sparse graphs is a hereditary class, we can inductively show that for every P_4-sparse graph G, every induced subgraph H of G, and every $t \geq \chi(G)$, $\text{dom}_H[t] \leq \text{dom}_G[t]$ holds. Let G be a P_4-sparse graph, and let H be an induced subgraph of G. If $\chi_b(H) < \chi(G)$, then $\chi_b(H) < \chi_b(G)$. Otherwise, $\chi_b(H) = \text{dom}_H[\chi_b(H)] \leq \text{dom}_G[\chi_b(H)]$ and by Corollary 2, $\text{dom}_G[\chi_b(H)] \leq \text{dom}_G[\chi_b(G)] = \chi_b(G)$ implying that $\chi_b(G) \geq \chi_b(H)$. □

2.3 Conclusions

In Section 2.1, we have shown that the b-chromatic number of a graph is hard to approximate in polynomial time within a factor of $120/113 - \varepsilon$, for any $\varepsilon > 0$, unless P = NP. This is the first and only hardness result for approximating the b-chromatic number. An interesting open problem is the existence of a constant-factor approximation algorithm for the b-chromatic number in general graphs.

In Section 2.2, we have proved that cographs and P_4-sparse graphs are b-continuous and b-monotonic. Besides, we have designed a dynamic programming algorithm to compute the b-chromatic number in polynomial time within these graph classes. One interesting problem is to extend our results to superclasses of these graph families, as for example, the class of distance-hereditary graphs. Finally, it would be an interesting problem to characterize b-monotonic graphs by forbidden induced subgraphs.

Chapter 3

Direct product of some vertex-transitive graphs

The *direct product* $G \times H$ of two graphs G and H is defined by $V(G \times H) = V(G) \times V(H)$, and where two vertices $(u_1, u_2), (v_1, v_2)$ are joined by an edge in $E(G \times H)$ if $\{u_1, v_1\} \in E(G)$ and $\{u_2, v_2\} \in E(H)$. This product is commutative and associative in a natural way (see reference [44] for a detailed description on product graphs). A coloring of $G \times H$ can be easily derived from a coloring of any of its factors, hence $\chi(G \times H) \leq \min\{\chi(G), \chi(H)\}$. One of the outstanding problems in graph theory is a formula concerning the chromatic number of the direct product of any two graphs G and H, called the *Hedetniemi conjecture* [39] (see also [32, 33, 22] and ref.), which states $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$. The inherent difficulty of Hedetniemi's conjecture lies in finding lower bounds for $\chi(G \times H)$. In this paper we prove the Hedetniemi's conjecture to be true in some classes of vertex-transitive graphs.

On the other hand, if I is an independent set of one factor, the pre-image of I under the projection is an independent set of the product. Then, $\alpha(G \times H) \geq \max\{\alpha(G)|H|, \alpha(H)|G|\}$. In this case it is known that the equality does not hold in general. In fact, Jha and Klavžar show in [51] that for any graph G with at least one edge and for any $j \in \mathbb{N}$ there is a graph H such that $\alpha(G \times H) > \max\{\alpha(G).|V(H)|, \alpha(H).|V(G)|\} + j$. In [77], Tardif asks whether $\alpha_k(G \times H) = \max\{\alpha_k(G)|H|, \alpha_k(H)|G|\}$ always holds for vertex-transitive graphs, where $\alpha_k(G)$ is the maximal size of an induced k-colourable subgraph of G.

In other related work, Larose and Tardif investigate in [60] the relationship between projectivity and the structure of maximal independent sets of finite direct products of several copies of the same graph G, being G a circular graph, a Kneser graph or a truncated simplices.

Independence and chromatic properties of circular graphs and Kneser graphs are analyzed using graph homomorphism. An edge-preserving map from $\phi : V(G) \to V(H)$ is called a *homomorphism* from G to H and it is denoted by $\phi : G \to H$. We say that G and H are *homomorphically equivalent* if there exist $\phi : G \to H$ and $\psi : H \to G$. Notice that if there is $\phi : G \to H$ then $\chi(G) \leq \chi(H)$. In particular if G and H are homomorphically equivalent then $\chi(G) = \chi(H)$. The following result is direct.

Lemma 9 *Let G be a graph and let H be an induced subgraph of G. Then, $G \times H$ and H are homomorphically equivalent and therefore, $\chi(G \times H) = \chi(H)$.*

In the context of vertex transitive graphs The "No-Homomorphism" lemma of Albertson and Collins is useful to get bounds on the size of independent sets.

Lemma 10 *(Albertson-Collins [2]) Let G, H be graphs such that H is vertex-transitive and there is a homomorphism $\phi : G \to H$. Then,*

$$\frac{\alpha(G)}{|V(G)|} \geq \frac{\alpha(H)}{|V(H)|}.$$

The chromatic number of a graph G and its independence number are closely related via the inequality

$$\chi(G) \geq \lceil |V(G)|/\alpha(G) \rceil.$$

Let n be a positive integer. We denote by $[n]$ the set $\{0, 1, \ldots, n-1\}$. The complete graph K_n will usually be on the vertex set $[n]$. By using this relation, Lemma 10, and Lemma 9, we can deduce the following well known result.

Corollary 3 *Let $k \geq 2$ be an integer and let $n_1, n_2, \ldots, n_k$ be positive integers. Then,*

$$\alpha\left(\textstyle\prod_i K_{n_i}\right) = \max_i \left\{ \left(\textstyle\prod_j n_j\right)/n_i \right\} \text{ and } \chi(\textstyle\prod_i K_{n_i}) = \min_i \{n_i\},$$

where $1 \leq i, j \leq k$.

A set $S \subseteq V$ is called a *dominating set* if for every vertex $v \in V \setminus S$ there exists a vertex $u \in S$ such that u is adjacent to v. The minimum cardinality of a dominating set in G is called the *domination number* of G and is denoted $\gamma(G)$. A set $S \subseteq V$ is called *independent* if no two vertices in S are adjacent. A set $S \subseteq V$ is called an *independent dominating set* of G if it is both independent and dominating set of G. The minimum cardinality of an independent dominating set in G is called the *independent domination number* of G and is denoted $i(G)$. The *domatic number* $d(G)$ is the maximum order of a partition of V into dominating sets. The domatic number of a graph was introduced by Cockayne and Hedetniemi [16]. A partition of the vertex set V into independent dominating sets is called an *idomatic partition* of G [15, 16]. Clearly, an idomatic partition of a graph G represents a proper coloring of the vertices of G. The maximum order of an idomatic partition of G is called the *idomatic number* $id(G)$. An idomatic partition of a graph G into k parts is called an *idomatic k-partition of* G. Notice that not every graph has an idomatic k-partition, for any k. For example, the cycle graph on five vertices C_5 has no an idomatic k-partition for any k.

Let Γ be a group and C a subset of Γ closed under inverses and identity free. The Cayley graph Cay(Γ, C) is the graph with Γ as its vertex set, two vertices u and v being joined by an edge if and only if $u^{-1}v \in C$. The set C is then called the *connector set* of Cay(Γ, C). Simple examples of Cayley graphs include the cycles, which are Cayley graphs of cyclic groups, and the complete graphs K_n which are Cayley graphs of any group of order n. Cayley graphs constitute a rich class of vertex-transitive graphs (see [32, 33] and references therein).

Let $t \geq 1$ be an integer and let $n_1, n_2, \ldots, n_t$ be positive integers. Notice that the direct product graph $G = K_{n_1} \times K_{n_2} \times \ldots \times K_{n_t}$ can be seen as the Cayley graph of the direct product group $\mathcal{G} = Z_{n_1} \times Z_{n_2} \times \ldots \times Z_{n_t}$ with connector set $[n_1] \setminus \{0\} \times \ldots \times [n_t] \setminus \{0\}$, where Z_{n_i} denotes the additive cyclic group of integers modulo n_i.

Some recent results concerning independence parameters in graphs with connection to direct products graphs and Cayley graphs can be found in [11, 61, 20] (see also references therein).

Idomatic partitions of graphs were studied in [20] as an special coloring problem on graphs defined as *fall colorings*. In this work, the authors show the following result.

Theorem 17 ([20]) *Let $n_1 > 1$ and $n_2 > 1$ be two integers. The direct product graph $K_{n_1} \times K_{n_2}$ admits an idomatic n_1-partition and an idomatic n_2-partition. Furthermore, if $t > 1$ is an integer such that $t \notin \{n_1, n_2\}$, then $K_{n_1} \times K_{n_2}$ has no idomatic t-partition.*

Moreover, in [20] is posed the question of characterizing the idomatic partitions of the direct product of three or more complete graphs.

In this chapter, we study in Section 3.1 the independence and chromatic numbers of finite direct products graphs of circular graphs, Kneser graphs and powers of cycles. In the case of circular and Kneser graphs, this is done via classical homomorphisms. For the direct product graph of powers of cycles, we first analyze its independence number and then we use such a result to compute its chromatic number. This work has been done with the collaboration of Juan Vera (University of Waterloo, Canada) (see reference [80]).

In Section 3.2, we give a full characterization of the idomatic partitions of the direct product of three complete graphs by using an standard algebraic approach, and we show how to use such a characterization in order to construct idomatic partitions of the direct product of four or more complete graphs (see reference [78]).

3.1 Independence and chromatic properties

3.1.1 Circular graphs

Let m, n be integers such that $m \geq 2n > 0$. The *circular graph* C_n^m is the Cayley graph for the cyclic group $\mathbb{Z}_m$ with connector set $\{n, n+1, n+2, \ldots, m-n\}$. These graphs play an important role in the definition of the star chromatic number defined by Vince in [81]. The following result can be easily deduced.

Lemma 11 *Let m, n be integers with $m \geq 2n > 0$. Then, $\alpha(C_n^m) = n$ and $\chi(C_n^m) = \lceil \frac{m}{n} \rceil$.*

Concerning homomorphisms between circular graphs, Bondy and Hell show in [7] the following result.

Lemma 12 *(Bondy-Hell [7]) Let m, n, k be positive integers such that $m \geq 2n$. Then, C_n^m and C_{kn}^{km} are homomorphically equivalent.*

Lemma 13 *Let r, m be positive integers and let $n_1, n_2, \ldots, n_r$ be positive integers such that $n_1 \leq n_2 \leq \ldots \leq n_r$ and $m \geq 2n_i$, for each $i \in [r]$. Then, $C_{n_r}^m$ is a subgraph of the graph $C_{n_1}^m \times C_{n_2}^m \times \ldots \times C_{n_r}^m$.*

Proof : Let $\phi : C_{n_r}^m \to \prod_i C_{n_i}^m$ be the map defined by $x \mapsto (x, x, \ldots, x)$ for all $x \in V(C_{n_r}^m)$. It is easy to deduce that this map is an injective graph homomorphism. □

By Lemma 13 and Lemma 9 we have the following result.

Corollary 4 *Let r, m be positive integers and let $n_1, n_2, \ldots, n_r$ be positive integers such that $m \geq 2n_i$, for each $i \in [r]$. Then, $\chi\left(\prod_i C_{n_i}^m\right) = \min_i\{\chi(C_{n_i}^m)\} = \min_i\left\{\left\lceil \frac{m}{n_i} \right\rceil\right\}$.*

Let $m_1, m_2, \ldots, m_r$ be positive integers, with $r \geq 1$. We denote by $[m_1, m_2, \ldots, m_r]$ the least common multiple of $m_1, m_2, \ldots, m_r$.

Theorem 18 *Let r be a positive integer, and let $m_1, m_2, \ldots, m_r, n_1, n_2, \ldots, n_r$ be positive integers such that $m_i \geq 2n_i$, for each $i \in [r]$. Then, $\chi\left(\prod_i C_{n_i}^{m_i}\right) = \min_i\{\chi(C_{n_i}^{m_i})\}$.*

Proof : Let $m = [m_1, m_2, \ldots, m_r]$ and $k_i = m/m_i$ for each $i \in [r]$. By Lemma 12, for each i, we have $C_{n_ik_i}^m$ homomorphically equivalent to $C_{n_i}^{m_i}$. Therefore $\prod_i C_{n_ik_i}^m$ is homomorphically equivalent to $\prod_i C_{n_i}^{m_i}$. By Corollary 4, we have

$$\chi\left(\prod_i C_{n_i}^{m_i}\right) = \chi\left(\prod_i C_{n_ik_i}^m\right) = \min_i\left\{\chi(C_{n_ik_i}^m)\right\} = \min_i\left\{\left\lceil \frac{m_i}{n_i} \right\rceil\right\} = \min_i\left\{\chi(C_{n_i}^{m_i})\right\}.$$

□

Lemma 14 *Let r, m be positive integers and let $n_1, n_2, \ldots, n_r$ be positive integers such that $m \geq 2n_i$, for each $i \in [r]$. Then, $\alpha\left(\prod_i C_{n_i}^m\right) = m^{r-1}\max_i\{\alpha(C_{n_i}^m)\} = m^{r-1}\max_i\{n_i\}$.*

Proof : W.l.o.g. we can assume that $n_1 \leq n_2 \leq \ldots \leq n_r$. By Lemma 13, the graph $C_{n_r}^m$ is a subgraph of the graph $C_{n_1}^m \times C_{n_2}^m \times \ldots \times C_{n_r}^m$ and thus, there is a natural homomorphism (i.e. the inclusion map) from $C_{n_r}^m$ to $\prod_i C_{n_i}^m$. Moreover, as $\prod_i C_{n_i}^m$ is vertex-transitive, by Lemma 10 we have $\alpha(C_{n_r}^m)/m \geq \alpha\left(\prod_i C_{n_i}^m\right)/m^r$. Therefore,

$$\alpha\left(\prod_i C_{n_i}^m\right) \leq m^{r-1}\alpha(C_{n_r}^m) = m^{r-1}n_r = m^{r-1}\max_i\{n_i\}.$$ □

Theorem 19 *Let r be a positive integer, and let $m_1, m_2, \ldots, m_r, n_1, n_2, \ldots, n_r$ be positive integers such that $m_i \geq 2n_i$, for each $i \in [r]$. Let $M = m_1m_2\ldots m_r$. Then, $\alpha\left(\prod_i C_{n_i}^{m_i}\right) = \max_i\{\alpha(C_{n_i}^{m_i})M/m_i\} = \max_i\{n_iM/m_i\}$.*

Proof : Let $m = [m_1, m_2, \ldots, m_r]$ and let $k_i = m/m_i$ for each $i \in [r]$. By Lemma 12, $\prod_i C_{n_ik_i}^m$ is homomorphically equivalent to $\prod_i C_{n_i}^{m_i}$. Moreover, as $\prod_i C_{n_ik_i}^m$ and $\prod_i C_{n_i}^{m_i}$ are vertex-transitive, by Lemma 10, we have $\alpha(\prod_i C_{n_ik_i}^m)/m^r = \alpha(\prod_i C_{n_i}^{m_i})/M$. Now, by Lemma 14, we have $\alpha(\prod_i C_{n_ik_i}^m) = m^{r-1}\max_i\{n_ik_i\}$. W.l.o.g. we can assume that $n_1k_1 \leq n_2k_2 \leq \ldots \leq n_rk_r$. Therefore, $\alpha(\prod_i C_{n_i}^{m_i}) = n_rk_rM/m = m_1m_2\ldots m_{r-1}n_r = \max_i\{n_iM/m_i\} = \max_i\{\alpha(C_{n_i}^{m_i})M/m_i\}$. □

3.1.2 Kneser graphs

Let m, n be positive integers such that $m \geq 2n$. The *Kneser graph* K_n^m is the graph whose vertices are the n-subsets of $\{0, 1, \ldots, m-1\}$, where two vertices are adjacent if they are disjoint. In a celebrated paper, Lovász shows the following result.

Theorem 20 *(Lovász [62]) The chromatic number of K_n^m is $m - 2n + 2$.*

The independence number of Kneser graphs is related to the following classical inequality.

Theorem 21 *(Erdös-Ko-Rado, [23]) Let m, n be positive integers such that $n < m/2$, and $\mathbb{F}$ a family of pairwise intersecting n-subsets of $[m]$. Then $|\mathbb{F}| \leq \binom{m-1}{n-1}$.*

Theorem 21 implies that the sets $I_k = \{A \in V(K_n^m) : k \in A\}$ are independent sets of maximal cardinality in K_n^m, for $k = 0, 1, \ldots, m-1$. Hilton-Milner [40], show that those are the only independent sets of maximal cardinality in K_n^m. Recently, the diameter of Kneser graphs has been computed in [79].

Concerning homomorphisms between Kneser graphs, Stahl shows the following useful result.

Theorem 22 *(Stahl [75]) Let m, n be integers such that $n > 1$ and $m \geq 2n$. Then, there is an homomorphism from K_n^m to K_{n-1}^{m-2}.*

Lemma 15 *Let n, r be positive integers and let $m_1 \leq m_2 \leq \ldots \leq m_r$ be positive integers such that $m_i \geq 2n$, for $i \in [r]$. Then, $K_n^{m_1}$ is a subgraph of the graph $K_n^{m_1} \times K_n^{m_2} \times \ldots \times K_n^{m_r}$.*

Proof : Let $\Phi : K_n^{m_1} \to \prod_i K_n^{m_i}$ be the map defined by $\Phi(A) = (A, A, \ldots, A)$ for all $A \in V(K_n^{m_1})$. It is clear that this map is an injective homomorphism. □

By Lemma 15, Lemma 9 and Theorem 20 we can deduce the following result.

Corollary 5 *Let n, r be positive integers and let $m_1, m_2, \ldots, m_r$ be positive integers such that $m_i \geq 2n$, for $i \in [r]$. Then, $\chi(\prod_i K_n^{m_i}) = \min_i\{\chi(K_n^{m_i})\} = \min_i\{m_i\} - 2n + 2$.*

Lemma 16 *Let r be a positive integer, and let $m_1, m_2, \ldots, m_r, n_1, n_2, \ldots, n_r$ be positive integers such that $m_i \geq 2n_i$, for $i \in [r]$, and assume that $n_1 \leq n_2 \leq \ldots \leq n_r$, with $n_r > 1$. Then, there is a graph homomorphism $\Phi : \prod_i K_{n_r}^{m_i+2(n_r-n_i)} \to \prod_i K_{n_i}^{m_i}$.*

Proof : By Theorem 22, for each $i \in [r]$, there is a graph homomorphism $\phi_i : K_{n_r}^{m_i+2(n_r-n_i)} \to K_{n_i}^{m_i}$. Therefore, there is a graph homomorphism $\Phi : \prod_i K_{n_r}^{m_i+2(n_r-n_i)} \to \prod_i K_{n_i}^{m_i}$. □

Theorem 23 *Let r be a positive integer, and let $m_1, m_2, \ldots, m_r, n_1, n_2, \ldots, n_r$ be positive integers such that $m_i \geq 2n_i$, for $i \in [r]$. Then, $\chi\left(\prod_i K_{n_i}^{m_i}\right) = \min_i\{\chi(K_{n_i}^{m_i})\}$.*

Proof : W.l.o.g. we can assume that $n_1 \leq n_2 \leq \ldots \leq n_r$, and assume that $n_r > 1$. Then, by Lemma 16, there is a graph homomorphism $\Phi : \prod_i K_{n_r}^{m_i+2(n_r-n_i)} \to \prod_i K_{n_i}^{m_i}$, which implies that $\chi(\prod_i K_{n_i}^{m_i}) \geq \chi\left(\prod_i K_{n_r}^{m_i+2(n_r-n_i)}\right)$. By Corollary 5 we have $\chi\left(\prod_i K_{n_r}^{m_i+2(n_r-n_i)}\right) = \min_i\{m_i + 2(n_r - n_i) - 2n_r + 2\} = \min_i\{m_i - 2n_i + 2\} = \min_i\{\chi(K_{n_i}^{m_i})\}$. □

Let m, n be positive integers such that $m \geq 2n$. The circular graph C_n^m is a subgraph of the Kneser graph K_n^m. More precisely the map $\phi : C_n^m \to K_n^m$ defined by $\phi(u) = \{u, u+1, \ldots, u+n-1\}$ (arithmetic operations are taken modulo m) is an injective graph homomorphism. Notice that the Erdös-Ko-Rado inequality (Theorem 21) can be easily deduced by using the fact that C_n^m is a subgraph of K_n^m, and then, using the No-Homomorphism-Lemma (Lemma 10). In the same way, we can deduce the independence number of the direct product of Kneser graphs, which is a particular case of a more general result of Ahlswede, Aydinian, and Khachatrian [1] in extremal set theory.

Theorem 24 *Let r be a positive integer, and let $m_1, m_2, \ldots, m_r, n_1, n_2, \ldots, n_r$ be positive integers such that $m_i \geq 2n_i$, for $i \in [r]$. Let $N = \prod_i \binom{m_i}{n_i}$. Then,*

$$\alpha\left(\prod_i K_{n_i}^{m_i}\right) = \max_i \left\{\alpha(K_{n_i}^{m_i})N/\binom{m_i}{n_i}\right\}.$$

Proof : We know that for each $i \in [r]$, we have that $C_{n_i}^{m_i}$ is a subgraph of $K_{n_i}^{m_i}$. Therefore, there is a homomorphism from $\prod_i C_{n_i}^{m_i}$ to $\prod_i K_{n_i}^{m_i}$. Let $M = \prod_i m_i$. By Lemma 10, we have $\alpha(\prod_i C_{n_i}^{m_i})/M \geq \alpha(\prod_i K_{n_i}^{m_i})/N$. Moreover, by Theorem 19, $\alpha\left(\prod_i C_{n_i}^{m_i}\right) = \max_i\{n_i M/m_i\}$. Thus, $\alpha(\prod_i K_{n_i}^{m_i}) \leq N \max_i\{n_i/m_i\} = \max_i\{\binom{m_i-1}{n_i-1}N/\binom{m_i}{n_i}\} = \max_i\{\alpha(K_{n_i}^{m_i})N/\binom{m_i}{n_i}\}$, which proves this theorem. □

3.1.3 Powers of cycles

For positive integers n and a such that $n \geq 2a$, we denote by $C(n, a)$ the graph with vertex set $\{0, 1, \ldots, n-1\}$ and edge set $\{ij : i - j \equiv \pm k \mod n, 1 \leq k \leq a\}$; the graph $C(n, a)$ is the a-th power of the n-cycle $C(n, 1)$. Notice that graph $C(n, a)$ is the complement graph of the circular graph C_{a+1}^n. Prowse and Woodall analyze in [72] a restricted coloring problem (the list-coloring problem) on powers of cycles. In particular, they show the following result.

Theorem 25 *(Prowse-Woodall [72]) Let n, a be positive integers such that $a \leq n/2$ and $n = q(a+1) + r$, where $q \geq 1$ and $0 \leq r \leq a$. Then, $\alpha(C(n, a)) = \lfloor \frac{n}{a+1} \rfloor = q$ and $\chi(C(n, a)) = \lceil \frac{n}{\alpha(C(n,a))} \rceil = a + 1 + \lceil \frac{r}{q} \rceil$.*

Let $V_1, V_2, \ldots, V_j$ be a vertex decomposition (i.e. a partition of the vertex set V) of the graph G. Then, it is easy to deduce that $\alpha(G) \leq \sum_i \alpha(G[V_i])$, where, for $1 \leq i \leq j$, $G[V_i]$ denotes the subgraph of G induced by V_i.

Lemma 17 *Let m, n, a be positive integers such that $a \leq n/2$, and let $\alpha = \alpha(C(n,a))$. Then,*

$$\alpha(K_m \times C(n,a)) = \max\{n, m\alpha\}.$$

Proof : Let $n = q(a+1) + r$, with $q \geq 1$ and $0 \leq r \leq a$. By Theorem 25 we have that $\alpha = q$, and thus we need to prove that $\alpha(K_m \times C(n,a)) \leq \max\{n, mq\}$. Let I be a maximal independent set of $K_m \times C(n,a)$. We can assume that $|I| > n$. Otherwise, the lemma trivially holds. Thus, there exists $j \in \{0, \ldots, n-1\}$ such that there are at least two vertices in I with the second coordinate equal to j. As $C(n,a)$ is vertex transitive, we can assume that $j = 0$. As I is an independent set, there is no vertex in I having as second coordinate an integer i, such that $0 < i \leq a$ or such that $n - a \leq i \leq n-1$. Thus, as $0 \leq r \leq a$, we can assume that the remaining vertices of I form an independent set in the induced subgraph $K_m \times C(n,a)[\{a+1, a+2, \ldots, n-r-1\}]$. This induced subgraph admits a vertex decomposition into $q-1$ subgraphs all of them isomorphic to $K_m \times K_{a+1}$. Therefore, by using Corollary 3, we have that $|I| \leq m + (q-1)\alpha(K_m \times K_{a+1}) = m + (q-1)\max\{m, a+1\}$. If $m \geq a+1$ then $|I| \leq mq$. Otherwise, $|I| \leq (a+1)q \leq n$. $\square$

Theorem 26 *For $i = 1, 2$, let n_i, a_i be positive integers such that $n_i \geq 2a_i$, and let $\alpha_i = \alpha(C(n_i, a_i))$. Then,*

$$\alpha(C(n_1, a_1) \times C(n_2, a_2)) = \max\{\alpha_1 n_2, \alpha_2 n_1\}.$$

Proof : For $i = 1, 2$, arithmetic operations on the vertex set of $C(n_i, a_i)$ will be taken modulo n_i. Let $n_i = q_i(a_i+1)+r_i$, with $q_i \geq 1$ and $0 \leq r_i \leq a_i$. By Theorem 25, $\alpha_i = q_i$, for $i = 1, 2$. Let I be a maximal independence set in the graph $C(n_1, a_1) \times C(n_2, a_2)$. We should prove that $|I| \leq \max\{q_1 n_2, q_2 n_1\}$. We define $I_1 = \{x \in I : (x_1 - 1, x_2) \in I$ or $(x_1 + 1, x_2) \in I\}$ and $I_2 = I \setminus I_1$. For $x \in I$ we define $S_x = \{(x_1, x_2 + i) : i = 0, \ldots, a_2\}$ if $x \in I_1$ and $S_x = \{(x_1 + i, x_2) : i = 0, \ldots, a_1\}$ if $x \in I_2$.

Claim 1 *Let $x, y \in I$. If $x \neq y$ then $S_x \cap S_y = \emptyset$.*

Let $x, y \in I$ be such that $x \neq y$. First we show $y \notin S_x$ and $x \notin S_y$. W.l.o.g. assume $y \in S_x$. If $x \in I_1$, then $x_1 = y_1$ and $0 < y_2 - x_2 \leq a_2$. By the maximality of I, $\{(x_1, x_2 + i) : i = 1, \ldots, y_2 - x_2\} \subset I$, contradicting $x \in I_1$. By a similar argument $x \notin I_2$. Now, assume $S_x \cap S_y \neq \emptyset$. Note that if $x, y \in I_1$ or $x, y \in I_2$, then $x \in S_y$ or $y \in S_x$. Therefore, $x \in I_1$ if and only if $y \in I_2$. W.l.o.g. assume $x \in I_1$ and $y \in I_2$. Let $z \in S_x \cap S_y$. Then $z_1 = x_1$ and $z_2 = y_2$. Thus, $0 \leq y_2 - x_2 \leq a_2$ and $0 \leq x_1 - y_1 \leq a_1$, contradicting $x, y \in I$, proving this Claim.

Now, w.l.o.g. assume that $a_1 \leq a_2$ and $|I| > n_2 q_1$; and let $A = \cup_{x \in I} S_x$. By Claim 1, we have $|A| = |I_1|(a_2+1) + |I_2|(a_1+1) \geq |I|(a_1+1) > n_2 q_1(a_1+1)$. Then there is $0 \leq j < n_2$ such that $A_j = \{0 \leq x < n_1 : (x, j) \in A\}$ has size larger than $q_1(a_1+1)$. Given $x \in A_j$ let $\hat{x}$ be defined as the only point in I such that $(x, j) \in S_{\hat{x}}$. Also, for $i = 1, 2$, let $B_i = \{x \in A_j : \hat{x} \in I_i\}$ and let $B_2' = \{x \in A_j : (x, j) = \hat{x} \in I_2\}$. By Claim 1, we have

B'_2 is an independence set in $C(n_1, a_1)$ and $|A_j| = (a_1+1)|B'_2|+|B_1| \leq (a_1+1)q_1+|B_1|$. Therefore, B_1 is nonempty. As $C(n_1, a_1) \times C(n_2, a_2)$ is vertex-transitive we can assume $A_j = \{x^i : i = 1, \ldots, |A_j|\}$ ordered such that $x^{i+1} > x^i$ for all i and $x^1 = 0 \in B_1$. Notice that as $|A_j| > q_1(a_1+1)$, then $x^{i+1}-x^i \leq a_1$ for all i. Now we want to prove B_2 is empty. For this assume $B'_2 \neq \emptyset$ and let $k = \min_i \{x^i \in B'_2\} = \min_i \{x^i \in B_2\}$. Then $x^{k-1} \in B_1$. Now, $\hat{x}^{k-1}, \hat{x}^k \in I$, but $\hat{x}_1^k - \hat{x}_1^{k-1} = x^k - x^{k-1} \leq a_1$ and $\hat{x}_2^{k-1} - \hat{x}_2^k = j - \hat{x}_2^{k-1} \leq a_2$. Then $\hat{x}_2^{k-1} = j$ and by maximality of I we get $x^k = x^{k-1}+1$, but this contradicts $\hat{x}^k \in I_2$. Therefore B_2 is empty.

Finally, by a similar argument to the one above, for every $1 \leq i \leq |A_j|$ we have $\hat{x}_2^i = \hat{x}_2^{i+1}$ and $x^{i+1} = x^i + 1$. Therefore there is $0 \leq j' < n_2$ such that $[0, n_1 - 1] \times \{j'\} \subseteq I$. W.l.o.g assume $j' = 0$. The vertices in $I \setminus [0, n_1 - 1] \times \{0\}$ belong to the induced subgraph $C(n_1, a_1) \times C(n_2, a_2)[\{a_2 + 1, a_2 + 2, \ldots, n_2 - r_2 - 1\}]$, which admits a vertex decomposition into $q_2 - 1$ subgraphs all of them isomorphic to $C(n_1, a_1) \times K_{a_2+1}$. Therefore, by Lemma 17, we have that $|I| \leq n_1 + \alpha(C(n_1, a_1) \times K_{a_2+1})(q_2 - 1) = n_1 + (q_2 - 1)\max\{n_1, (a_2 + 1)q_1\} \leq \max\{q_2 n_1, q_1 n_2\}$. □

Theorem 27 *For $i = 1, 2$, let n_i, a_i be positive integers such that $n_i \geq 2a_i$, and let $\alpha_i = \alpha(C(n_i, a_i))$. Then,*

$$\chi(C(n_1, a_1) \times C(n_2, a_2)) = \min\{\chi(C(n_1, a_1)), \chi(C(n_2, a_2))\} = \min\left\{\left\lceil\frac{n_1}{\alpha_1}\right\rceil, \left\lceil\frac{n_2}{\alpha_2}\right\rceil\right\}.$$

Proof : For $i = 1, 2$, let $n_i = q_i(a_i + 1) + r_i$, with $q_i \geq 1$ and $0 \leq r_i \leq a_i$. By Theorem 25 we have that $\chi(C(n_i, a_i)) = \lceil\frac{n_i}{\alpha_i}\rceil$, where $\alpha_i = q_i$. Moreover, by Theorem 26, we have that $\alpha(C(n_1, a_1) \times C(n_2, a_2)) = \max\{n_1\alpha_2, n_2\alpha_1\}$. So, we have that $\chi(C(n_1, a_1) \times C(n_2, a_2)) \geq \lceil\frac{n_1 n_2}{\max\{n_1\alpha_2, n_2\alpha_1\}}\rceil$. Thus, if $n_1\alpha_2 \geq n_2\alpha_1$ then $\chi(C(n_1, a_1) \times C(n_2, a_2)) \geq \lceil\frac{n_1 n_2}{n_1\alpha_2}\rceil = \lceil\frac{n_2}{\alpha_2}\rceil = \chi(C(n_2, a_2))$. Otherwise, $\chi(C(n_1, a_1) \times C(n_2, a_2)) \geq \lceil\frac{n_1 n_2}{n_2\alpha_1}\rceil = \lceil\frac{n_1}{\alpha_1}\rceil = \chi(C(n_1, a_1))$. Therefore, $\chi(C(n_1, a_1) \times C(n_2, a_2)) \geq \min\{\chi(C(n_1, a_1)), \chi(C(n_2, a_2))\}$. □

We have not be able to generalize Theorem 26 for any finite product of powers of cycles graphs, and so it remains as an open problem.

3.2 Idomatic sets and idomatic partitions

3.2.1 Independent dominating sets

Lemma 18 *Let $G = K_{n_0} \times K_{n_1} \times K_{n_2}$ with $n_0, n_1, n_2 \geq 2$ and let I be an independent dominating set in G. If the set I contains at least two vertices agreeing in exactly two coordinates, then $I = pr_i^{-1}(k)$, where $i \in [3]$, pr_i is the projection of G on K_{n_i} and $k \in [n_i]$.*

Proof. As G is vertex-transitive and the direct product is commutative, we can assume w.l.o.g. that the vertices (x, i, j) and (y, i, j) of G belong to I, with i and j fix, and $x \neq y$.

First note that for all $z \in [n_0]$, with $z \notin \{x, y\}$, we have that $(z, i, j) \in I$. Otherwise, let $z \notin \{x, y\}$ such that $(z, i, j) \notin I$. As I is a dominating set, then there exists a vertex $(a, b, c) \in I$ such that $a \neq z$, $b \neq i$ and $c \neq j$. If $a \notin \{x, y\}$ then (a, b, c) is adjacent to vertices (x, i, j) and (y, i, j). If $a \in \{x, y\}$, say $a = x$ (the case $a = y$ is analogous), then (a, b, c) is adjacent to vertex (y, i, j). In both cases, we obtain a contradiction to the independence of I. Now, assume that there exists a vertex $(w, q, j) \notin I$, with $q \neq i$. Otherwise, $I = \mathrm{pr}_2^{-1}(j)$ and there is nothing to prove. As I is a dominating set, then there exists a vertex $(a, b, c) \in I$ with $a \neq w$, $b \neq q$ and $c \neq j$. As (z, i, j) belongs to I for any $z \in [n_0]$, then $b = i$, otherwise I is not an independent set. Thus, the vertices (a, i, j) and (a, i, c) belong to I. By using a similar argument as before, we can deduce that $(a, i, h) \in I$ for all $h \in [n_2]$. Therefore, we have that (z, i, j) and (a, i, h) belong to I for all $z \in [n_0]$ and for all $h \in [n_2]$ which implies, by the hypothesis that I is an independent dominating set of G, that $I = \mathrm{pr}_1^{-1}(i)$. □

Lemma 19 *Let $G = K_{n_0} \times K_{n_1} \times K_{n_2}$, with $n_0, n_1, n_2 \geq 2$, and let I be an independent set of G such that no two vertices in it agree in exactly two coordinates. Thus, the set I is a dominating set of G if and only if*

$$I = \{(\alpha_0, \alpha_1, \alpha_2), (\alpha_0, \beta_1, \beta_2), (\beta_0, \alpha_1, \beta_2), (\beta_0, \beta_1, \alpha_2)\},$$

for some $\alpha_i, \beta_i \in [n_i]$, with $\alpha_i \neq \beta_i$ and $i \in [3]$.

Proof. Assume first that such independent set I is also a dominating set of G. By hypothesis, I contains at least two vertices, and any pair of such vertices agreeing in exactly one coordinate. As G is vertex-transitive, we can assume w.l.o.g. that vertex $(0, 0, 0)$ belongs to I. By the commutativity of the direct product, we can assume that I contains also the vertex $(0, \beta_1, \beta_2)$, with $\beta_i \neq 0$ for $i = 1, 2$. Furthermore, by hypothesis, I contains no vertex of the form $(0, 0, z)$, for any $z \neq 0$. As I is a dominating set, then there exists $(\beta_0, b, c) \in I$ with $\beta_0 \neq 0$, $b \neq 0$ and $c \neq z$. If $c \neq 0$ then vertices $(0, 0, 0)$ and (β_0, b, c) are adjacent which is a contradiction to the independence of I. So $c = 0$ which implies that $b = \beta_1$, otherwise there is again a contradiction with the independence of I. Therefore, vertices $(0, 0, 0)$, $(0, \beta_1, \beta_2)$ and $(\beta_0, \beta_1, 0)$ belong to I. Similarly, by hypothesis, I contains no vertex of the form $(0, y, 0)$ for any $y \neq 0$. As I is a dominating set, there exists a vertex $(u, v, w) \in I$ with $u \neq 0$, $v \neq y$ and $w \neq 0$, which implies that vertex $(\beta_0, 0, \beta_2)$ belongs to I. By hypothesis, it is clear that no other vertex different to the previous four vertices can belong to I, otherwise there is a contradiction to the independence of I.
Conversely, let $I = \{(\alpha_0, \alpha_1, \alpha_2), (\alpha_0, \beta_1, \beta_2), (\beta_0, \alpha_1, \beta_2), (\beta_0, \beta_1, \alpha_2)\}$, for some $\alpha_i, \beta_i \in [n_i]$, with $\alpha_i \neq \beta_i$ and $i \in [3]$. Clearly, I is a maximal independent set w.r.t. the property that any pair of vertices in it agree in exactly one coordinate. Suppose that there is a vertex $(x_0, x_1, x_2) \in G \setminus I$ such that it is not adjacent to any vertex in I. Thus, $x_i = \alpha_i$ for some (but not for all) $i \in [3]$. So, assume that $x_2 \neq \alpha_2$ (the other cases can be proved similarly). If $x_0 = \alpha_0$ and $x_1 = \alpha_1$ then $(\beta_0, \beta_1, \alpha_2)$ is adjacent to it. Therefore, assume that $x_1 \neq \alpha_1$. As $x_0 = \alpha_0$, then it implies that $x_1 = \beta_1$, otherwise (x_0, x_1, x_2) is adjacent to $(\beta_0, \beta_1, \alpha_2)$. But, the last implies that $x_2 = \beta_2$, otherwise (x_0, x_1, x_2) is adjacent to

$(\beta_0, \alpha_1, \beta_2)$. Thus, $(x_0, x_1, x_2) = (\alpha_0, \beta_1, \beta_2) \in I$ that is a contradiction. Similarly, if we assume that $x_0 \neq \alpha_0$, $x_1 = \alpha_1$, and $x_2 \neq \alpha_2$ we obtain that $(x_0, x_1, x_2) = (\beta_0, \alpha_1, \beta_2) \in I$ that is a contradiction. Therefore, I is an independent dominating set of G. □

Definition 2 *Let $G = K_{n_0} \times K_{n_1} \times K_{n_2}$, with $n_i \geq 2$, and let I be an independent dominating set in G. The set I is said to be of* **Type A** *if it verifies the hypothesis in Lemma 18, and it is said to be of* **Type B** *if it verifies the hypothesis in Lemma 19.*

The following result is a consequence of Lemmas 18 and 19.

Theorem 28 *Let $G = K_{n_0} \times K_{n_1} \times K_{n_2}$, with $n_i \geq 2$, and let I be an independent set in G. Then, I is also a dominating set in G if and only if it is of Type A or Type B.*

3.2.2 Idomatic partitions

Definition 3 *Let $G = K_{n_0} \times K_{n_1} \times K_{n_2}$, with $n_i \geq 2$, and let $G_1, G_2, \ldots, G_t$ be an idomatic t-partition of G, with $t > 1$. Such an idomatic partition is called*

- *of* **Type A***: If all independent dominating sets G_i are of Type A.*
- *of* **Type B***: If all independent dominating sets G_i are of Type B.*
- *of* **Type C***: If there is at least one independent dominating set G_i of Type A, and at least one independent dominating set G_j of Type B, with $i \neq j$.*

Theorem 29 *Let $G = K_{n_0} \times K_{n_1} \times K_{n_2}$, with $n_i \geq 2$. Then, G has an idomatic n_i-partition of Type A for each $i \in [3]$. Moreover, such partitions are the only idomatic partitions of Type A of G.*

Proof. Let pr_i be the projection of G on K_{n_i}, for $i \in [3]$. It is easy to deduce that $\mathrm{pr}_i^{-1}(0), \mathrm{pr}_i^{-1}(1), \ldots, \mathrm{pr}_i^{-1}(n_i - 1)$ is an idomatic n_i-partition of G. In order to proof the second part, assume that G has an idomatic partition of Type A containing two different independent dominating sets I_i and I_j such that $I_i = K_{n_k} \times K_{n_j} \times \{\alpha_i\}$ for some fixed $\alpha_i \in [n_i]$ and $I_j = K_{n_k} \times \{\alpha_j\} \times K_{n_i}$ for some fixed $\alpha_j \in [n_j]$, where $i, j, k \in [3]$ and i, j, k pairwise different. Clearly, $I_i \cap I_j \neq \emptyset$ that is a contradiction. □

Proposition 4 *Let $G = K_{n_0} \times K_{n_1} \times K_{n_2}$, with $n_i \geq 2$. If G has an idomatic partition of Type B then there exist $j, k \in [3]$, with $j \neq k$, such that n_j and n_k are both even.*

Proof. By Lemma 19, we know that each part in an idomatic partition of Type B has four vertices, and thus 4 is a divisor of $n_0.n_1.n_2$. That is, there is at least one n_j, with $j \in [3]$ such that $2|n_j$. By the commutativity of the direct product, we can assume w.l.o.g. that $j = 2$. Let G_k be a part of the idomatic partition of Type B. By definition, G_k is an independent dominating set of Type B. So, let $G_k = \{(\alpha_0, \alpha_1, \alpha_2), (\alpha_0, \beta_1, \beta_2), (\beta_0, \alpha_1, \beta_2), (\beta_0, \beta_1, \alpha_2)\}$, where $\alpha_i, \beta_i \in [n_i]$ with $\alpha_i \neq \beta_i$. Fix the element $\alpha_2 \in [n_2]$. The number of vertices (x, y, α_2) in G is exactly $n_0.n_1$. Moreover, as $\alpha_i \neq \beta_i$ then, there are exactly $\frac{n_0.n_1}{2}$ parts in any idomatic partition of Type B each one containing exactly two different vertices (x, y, α_2) and (x', y', α_2), with $x \neq x'$ and $y \neq y'$. Therefore, $2|n_0.n_1$, which implies that $2|n_0$ or $2|n_1$. □

Proposition 5 *Let $G = K_{n_0} \times K_{n_1} \times K_{n_2}$, with $n_i \geq 2$. If there exist $j, k \in [3]$, with $j \neq k$, such that n_j and n_k are both even, then G has an idomatic partition of Type B of order $\frac{n_0.n_1.n_2}{4}$.*

Proof. As mentioned previously, the graph $G = K_{n_0} \times K_{n_1} \times K_{n_2}$ can be seen as the Cayley graph associated with the direct product group $\mathcal{G} = Z_{n_0} \times Z_{n_1} \times Z_{n_2}$ with connector set $[n_0] \setminus \{0\} \times [n_1] \setminus \{0\} \times [n_2] \setminus \{0\}$, where Z_{n_i} denotes the additive cyclic group of the integers modulo n_i. By the commutativity of the direct product, we can assume w.l.o.g. that $2|n_1$ and $2|n_2$. Let a_j be an element of order $\frac{n_j}{2}$ in the group Z_{n_j}, for $j \in \{1, 2\}$. Let $H_0 =< (1, 0, 0) >$ be the cyclic subgroup of $\mathcal{G}$ generated by the element $(1, 0, 0)$. Similarly, let $H_1 =< (0, a_1, 0) >$ and $H_2 =< (0, 0, a_2) >$ be cyclic subgroups of $\mathcal{G}$. It is easy to deduce that $H_i \cap H_j = \{(0, 0, 0)\}$ for all $i, j \in [3]$, with $i \neq j$. As $\mathcal{G}$ is an Abelian group then, by using standard group theoretic concepts, it can be deduced that the set $H_0.H_1.H_2 = \{h_0 + h_1 + h_2 : h_i \in H_i \text{ for } i \in [3]\}$ is a subgroup of order $\frac{n_0.n_1.n_2}{4}$ in $\mathcal{G}$. Let P denotes the subgroup $H_0.H_1.H_2$ and let $r = \frac{n_0.n_1.n_2}{4}$. Moreover, let $P = \{p_1, p_2, \ldots, p_r\}$, where $p_1 = (0, 0, 0)$ is the identity element. The following claim can be obtained by using standard arguments in group theory.

Claim 1 *Let P be the subgroup of $\mathcal{G} = Z_{n_0} \times Z_{n_1} \times Z_{n_2}$ defined previously. For $j = 1, 2$, let a_j be the element of order $n_j/2$ in Z_{n_j} chosen in order to construct the subgroup H_j of $\mathcal{G}$. Let β_0 be any element in Z_{n_0}, with $\beta_0 \neq 0$. Moreover, for $j = 1, 2$, let β_j be any element in Z_{n_j} such that $\beta_j \notin < a_j >$. Then, $P, (0, \beta_1, \beta_2) + P, (\beta_0, 0, \beta_2) + P, (\beta_0, \beta_1, 0) + P$ is a partition of $\mathcal{G}$ into left cosets of P.*

In fact, let $D = \{(0, \beta_1, \beta_2), (\beta_0, 0, \beta_2), (\beta_0, \beta_1, 0)\}$. By construction, no element in the set D belongs to the subgroup P. Moreover, let x, y be any two different elements in D. It is easy to show that there exists no element $z \in P$ such that $x + z = y$. Otherwise, $z = (p_0, p_1, p_2) \in P$ is such that $p_1 = \pm\beta_1$ or $p_2 = \pm\beta_2$ that is a contradiction. Therefore, Claim 1 holds.

Now, for each $1 \leq i \leq r$, let $C_i = \{p_i, (0, \beta_1, \beta_2) + p_i, (\beta_0, 0, \beta_2) + p_i, (\beta_0, \beta_1, 0) + p_i : p_i \in P\}$. We want to show that $C_1, C_2, \ldots, C_r$ is an idomatic r-partition of the graph $G = K_{n_0} \times K_{n_1} \times K_{n_2}$. By using the fact that G is the Cayley graph Cay$(\prod Z_{n_i}, \prod([n_i] \setminus \{0\}))$, we obtain the following claim.

Claim 2 *Let x, y, z be three vertices of G. Then, vertices $x + y$ and $x + z$ are adjacent in G if and only if vertices y and z are adjacent in G.*

Notice that, by Claim 2, each part C_i is an independent set of the graph G. Moreover, by Lemma 19, each set C_i is an independent dominating set of Type B, which completes the proof. □

By Propositions 4 and 5, we obtain the following theorem.

Theorem 30 *Let $G = K_{n_0} \times K_{n_1} \times K_{n_2}$, with $n_i \geq 2$. Then, G has an idomatic partition of Type B if and only if there exist $j, k \in [3]$, with $j \neq k$, such that n_j and n_k are both even.*

Example 1 *Let $G = K_2 \times K_3 \times K_4$. An idomatic 6-partition of Type B of G can be constructed as follows : let $P =< (0,0,0) > . < (0,1,0) > . < (0,0,2) >= \{p_0, p_1, p_2, p_3, p_4, p_5\}$ be a subgroup of the group $Z_2 \times Z_3 \times Z_4$, where $p_0 = (0,0,0)$, $p_1 = (0,1,0)$, $p_2 = (0,2,0)$, $p_3 = (0,0,2)$, $p_4 = (0,1,2)$, and $p_5 = (0,2,2)$. Let $x_1 = (0,1,1)$, $x_2 = (1,0,1)$, and $x_3 = (1,1,0)$. Then, $C_i = \{p_i, p_i + x_1, p_i + x_2, p_i + x_3\}$, for $i = 0,1,\ldots,5$, is an idomatic 6-partition of Type B of G .*

Theorem 31 *Let $G = K_{n_0} \times K_{n_1} \times K_{n_2}$, with $n_i \geq 2$, and let q_1, q_2 be two positive integers. Then, G has an idomatic $(q_1 + q_2)$-partition of Type C if and only if there exists $i \in [3]$ such that $n_i - q_1 > 1$ and $K_{n_j} \times K_{n_k} \times K_{n_i - q_1}$ admits an idomatic q_2-partition of Type B, with $j, k, i \in [3]$ and j, k, i pairwise different.*

Proof. Assume first that G has an idomatic $(q_1 + q_2)$-partition of Type C, where q_1 (resp. q_2) denotes the number of independent dominating sets of Type A (resp. Type B) in such a partition. By Theorem 29, it can be deduced that the q_1 dominating sets of Type A must be all of the form $K_{n_j} \times K_{n_k} \times \{s\}$ for some $s \in K_{n_i}$ with i fix, where $j, k, i \in [3]$ and j, k, i pairwise different. So, by permuting (if necessarily) the elements in the factor K_{n_i}, we can assume w.l.o.g. that the q_1 independent dominating sets of Type A are the sets $K_{n_j} \times K_{n_k} \times \{s\}$, for $s = n_i - q_1, \ldots, n_i - 1$. Clearly, the remaining q_2 independent dominating sets of Type B induce an idomatic q_2-partition of Type B of the direct product graph $K_{n_j} \times K_{n_k} \times K_{n_i - q_1}$. Finally, note that if $n_i - q_1 = 1$, then all the independent dominating sets in the idomatic partition are of Type A, which is a contradiction, and thus, $n_i - q_1 > 1$. The other direction of the proof is trivial. □

Example 2 *Let $G = K_2 \times K_3 \times K_4$. An idomatic 5-partition of Type C of G can be constructed as follows : consider first the graph $G' = K_2 \times K_2 \times K_4$ and let $P = < (0,0,0) > . < (0,0,0) > . < (0,0,1) >= \{p_0, p_1, p_2, p_3\}$ be a subgroup of the group $Z_2 \times Z_2 \times Z_4$, where $p_0 = (0,0,0)$, $p_1 = (0,0,1)$, $p_2 = (0,0,2)$, and $p_3 = (0,0,3)$. Let $x_1 = (0,1,1)$, $x_2 = (1,0,1)$, and $x_3 = (1,1,0)$. Then, $C'_i = \{p_i, p_i + x_1, p_i + x_2, p_i + x_3\}$, for $i = 0,1,2,3$ is an idomatic 4-partition of G' of Type B. Then, $(K_2 \times \{2\} \times K_4) \cup (\cup C'_i)$ is an idomatic 5-partition of Type C for G.*

From Theorems 29, 30 and 31, we have a full characterization of the idomatic partitions of the direct product of three complete graphs as follows.

Theorem 32 *Let $G = K_{n_0} \times K_{n_1} \times K_{n_2}$, with $n_i \geq 2$. If $\mathcal{I}$ is an idomatic partition of G, then $\mathcal{I}$ must be of Type A, B or C.*

By Theorem 17 (see [20]) we know that the idomatic number of the graph $G = K_{n_0} \times K_{n_1}$, with $n_0, n_1 \geq 2$, is equal to $\max\{n_0, n_1\}$. Now, having the characterization of the idomatic partitions of the direct product of three complete graphs then, by using Theorems 29, 30, 31, and Proposition 5, we can easily deduce the following corollary.

Corollary 6 *Let $G = K_{n_0} \times K_{n_1} \times K_{n_2}$, with $n_0, n_1, n_2 \geq 2$, and let $id(G)$ denote the idomatic number of graph G. Let $t = \max\{n_0, n_1, n_2\}$. Then,*

1. *If n_i is an odd integer for all $i \in [3]$, then $id(G) = t$.*
2. *If n_i is an even integer and $n_j \leq n_k$ are odd integers, with $i, j, k \in [3]$ and i, j and k pairwise different, then $id(G) = \max\{t, \frac{n_i.n_j.(n_k-1)}{4} + 1\}$.*
3. *If n_i and n_j are even integers, with $i, j \in [3]$ and $i \neq j$, then $id(G) = \frac{n_i.n_j.n_k}{4}$.*

3.2.3 Some General results

Theorem 33 *Let $G \times H$ be the direct product graph of graphs G and H respectively. If G admits an idomatic r-partition for some $r > 0$, and if H has no isolated vertices, then $G \times H$ admits an idomatic r-partition.*

Proof. Assume that G admits an idomatic r-partition, for some positive integer r. Let $G_1, G_2, \ldots, G_r$ be such an idomatic r-partition of G. Set $S_i = G_i \times H$, for $1 \leq i \leq r$. Clearly, $\bigcup_{i=1}^{r} S_i$ is a vertex partition of the graph $G \times H$. As for each $1 \leq i \leq r$, we have that G_i is an independent dominating set in G, it follows, by the definition of direct product graph and by the hypothesis that H has at least one edge, that S_i is an independent dominating set in $G \times H$, and therefore $\bigcup_{i=1}^{r} S_i$ is an idomatic r-partition of $G \times H$. □

So, by using Theorem 33, we can directly deduce the following result.

Proposition 6 *Let $G = K_{n_0} \times K_{n_1} \times \ldots \times K_{n_t}$, with $t \geq 3$ and $n_i \geq 2$ for any $i \in [t+1]$. Let $\mathcal{J}$ be any subset of $[t+1]$. If $\prod_{i \in \mathcal{J}} K_{n_i}$ has an idomatic partition of size r, then G has an idomatic r-partition.*

Notice that Theorem 29 can be generalized as follows.

Theorem 34 *Let $G = K_{n_0} \times K_{n_1} \times \ldots \times K_{n_t}$, with $t \geq 3$ and $n_i \geq 2$ for any $i \in [t+1]$. Then, G has an idomatic n_i-partition of Type A for each $i \in [t+1]$. Moreover, such partitions are the only idomatic partitions of Type A of G.*

3.3 Conclusions

In Section 3.1, we have shown that for circular graphs, Kneser graphs, and powers of cycles graphs, the Hedetniemi's conjecture on the chromatic number of direct products of these graphs is true. Moreover, we have also computed the independence number of direct products of these graphs. However, it is unknown if for any vertex-transitive graphs G and H, the equalities $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$ and $\alpha(G \times H) = \max\{\alpha(G)|H|, \alpha(H)|G|\}$ always hold.

In Section 3.2, we are obtained a full characterization of the idomatic partitions of the direct product of three complete graphs. Moreover, From Theorem 34 and Proposition 6 in Section 3.2, we are able to construct many idomatic partitions for a direct product of four or more complete graphs. However, we do not know if there exist other different types of idomatic partitions. Therefore, a full characterization of such idomatic partitions for the direct product of finitely many complete graphs remains an open question.

Chapter 4

Sum-coloring of graphs

Given a vertex coloring of a graph G, the *sum* of the coloring is the sum of the colors assigned to the vertices. The *chromatic sum* $\Sigma(G)$ of G is the smallest sum that can be achieved by any proper coloring of G. In the *Minimum Sum Coloring* (MSC) problem we have to find a coloring of G with sum $\Sigma(G)$.

The MSC problem was introduced by Kubicka [58]. The problem is motivated by applications in scheduling [3, 4, 35, 36] and VLSI design [69, 76]. The computational complexity of determining the vertex chromatic sum of a simple graph has been studied extensively since then. In [59] it is shown that the problem is NP-hard in general, but polynomial-time solvable for trees. The dynamic programming algorithm for trees can be extended to partial k-trees and block graphs [49]. Furthermore, the MSC problem is NP-hard even when restricted to some classes of graphs for which finding the chromatic number is easy, such as bipartite or interval graphs [4, 76]. A number of approachability results for various classes of graphs were obtained in the last ten years [3, 30, 35, 36, 26].

Jansen has shown in [49] that a more general optimization problem where each color has an integer cost, but this cost is not necessarily equal to the color itself, *the Optimal Cost Chromatic Partition (OCCP) problem*, can be solved in polynomial-time for cographs and block graphs, but it remains NP-hard for permutation graphs. Salavatipour has shown in [73] that the OCCP problem can be solved in polynomial-time for the family of P_4-reducible graphs, a superclass containing the family of cographs. P_4-sparse graphs were introduced in [41]. They generalize cographs and P_4-reducible graphs, can be recognized in linear time [46], and are a subclass of perfect graphs [41]. In Section 4.1, we study the Minimum Sum Coloring (MSC) problem on P_4-sparse graphs. First, we introduce the concept of maximal sequence associated with an optimal solution of the MSC problem of any graph. Next, based in such maximal sequences, we show that there is a large sub-family of P_4-sparse graphs for which the MSC problem can be solved in polynomial-time. This work has been done with the collaboration of Flavia Bonomo (Universidad de Buenos Aires, Argentina) (see reference [10]).

In an analogous way, it has been defined the edge coloring version of the MSC problem : the *Minimum Sum Edge Coloring* (MSEC) problem. The *edge-chromatic sum* of a graph G is denoted by $\Sigma'(G)$.The MSEC problem is NP-hard for bipartite graphs [31], even if the graph is also planar and has maximum degree 3 [64]. Furthermore, in [64] is also shown that the MSEC is NP-hard for 3-regular planar graphs and for partial 2-trees. Independently of the result given by Jansen in [49] concerning the polynomial-time complexity of a more general optimization problem (i.e. the General Optimal Cost Chromatic Partition Problem) containing the MSC problem on block graphs, it has been shown in [31, 74, 83] that the MSEC problem can be solved in polynomial time by a dynamic programming algorithm that uses weighted bipartite matching as a subroutine (in fact, notice that the family of block graphs includes trees and line graphs of trees). For general multigraphs, a 1.829-approximation algorithm for the MSEC problem is presented in [36]. For bipartite graphs there exist better approximation ratios : a 1.796-approximation algorithm is given in [35], and a 1.414-approximation algorithm is proposed recently in [28]. In Section 4.2, we show that the MSEC problem is polynomial-time solvable for multicycles (resp. multipaths), i.e. cycles (resp. paths) with parallel edges. This work has been done with the collaboration of Jean Cardinal (Université Libre de Bruxelles, Belgique) and Vlady Ravelomanana (Université Paris-Nord, France) (see references [13, 14]).

We also define the minimum number of colors needed in a minimum sum coloring of G. This number is called the *strength* $s(G)$ of the graph G in the case of vertex colorings, and the *edge strength* $s'(G)$ in the case of edge colorings. Clearly, $s(G) \geq \chi(G)$ and $s'(G) \geq \chi'(G)$. Hardness results were also given for the vertex and edge strength of a simple graph by Salavatipour [74], and Marx [66].

Some results concern the relations between the chromatic number $\chi(G)$ and the strength $s(G)$ of a graph. It has been known for long that the vertex strength can be arbitrarily larger than the chromatic number [24]. However, if G is a proper interval graph, then $s(G) = \chi(G)$ [69], and $s(G) \leq \min\{n, 2\chi(G) - 1\}$ if G is an interval graph [68]. Hajiabolhassan, Mehrabadi, and Tusserkani [34] proved an analog of Brooks' theorem for the vertex strength of simple graphs: $s(G) \leq \Delta(G)$ for every simple graph G that is neither an odd cycle nor a complete graph, where $\Delta(G)$ is the maximum degree in G.

Concerning the relation between the chromatic index and the edge strength, Mitchem, Morriss, and Schmeichel [67] proved an inequality similar to Vizing's theorem : $s'(G) \leq \Delta(G) + 1$ for every simple graph G. Harary and Plantholt [82] have conjectured that $s'(G) = \chi'(G)$ for every simple graph G, but this was later disproved by Mitchem *et al.* [67], and Hajiabolhassan *et al.* [34].

An interesting application of the MSEC problem is to model dedicated scheduling of biprocessor jobs. The vertices correspond to the processors and each edge $e = uv$ corresponds to a job that requires a time unit of simultaneous work on the two preassigned processors u and v. The colors correspond to the available time slots. A processor cannot work on two jobs at the same time, this corresponds to the requirement that a color can appear at most once on the edges incident to a vertex. The objective is to minimize the average time before a job is completed. When there can be $\omega(e)$ instances of the same job, it arises the notion of *set-coloring* of the corresponding conflict graph. Formally, given a simple graph $G = (V, E)$ and a demand function $\omega : V \rightarrow Z^+$, a *vertex set-coloring* of (G, ω) consists in assigning to each vertex $v \in V$ a set of $\omega(v)$ colors in such a way that adjacent vertices will be assigned disjoint sets of colors. Given a vertex set-coloring of a graph G with demand function ω, the *sum* of the set-coloring is the sum of the colors in the set assigned to each one of the vertices. The *chromatic set-sum* $\Sigma(G, \omega)$ of (G, ω) is the smallest sum that can be achieved by any proper set-coloring of (G, ω). In the *Minimum Sum Set Coloring* (MSSC) problem we have to find a set-coloring of (G, ω) with sum $\Sigma(G, \omega)$. Clearly, when $\omega(v) = 1$ for each vertex v of the graph, the MSSC problem becomes the MSC problem. The dedicated scheduling of biprocessor jobs with multiple instances can be modeled as a MSSC problem on the line graph of the conflict graph. A similar problem where each job e requires $\omega(e)$ time units of dedicated biprocessors, thus leading to a different objective function, was studied in [65]. In this case, sometimes it is allowed that a job is interrupted and continue later : the set of colors assigned to a vertex does not have to be consecutive. This type of scheduling is called *preemptive* (assuming that preemptions can happen only at integer times). Otherwise, if the set of colors assigned to each vertex needs to be consecutive, then the scheduling is called *non-preemptive*. In our case, the non-preemptive case arises when each job requires a high cost setup on the processors, and thus the objective is to minimize the average time before a job is completed, within the solutions minimizing the setup costs. Therefore, we have two variants of the MSSC problem : the preemptive and the non-preemptive one. It will be show in Section 4.3 that the computational complexity of the MSC problem and the MSSC problem (resp. preemptive MSSC problem and the non-preemptive MSSC problem) can be different for the same family of graphs : *block graphs*. A maximal subgraph 2-connected of a graph is called a *block*. A *block graph* is a graph for which each block is a clique. The family of block graphs includes as special cases trees and line graphs of trees. This work has been done with the collaboration of Flavia Bonomo and Guillermo Dúran (Universidad de Buenos Aires, Argentina), and Javier Marenco (Universidad Nacional de General Sarmiento, Argentina) (see reference [9]).

4.1 MSC problem in P_4-sparse graphs

By Theorem 7 (see Chapter 2, Section 2.2.1), P_4-sparse graphs have a nice decomposition property : if G is a non-trivial P_4-sparse graph, then either G or $\overline{G}$ is not connected, or G is a spider.

In fact, to each P_4-sparse graph G one can associate a corresponding decomposition rooted tree T in the following way. Each non-leaf node in the tree is labeled with either "$\cup$" (union-nodes), or "$\vee$" (join-nodes) or "SP" (spider-partition-nodes), and each leaf is labeled with a vertex of G. Each non-leaf node has two or more children. Let T_x be the subtree of T rooted at node x and let V_x be the set of vertices corresponding to the leaves in T_x. Then, each node x of the tree corresponds to the graph $G_x = (V_x, E_x)$. An union-node (join-node) corresponds to the disjoint union (join) of the P_4-sparse graphs associated with the children of the node. A spider-partition-node corresponds to a spider-partition (S, C, R) of the P_4-sparse graphs associated with the children of the node. Finally, the P_4-sparse graph that is associated with the root of the tree is just G, the P_4-sparse graph represented by this decomposition tree. The decomposition tree associated with a P_4-sparse graph can be computed in linear time [47].

4.1.1 Maximal sequences and optimal solutions of the MSC problem

A k-coloring of a graph $G = (V, E)$ is a partition of the vertex set V into k independent sets $S_1, \ldots, S_k$, where each vertex in S_i is colored with color i, for $1 \leq i \leq k$. So, for any such k-partition of V into independent sets, we can associate a non-negative *sequence* p such that $p[i] = |S_i|$ for $i = 1, \ldots, k$ and $p[i] = 0$ for $i > k$. In the sequel, we deal only with finite support non-negative integer sequences. Let $|p| = \max\{i : p[i] > 0\}$.

Definition 4 *Let p and q be two integer sequences. We say that p dominates q, denoted by $p \succeq q$, if for all $t \geq 1$ it holds that $\sum_{1\leq i\leq t} p[i] \geq \sum_{1\leq i\leq t} q[i]$.*

Definition 5 *Let p be a sequence. We denote by $\widetilde{p}$ the sequence that results from p when we order it in a non-decreasing way.*

The following two lemmas are direct consequences of Definition 4.

Lemma 20 *The dominance relation $\succeq$ is a partial order.*

Lemma 21 *Let p be a sequence. Then, $\widetilde{p} \succeq p$.*

The following lemma will be very useful in order to study the MSC problem on graphs.

Lemma 22 *Let p and q be two sequences and let $n = \max\{|p|, |q|\}$. If $p \succeq q$ and $\sum_{1\leq i\leq n} p[i] = \sum_{1\leq i\leq n} q[i]$, then it holds that $\sum_{1\leq i\leq n} i.p[i] \leq \sum_{1\leq i\leq n} i.q[i]$.*

Proof. Let $N = \sum_{1\leq i\leq n} p[i] = \sum_{1\leq i\leq n} q[i]$.. Let P and Q be two sequences obtained from p and q such that $|P| = |Q| = N$, and defined by $P[j] = \min\{k : \sum_{1\leq i\leq k} p[i] \geq j\}$ (resp. $Q[j] = \min\{k : \sum_{1\leq i\leq k} q[i] \geq j\}$) for $j = 1, \ldots, N$. By hypothesis, $p \succeq q$, and so, $P[j] \leq Q[j]$ for all $1 \leq j \leq N$. Therefore, $\sum_{1\leq i\leq n} i.p[i] = \sum_{1\leq j\leq N} P[j] \leq \sum_{1\leq j\leq N} Q[j] = \sum_{1\leq i\leq n} i.q[i]$. □

Notice that if the sequences represent partitions of the vertex set of a graph into independent sets, where the value of the ith element of the sequence represents the size of the ith independent set in the partition, then for the sum-coloring problem on graphs we can restrict us to study maximal sequences w.r.t. the partial order $\succeq$. Notice also that maximal sequences are non-increasing sequences. In the following, we define some operations between sequences.

Definition 6 *Let p and q be two sequences. The join of p and q, denoted by $p \star q$, is the sequence that results by ordering in a non-increasing way the concatenation of sequences p and q.*

Definition 7 *Let p and q be two sequences. The sum of p and q, denoted by $p + q$, is the sequence such that its i-th value is equal to $p[i] + q[i]$, for $i = 1, \ldots, \max\{|p|, |q|\}$.*

Definition 8 *Let p and q be two sequences. We say that p and q are non-comparable, denoted by $p||q$, if $p \not\succeq q$ and $q \not\succeq p$.*

The following two lemmas will be useful in order to study the MSC problem on P_4-sparse graphs.

Lemma 23 *Let p, p' and q be sequences. If $\widetilde{p} \succeq \widetilde{p'}$ then $p \star q \succeq p' \star q$.*

Proof. Let's consider the sequence $p' \star q$. By definition of join, $\widetilde{p'}$ is a subsequence of $p' \star q$. Let s be the sequence that results from $p' \star q$ by replacing each element $\widetilde{p'}[i]$ for $\widetilde{p}[i]$. As by hypothesis, $\widetilde{p} \succeq \widetilde{p'}$, then we have that $s \succeq p' \star q$. But now, note that $p \star q = \widetilde{s}$ and thus, $p \star q \succeq s \succeq p' \star q$. □

Lemma 24 *Let p, p' and q be sequences. Then, $p||p'$ if and only if $p + q||p' + q$.*

Proof. Note that $p||p'$ if and only if there exist two different positive integers j_1 and j_2 such that $\sum_{i=1}^{j_1} p[i] > \sum_{i=1}^{j_1} p'[i]$ and $\sum_{i=1}^{j_2} p[i] < \sum_{i=1}^{j_1} p'[i]$. Therefore, $\sum_{i=1}^{j_1}(p[i] + q[i]) > \sum_{i=1}^{j_1}(p'[i] + q[i])$ and $\sum_{i=1}^{j_2}(p[i] + q[i]) < \sum_{i=1}^{j_1}(p'[i] + q[i])$ that it is equivalent to $p + q||p' + q$. □

A direct consequence of the previous lemma is the following result.

Corollary 7 *Let p, p' and q be sequences. Then, $p \succeq p'$ if and only if $p + q \succeq p' + q$.*

Let G be a graph. Suppose that $G = G_1 \cup G_2$ and let p be a sequence representing a partition of the vertex set of G into independent sets. Clearly, $p = p_1 + p_2$ where p_1 (resp. p_2) is a sequence representing a partition of the vertex set of G_1 (resp. G_2) into independent sets. In an analogous way, suppose that $G = G_1 \vee G_2$ and let p be a sequence representing a partition of the vertex set of G into independent sets. Clearly, $p = p_1 \star p_2$ where p_1 (resp. p_2) is a sequence representing a partition of the vertex set of G_1 (resp. G_2) into independent sets. Therefore, by Corollary 7 and Lemma 23, if we are looking for maximal sequences of G representing partitions of its vertex set into independent sets then, in both cases it is sufficient to consider maximal partitions of the graphs G_1 and G_2.

4.1.2 Maximal sequences of P_4-sparse graphs

In the sequel, sequences of a graph will represent partitions of its vertex set into independent sets.

Lemma 25 *Let $G = (S, C, R)$ be a thin spider. Then,*

1. *If $R = \emptyset$ then, G has only one maximal sequence p, with $|p| = |C|$, where $p[1] = |C|$, $p[2] = 2$, and $p[i] = 1$ for $3 \leq i \leq |C|$.*

2. *If $R \neq \emptyset$ then, the number of maximal sequences of G is equal to the number of maximal sequences of $G[R]$. Moreover, for each maximal sequence q of $G[R]$ there exists only one maximal partition q' of G with $|q'| = |q| + |C|$ and where $q'[1] = q[1] + |C|$, $q'[i] = q[i]$ for $2 \leq i \leq |q|$, and $q'[i] = 1$ for $|q| + 1 \leq i \leq |q| + |C|$.*

Proof. Let $S = \{s_1, \ldots, s_k\}$ and $C = \{c_1, \ldots, c_k\}$, with $k \geq 2$. Let $S_1, \ldots, S_t$ be a partition of the vertex set of G into independent sets, with $t \geq 1$, such that its associated sequence p is maximal. Then,

1. By hypothesis, we have that $R = \emptyset$. Note first that each vertex $c_i \in C$ must belong to a different independent set S_j and so, $t \geq k$. Now, by definition of a thin spider, each vertex s_i is adjacent to vertex c_j if and only if $i = j$. We claim that there is S_i such that $S_i = S$ or there are S_i and S_j, with $i \neq j$, such that $S_i = (S \setminus \{s_n\}) \cup \{c_n\}$ and $S_j = \{s_n, c_m\}$, for some $n, m \in \{1, \ldots, k\}$, with $m \neq n$. Assume that it is not true, that is, suppose that there are S_i, S_j, S_l, with $i < j < l$, such that each one of them contains at least one vertex of S. Let $s_q \in S$ be a vertex in S_l. Then vertex $c_q \in C$ belongs at most to one of S_i or S_j but not to both. Thus, vertex s_q must migrate to one of S_i or S_j who contains no vertex c_q which gives a sequence that dominates p, that is a contradiction. Therefore, vertices in S belong to only one set S_i or to two different sets S_i and S_j. As p is maximal, then p is such that : ***(i)*** $p[1] = k$ and $p[i] = 1$ for $2 \leq i \leq k+1$, that is, $S_1 = S$ and $S_j = \{c_{j-1}\}$ for $2 \leq j \leq k+1$; or ***(ii)*** $p[1] = k$, $p[2] = 2$, and $p[i] = 1$ for $3 \leq i \leq k$, that is, sequence p is associated with the partition $S_1 = (S \setminus \{s_1\}) \cup \{c_1\}$, $S_2 = \{s_1, c_2\}$, and $S_l = \{c_l\}$ for $3 \leq l \leq k$. Clearly, the sequence of Case (ii) dominates the one of Case (i), and it is the only maximal sequence.

2. By hypothesis, we have that $R \neq \emptyset$. Let p be a maximal sequence of G and let p_1 and p_2 be sequences associated with $S \cup C$ and defined as : $p_1[1] = k$ and $p_1[i] = 1$ for $2 \leq i \leq k+1$; and $p_2[1] = k$, $p_2[2] = 2$, and $p_2[i] = 1$ for $3 \leq i \leq k$. In fact, p_1 (resp. p_2) represents the partition of the Case (i) (resp. (ii)) in Case (1) above. Let $S_1, \ldots, S_t$ be the partition of the vertex set of G into independent sets associated with the sequence p, with $t \geq k$. Let $S'_1, \ldots, S'_r$ (resp. $S''_1, \ldots, S''_{r'}$) be the partition that results from $S_1, \ldots, S_t$ after eliminating on it the sets S_w such that $S_w \subseteq R$ (resp. $S_w \subseteq S \cup C$), and let $Q_1, \ldots, Q_r$ (resp. $R_1, \ldots, R_{r'}$) be a partition of the vertex set $S \cup C$ (resp. R), where $Q_j = S'_j \setminus S'_j \cap R$ (resp. $R_j = S''_j \setminus S''_j \cap (S \cup C)$). By definition of spider partition, all vertices in R are adjacent to all vertices in C and non-adjacent to all the vertices in S. First, we will show that the sequence q' associated with the partition $Q_1 \ldots, Q_r$ of $S \cup C$ is equal either to p_1 or to p_2. For this, let j be the minimum positive integer such S_j contains vertices of R. By definition of thin spider and by the maximality of the sequence p, it suffices to consider the following cases:

   - Case $j = 1$. In this case, $S \subset S_1$ which implies that sequence q' is equal to sequence p_1.
   - Case $j = 2$. In this case, set S_1 must contains a vertex in C. Let $s_1 \in S$ be the vertex not in S_1. Note that $S_2 \not\subseteq R$, otherwise $S_2 \cup \{s_1\}$ leads to a partition of G such that its associated sequence dominates p, that is a contradiction. Therefore, $s_1 \in S_2$ and so, $c_1 \in S_1$. Let $T_1, \ldots, T_t$ be a partition of the vertex set of G into independent sets such that : $T_1 = S_1 \setminus \{c_1\} \cup S_2$, $T_i = S_{i+1}$ for $2 \leq i < t$, and $T_t = \{c_1\}$. Let p' be the sequence associated with the partition

$T_1, \ldots, T_t$. Clearly, p' dominates p that is a contradiction. Therefore, this case can no exists.

- Case $j \geq 3$. Notice that each set S_i must contains a vertex in C, for $1 \leq i < j$. As p is maximal, it implies that $S_1 = S \setminus \{s_1\} \cup \{c_1\}$ and $S_2 = \{s_1, c_2\}$, which implies that sequence q' is equal to sequence p_2.

Now, as sequence q' associated with the partition $Q_1 \ldots, Q_r$ of $S \cup C$ is equal either to p_1 or to p_2 then, it is not difficult to show that sequence q associated with the partition $R_1 \ldots, R_{r'}$ of R must be a maximal sequence for the graph $G[R]$, otherwise there is a contradiction with the maximality of p. Now, for the sequence q we define the following parameters : let $r_1 = |\{i : q[i] \geq k, 1 \leq i \leq |q|\}|$; $r_2 = |\{i : 1 < q[i] < k, 1 \leq i \leq |q|\}|$; and $r_3 = |\{i : q[i] = 1, 1 \leq i \leq |q|\}|$. By the observations made above, we have only two possibilities for the sequence p :

(a) The sequence q' associated with the partition $Q_1 \ldots, Q_r$ of $S \cup C$ is equal to p_1. As p is maximal, then we have that : $p[1] = q[1] + k$ (all vertices in $S \cup R_1$); $p[i] = q[i]$ for $2 \leq i \leq |q|$ (vertices in $R \setminus R_1$); and $p[i] = 1$ for $|q| + 1 \leq i \leq |q| + k$ (vertices in C).

(b) The sequence q' associated with the partition $Q_1 \ldots, Q_r$ of $S \cup C$ is equal to p_2. As p is maximal, then we have that : $p[i] = q[i]$ for $1 \leq i \leq r_1$ (vertices in $R_1, \ldots, R_{r_1}$); $p[r_1+1] = k$ (where there are $k-1$ vertices in S et one vertex in C); $p[i] = q[i-1]$ for $r_1+2 \leq i \leq r_1+r_2+1$ (vertices in $R_{r_1+1}, \ldots, R_{r_1+r_2}$); $p[r_1 + r_2 + 2] = 2$ (the remaining vertex in S and one vertex in C); $p[i] = 1$ for $r_1+r_2+3 \leq i \leq |q|+k$ (the remaining r_3 vertices in R and the remaining $k-2$ vertices in C).

It is not difficult to show that the sequence of the Case (a) dominates the one of Case (b). Therefore, it is the only maximal sequence of G.

□

Lemma 26 *Let $G = (S, C, R)$ be a thick spider. Then,*

1. *If $R = \emptyset$ then, G has only two maximal sequences p_1 and p_2, with $|p_1| = |C|$ and $|p_2| = |C| + 1$, where $p_1[i] = 2$ for $1 \leq i \leq |C|$, and $p_2[1] = |C|$ and $p_2[i] = 1$ for $2 \leq i \leq |C| + 1$.*

2. *If $R \neq \emptyset$ then, the number of maximal sequences of G is equal to the number of maximal sequences of $G[R]$. Moreover, for each maximal sequence q of $G[R]$ there exists only one maximal partition q' of G with $|q'| = |q| + |C|$ and where $q'[1] = q[1] + |C|$, $q'[i] = q[i]$ for $2 \leq i \leq |q|$, and $q'[i] = 1$ for $|q| + 1 \leq i \leq |q| + |C|$.*

Proof. Let $S = \{s_1, \ldots, s_k\}$ and $C = \{c_1, \ldots, c_k\}$, with $k \geq 3$. Let $S_1, \ldots, S_t$ be a partition of the vertex set of G into independent sets, with $t \geq 1$, such that its associated sequence p is maximal. Then,

1. By hypothesis, we have that $R = \emptyset$. As each vertex $c_i \in C$ must belong to a different independent set S_j then $t \geq k$. Now, by definition of a thick spider, each vertex s_i is adjacent to vertex c_j if and only if $i \neq j$. We claim that the sequence p is equal either to (i) $p[1] = k$ (i.e., $S_1 = S$) and $p[i] = 1$ for $2 \leq i \leq k+1$ (i.e., each S_i if formed by only one vertex in C, with $i > 1$); or (ii) $p[i] = 2$ for $1 \leq i \leq k$ (each S_i if formed by a vertex $s_i \in S$ and its only vertex non-adjacent $c_i \in C$). Assume that sequence p is not as claimed. Thus, there are positive integers $i_1 < i_2$ such that : (a) $S_{i_1}, S_{i_2} \subset S$ or (b) $S_{i_u} \subset S$ and $S_{i_v} = \{s_w, c_w\}$, where $u, v \in \{1, 2\}$, $u \neq v$, $s_w \in S$, and $c_w \in C$. Clearly, Case (a) gives a contradiction to the maximality of p. In the Case (b), if $u = 1$ and $v = 2$ then we can construct a partition $S'_1, \ldots, S'_t$ such that $S'_i = S_i$ for all $i \in \{1, \ldots, t\} \setminus \{i_1, i_2\}$, and where $S'_{i_1} = S_{i_1} \cup \{s_w\}$ and $S'_{i_2} = S_{i_2} \setminus \{s_w\}$. Now, by reordering (if necessarily) in a non increasing size the sets S'_i, for $i_2 \leq i \leq t$, we have that the sequence associated with this new partition dominates p that is a contradiction. Thus for Case (b), we have that $S_{i_1} = \{s_w, c_w\}$ and $S_{i_2} \subset S$. As p is maximal, then $1 \leq |S_{i_2}| \leq 2$ and thus, there exist $i_3 > i_2$, $s_l \in S_{i_2}$, and $c_l \in C$ such that $S_{i_3} = \{c_l\}$. Notice that $|S_{i_2}| = 2$, otherwise we can migrate vertex c_l into S_{i_2} removing in this way the part S_{i_3} and obtaining again a sequence that dominates p, that is a contradiction. So, let $s_y \in S_{i_2}$, with $s_y \neq s_l$. Swapping vertices c_l and s_y we must obtain a new partition such that its associated sequence is equal to p and it verifies the hypothesis of Case (b), but where i_2 is greater that the previous one. Now, repeating the same process to the current partition, we will obtain a partition such that its associated sequence dominates p, that is again a contradiction.

   Therefore, there are only two maximal sequences p_1 and p_2 for G such that $p_1[1] = k$ and $p_1[i] = 1$ for $2 \leq i \leq k+1$, and $p_2[i] = 2$ for $1 \leq i \leq k$. Moreover, we have that $p_1 || p_2$. In fact, let $j_1 = k-2$ and $j_2 = k$. Then, $\sum_{i=1}^{j_1} p_1[i] = 2k-3 > 2k-4 = \sum_{i=1}^{j_1} p_2[i]$ and $\sum_{i=1}^{j_2} p_1[i] = 2k-1 < 2k = \sum_{i=1}^{j_2} p_2[i]$.

2. By hypothesis, we have that $R \neq \emptyset$. Let p be a maximal sequence of G and let p_1 and p_2 be sequences associated with $S \cup C$ and defined as : $p_1[1] = k$ and $p_1[i] = 1$ for $2 \leq i \leq k+1$; and $p_2[i] = 2$ for $1 \leq i \leq k$. In fact, p_1 (resp. p_2) represents the only two maximal partitions of the Case (1) when $R = \emptyset$. Let $S_1, \ldots, S_t$ be the partition of the vertex set of G into independent sets associated with the sequence p, with $t \geq k$. Let $S'_1, \ldots, S'_r$ (resp. $S''_1, \ldots, S''_{r'}$) be the partition that results from $S_1, \ldots, S_t$ after eliminating on it the sets S_w such that $S_w \subseteq R$ (resp. $S_w \subseteq S \cup C$), and let $Q_1, \ldots, Q_r$ (resp. $R_1, \ldots, R_{r'}$) be a partition of the vertex set $S \cup C$ (resp. R), where $Q_j = S'_j \setminus S'_j \cap R$ (resp. $R_j = S''_j \setminus S''_j \cap (S \cup C)$). By definition of a spider partition, all vertices in R are adjacent to all vertices in C and non-adjacent to all the vertices in S. First, we will show that the partition $S_1, \ldots S_t$ can be transformed (if necessarily) into another one having as associated sequence also p and where the sequence q' associated with the partition $Q_1 \ldots, Q_r$ of $S \cup C$ is equal either to p_1 or to p_2. For this, let j be the minimum positive integer such that S_j contains vertices of R. Consider the following two cases:

   - Case $j = 1$. If $S_1 \subseteq R$ then we can migrate all vertices of S into S_1 obtaining in this way a partition of G such that its associated sequence dominates p, that is a contradiction. Therefore, S_1 must contain vertices of R and of S. However, as no vertex in R is adjacent to any vertex in S, then $S \subset S_1$, which implies that $Q_1 = S$ and so, sequence q' is equal to sequence p_1.

- Case $j > 1$. By definition, every vertex in C is adjacent to all vertices in R. So, each independent set S_i must contains one vertex in C, for $1 \leq i < j$. Moreover, if vertex $c_l \in C$ is in S_i, for $1 \leq i < j$, then vertex $s_l \in S$ must be also in S_i, otherwise sequence p is not maximal. Now, as p is maximal, then $1 \leq |S_j| \leq 2$. If $|S_j| = 1$ then it is clear that j must be equal to $k+1$, otherwise p is not maximal, and so in this case, sequence q' is equal to p_2. Let $|S_j| = 2$. We have two possibilities for S_j : (i) $S_j \subseteq R$; (ii) S_j contains one vertex in R and one vertex in S. In Case (i), all vertices in S must be in the sets S_i, with $i < j$, otherwise p is not maximal. So, in this case, sequence q' is equal to p_2. In Case (ii), let $S_j = \{x, s_l\}$, where $x \in R$ and $s_l \in S$. Again, as p is maximal, there is no vertex of S in the sets S_i, with $i > j$. As vertex $c_l \in C$ must be alone in some independent set S_w, with $w > j$, then swapping vertices c_l and x such that S_j becomes the set $\{s_l, c_l\}$ and S_w becomes $\{x\}$, we obtain a new partition where its associated sequence is equal to p and the sequence q' is equal to p_2.

As we can assume that the sequence q' associated with the partition $Q_1 \ldots, Q_r$ of $S \cup C$ is equal either to p_1 or to p_2, then the sequence q associated with the partition $R_1 \ldots, R_{r'}$ of R must be a maximal sequence for the graph $G[R]$, otherwise there is a contradiction with the maximality of p.

Now, for the sequence q we define the following parameters : let $r_1 = |\{i : q[i] \geq k, 1 \leq i \leq |q|\}|$; $r_2 = |\{i : 1 < q[i] < k, 1 \leq i \leq |q|\}|$; and $r_3 = |\{i : q[i] = 1, 1 \leq i \leq |q|\}|$. By the observations made above, we have only two possibilities for the sequence p :

(a) The sequence q' associated with the partition $Q_1 \ldots, Q_r$ of $S \cup C$ is equal to p_1. As p is maximal, then we have that : $p[1] = q[1] + k$ (all vertices in $S \cup R_1$); $p[i] = q[i]$ for $2 \leq i \leq |q|$ (vertices in $R \setminus R_1$); and $p[i] = 1$ for $|q| + 1 \leq i \leq |q| + k$ (vertices in C).

(b) The sequence q' associated with the partition $Q_1 \ldots, Q_r$ of $S \cup C$ is equal to p_2. As p is maximal, then we have that : $p[i] = q[i]$ for $1 \leq i \leq r_1 + r_2$ (vertices in $R_1, \ldots, R_{r_1+r_2}$); $p[i] = 2$ for $r_1 + r_2 + 1 \leq i \leq r_1 + r_2 + k$ (all vertices in $S \cup C$); and $p[i] = 1$ for $r_1 + r_2 + k + 1 \leq i \leq |q| + k$ (vertices in $R_{|q|-r_3+1} \ldots, R_{|q|}$).

It is not difficult to show that the sequence of the Case (a) dominates the one of Case (b). In fact, let q_1 (resp. q_2) be the sequence of G in Case (a) (resp. in Case (b)). Then, $q_1[i] > q_2[i]$ for $1 \leq i \leq |q| + k - 3$ and $q_1[i] = q_2[i]$ for $|q| + k - 2 \leq i \leq |q| + k$. Therefore, it is the only maximal sequence of G.

$\square$

Notice also that the trivial graph has only one maximal sequence p, with $|p| = 1$, where $p[1] = 1$. Therefore, we have the following theorems.

Theorem 35 *Let G be a P_4-sparse graph such that in its modular decomposition there are no thick spiders (S, C, R) with $R = \emptyset$. Then,*

1. *$s(G) = \chi(G)$, and $\Sigma(G)$ and an optimal coloring of G can be computed from its modular decomposition in polynomial time.*

2. *In such an optimal coloring, each independent set S_i is a maximum independent set of $G \setminus \bigcup_{1\leq j<i} S_j$ which verifies $\chi(G \setminus \bigcup_{1\leq j\leq i} S_j) = \chi(G \setminus \bigcup_{1\leq j<i} S_j) - 1$.*

Proof. Let T be the decomposition rooted tree associated with G. It is well known that T can be computed in linear time [47].

1. Let n be the number of vertices in G. In order to compute an optimal coloring with $s(G) = \chi(G)$ and sum $\Sigma(G)$ for this case, we proceed from the leaves to the root in T as follows. If x is a leaf in T then its associated partition is $S_1 = \{x\}$ having as maximal sequence p, with $|p| = 1$ and $p[1] = 1$. If node $x \in T$ is a union-node (resp. join-node) then, by Corollary 7 (resp. Lemma 23), the unique maximal sequence and its corresponding optimal partition of the vertex-set of G_x into independent sets can be computed from the unique maximal sequences and their corresponding optimal partitions of the children of x. Moreover, by definition of union and join, it is clear that $S(G_x) = \chi(G_x)$ and the sum of the colors is equal to $\Sigma(G_x)$. If node $x \in T$ is a spider-partition node representing the spider $\sigma = (S, C, R)$ then, the unique maximal sequence and its corresponding optimal partition of the vertex-set of G_x into independent sets can be computed from the unique maximal sequences and their corresponding optimal partitions of the children of x, either as shown in either the Lemma 25 (if σ is a thin spider) or as shown in Case (2) of Lemma 26 (if σ is a thick spider with $R \neq \emptyset$). Moreover, by Lemmas 25 and 26, it is clear that $S(G_x) = \chi(G_x)$ and the sum of the colors is equal to $\Sigma(G_x)$. Finally, notice that each node $x \in T$ needs $O(n)$ time to compute its optimal partition. As there are at most $2n - 1$ nodes in T, then the complexity time of the algorithm is bounded by $O(n^2)$.

2. By using Theorem 7 and by using Corollary 7 (resp. Lemma 23) if G is a disjoint union (resp. join) of P_4-sparse graphs, and Cases (1) and (2) of Lemma 25 (resp. Case (2) of Lemma 26) if G is a spider partition then, by induction on the number of vertices of G, the result holds.

□

Theorem 36 *Let G be a P_4-sparse graph on n vertices. Let k be the number of thick spiders (S, C, R) with $R = \emptyset$ in the modular decomposition of G. Then, $s(G) \leq \chi(G)+k$, the number of maximal sequences of G is at most 2^k, and an optimal coloring of G can be computed in $2^k P(n)$ time, where $P(n)$ is a polynomial on n.*

Proof. By Case (1) of Lemma 26, each thick spider $\sigma_i = (S_i, C_i, \emptyset)$ in the decomposition tree T of G have exactly two maximal sequences with their corresponding optimal partitions, with $1 \leq i \leq k$. Clearly, there are 2^k ways of choosing maximal sequences (and their corresponding partitions) for the k thick spiders σ_i. Now, given a fixed choice for the thick spiders σ_i and by using the algorithm in the proof of Case (1) of Theorem 35, we can compute in $O(n^2)$ time a maximal sequence and its corresponding partition into independent sets for G. This shows that G has at most 2^k maximal sequences and that an optimal coloring with $s(G)$ colors and sum $\Sigma(G)$ can be computed in $O(2^k n^2)$ time. Finally, note that for each thick spider σ_i, one of its maximal sequences has length $\chi(\sigma_i)+1$ and thus, the number of colors used in an optimal solution for G is upper bounded by $\chi(G) + k$. □

4.2 MSEC problem in multicycles

The following well-known result has been proved by König in 1916.

Theorem 37 (König's theorem [55]) *Let $G = (V, E)$ be a bipartite multigraph and let Δ denotes its maximum degree. Then $\chi'(G) = \Delta$.*

Hajiabolhassan *et al.* [34] mention that $s'(G) = \chi'(G)$ for every bipartite graph G. In fact, by using the same technique as in the classical proof of König's theorem, it is easy to deduce that $s'(G) = \chi'(G)$ for every bipartite multigraph G.

Theorem 38 *Let $G = (V, E)$ be a bipartite multigraph and let Δ denote its maximum degree. Then $s'(G) = \chi'(G) = \Delta$.*

Multicycles are cycles in which we can have parallel edges between two consecutive vertices. We consider the chromatic edge strength of multicycles.

The chromatic edge strength $s'(G)$ of a graph G is bounded from below by both Δ and $\lceil \frac{m}{\tau} \rceil$, where Δ is the maximum degree in G and τ is the cardinality of a maximum matching in G. In this section, we show that the lower bound $\max\{\Delta, \lceil \frac{m}{\tau} \rceil\}$ is indeed tight for multicycles. We assume that the multiplicity of each edge in the multicycle is at least one, so that the size τ of a maximum matching is equal to $\lfloor n/2 \rfloor$.

We first give a closed-form expression for the chromatic index of multicycles.

Theorem 39 ([5]) *Let $G = (V, E)$ be a multicycle on n vertices with m edges and maximum degree Δ. Let τ denote the maximum cardinality of a matching in G. Then*

$$\chi'(G) = \begin{cases} \Delta, & \text{if } n \text{ is even,} \\ \max\left\{\Delta, \lceil \frac{m}{\tau} \rceil\right\}, & \text{if } n \text{ is odd.} \end{cases}$$

We now introduce some useful notations. Given C the set of colors used in an edge coloring of a multigraph G, we denote by C_x the subset of colors of C assigned to edges incident to vertex x of G. Given two colors α and β, we call a path an (α, β)-path if the colors of its edges alternate between α and β. We also denote by $d_G(x)$ the degree of vertex x in G. We now state our main result.

Theorem 40 *Let $G = (V, E)$ be a multicycle on n vertices with m edges and maximum degree Δ, and let τ denote the maximum cardinality of a matching in G. Then*

$$s'(G) = \chi'(G) = \begin{cases} \Delta, & \text{if } n \text{ is even,} \\ \max\left\{\Delta, \lceil \frac{m}{\tau} \rceil\right\}, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. If n is even, then the result follows from Theorem 38. Thus, we assume that $n = 2k+1$ for a positive integer k and let $s' = s'(G)$. Let $r = \max\left\{\Delta, \lceil \frac{m}{k} \rceil\right\}$. As $\tau = k$, then it is clear that $\chi'(G) = r$. Moreover, as $s' \geq \chi'(G)$ then, it suffices to prove that $s' \leq r$. Assume that $s' > r$ and G is a smallest counterexample. We claim that there exists a minimum sum edge coloring f of G in which there is only one edge colored with color s'. Otherwise, delete one of the edges with color s', say e. From the minimality of G, there exists a minimum sum edge coloring of $G \setminus e$ with χ' colors. Then we obtain the desired edge coloring of G by assigning the color $s' = \chi' + 1 = r + 1$ to e.

Let E_i denote the set of edges of G with color i and let $[a, b]_0$ be the only edge in G colored with color s'. Moreover, let $G' = G \setminus [a, b]_0$. By the minimality of G, we have that $s'(G') = \chi'(G') = \max\left\{\Delta', \lceil \frac{m-1}{k} \rceil\right\} \leq r$. Let $C = \{1, \ldots, r\}$. The following properties for the edge coloring of G' can be easily deduced:

(1) There exists a color $\sigma \in C$ such that $|E_\sigma| < k$.

(2) $|C_a \cup C_b| = r$.

(3) There exist at least two colors α and β in C such that $\alpha \in C_a \setminus C_b$ and $\beta \in C_b \setminus C_a$, with $\alpha \neq \beta$.

For (1), notice that if there is no color $\sigma \in C$ such that $|E_\sigma| < k$, then $m - 1 = \sum_{i=1}^{r} |E_i| = kr$, hence $r = \frac{m-1}{k} < \frac{m}{k}$, contradicting the definition of r. Property (2) holds, otherwise edge $[a, b]_0$ can be colored with a color in C which contradicts the fact that G' is a counterexample. Finally, notice that the degree of vertices a and b in G' is at most equal to $\Delta - 1$. Since $r \geq \Delta$, there is a color $\beta \notin C_a$ and a color $\alpha \notin C_b$ with $\alpha, \beta \in C$. Clearly $\alpha \neq \beta$, otherwise $[a, b]_0$ can be colored with such a color, contradicting the fact that G is a counterexample. Moreover, by (2), we have that $\alpha \in C_a \setminus C_b$ and $\beta \in C_b \setminus C_a$, which proves (3).

By Property(2), it is sufficient to analyze the cases $\sigma \in C_b \setminus C_a$ (or $\sigma \in C_a \setminus C_b$) and $\sigma \in C_a \cap C_b$. The two cases are illustrated on Figures 4.1 and 4.2, respectively.

If $\sigma \in C_b \setminus C_a$ then, by Property (3), there exists a color $\alpha \in C_a \setminus C_b$ with $\alpha \neq \sigma$. Let $G(\alpha, \sigma)$ denote the subgraph of G' induced by the edges of color α and σ. Let $G_b(\alpha, \sigma)$ denote the connected component of $G(\alpha, \sigma)$ containing b. Clearly, $G_b(\alpha, \sigma)$ is a simple (σ, α)-path having b as last vertex and not containing vertex a, otherwise we have a contradiction to Property (1). Hence we can recolor the edges of the path $G_b(\alpha, \sigma)$ by swapping colors α and σ in such a way that $\sigma \notin C_b$. Since $\sigma \notin C_a$, we assign color σ to $[a, b]_0$, and obtain an edge coloring f'' of G using r colors. Figure 4.1 provides an example of this case.

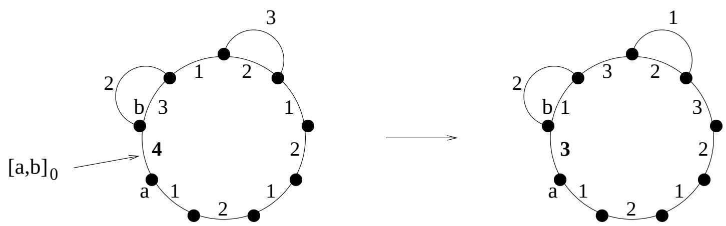


Figure 4.1: An illustration of the case $\sigma \in C_b \setminus C_a$ in the proof of Theorem 40, on a multicycle G with $s' = \chi' = 3$. The edge $[a,b]_0$ is the only edge colored with color $s'+1 = 4$. In this example, color $\sigma = 3$ and color $\alpha \in C_a \setminus C_b$ is equal to 1.

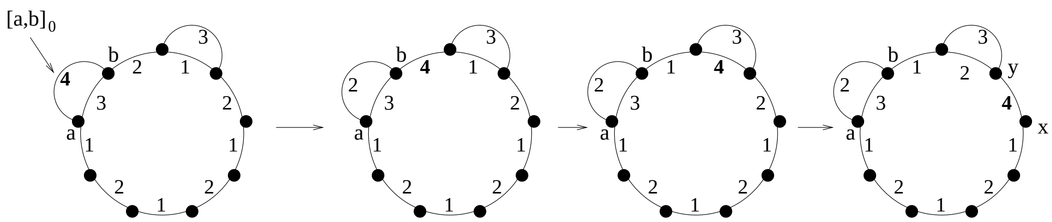


Figure 4.2: An illustration of the case $\sigma \in C_a \cap C_b$ in the proof of Theorem 40, on a multicycle G with $s' = \chi' = 3$. Again, edge $[a,b]_0$ is the only edge colored with color $s'+1 = 4$ and $\sigma = 3$.

We now want to show that

$$\sum_{e \in E} f''(e) < \sum_{e \in E} f(e), \qquad \textbf{(*)}$$

contradicting $s'(G) > r$. If the length of the path $G_b(\alpha, \sigma)$ is even, then $\sum_{e \in E} f''(e) - \sum_{e \in E} f(e) = \sigma - r - 1 \leq r - r - 1 < 0$. If the length of the path $G_b(\alpha, \sigma)$ is odd, say $2s+1$, with $s \geq 0$, then the difference is $(\sigma + (s+1)\alpha + s\sigma) - (r + 1 + (s+1)\sigma + s\alpha) = \alpha - r - 1 \leq r - r - 1 < 0$. Thus, inequality **(*)** always holds.

The other case is when $\sigma \in C_a \cap C_b$. By Property (3), there exist a color $\beta \in C_b \setminus C_a$. Let us assume that vertices are ordered clockwise and let b be the clockwise vertex of edge $[a,b]_0$. Recolor edge $[a,b]_0$ with color β and the edge of color β incident to b with color $s' = r+1$. This recoloring does change neither the value of the sum nor the number of colors. Let $[x,y]_0$ be the edge that is recolored with color s', with x being its counterclockwise vertex.

By Property (3) again, a color β_y such that $\beta_y \in C_y \setminus C_x$ exists. We can therefore repeat the above procedure until the edge $[x,y]_0$ is such that $\sigma \in C_x \setminus C_y$ or $\sigma \in C_y \setminus C_x$. This is always possible, because the cycle is odd, and $|E_\sigma| < k$; hence by moving around the cycle this way, we will eventually find an edge $[x,y]_0$ that is adjacent to only one edge of color σ. Assume, without loss of generality, that $\sigma \in C_y \setminus C_x$. Then letting $a = x$ and $b = y$ leads us back to the first case. Figure 4.2 gives an example of this case. $\square$

4.2.1 Algorithms

We now present algorithms for minimum sum coloring of multicycles. Our algorithms assume that the encoding of the multicycle given as input has size $\Theta(n+m)$. This does not allow for implicit representations consisting of, for instance, the number of vertices and the number of parallel edges between each pair of consecutive vertices. This assumption is natural since we expect the resulting coloring to be represented by an encoding of size linear in the number of edges.

The line graph of a multicycle is a proper circular arc graph. Hence the problem of coloring edges of multicycles is a special case of proper circular arc graph coloring. It is easy to realize that not all proper circular arc graphs are line graphs of multicycles, though. Proper circular arc graphs were shown by Orlin, Bonuccelli, and Bovet [70] to admit *equitable colorings*, that is, colorings in which the sizes of any two color classes differ by at most one, that only use χ colors. Therefore, a corollary of our results is that multicycles admit both equitable and minimum sum edge colorings with the same, minimum, number of colors, and that both types of colorings can be computed efficiently.

We first present a general algorithm, then focus on the case where n is even.

The general case

A natural idea for solving minimum cost coloring problems is to use a greedy algorithm that iteratively removes maximum independent sets (or maximum matchings in the case of edge coloring) [3, 26]. It can be shown that this approach fails here. Instead we use an algorithm in which the smallest color class, corresponding to color s', is removed iteratively.

We first consider the case where $\lceil m/k \rceil \geq \Delta$ and k divides m. Then the number of colors must be equal to m/k. But since each color class can contain at most k edges, every color class in a minimum sum coloring must have size exactly k. Such a coloring can be easily found in linear time by a sweeping algorithm that assigns each color $i \bmod \chi'$ in turn. This is a special case of the algorithm of Orlin *et al.* (Lemma 2, [70]) for circular arc graph coloring. In the remainder of this section, we refer to this case as the ”easy case”.

Algorithm MULTICYCLECOLOR**.**

1. $i \leftarrow s'(G)$, $G_i \leftarrow G$
2. **if** $\lceil |E(G_i)|/k \rceil \geq \Delta(G_i)$ and k divides $|E(G_i)|$ **then** apply the ”easy case” algorithm and terminate
3. **else**
   (a) Find a matching M of minimum size such that $s'(G_i \setminus M) = s'(G_i) - 1$
   (b) color the edges of M with color i
   (c) $G_{i-1} \leftarrow G_i \setminus M$, $i \leftarrow i - 1$
   (d) **if** $G_i \neq \emptyset$ **then** go to step 2

The correctness of the algorithm relies on the following lemma.

Lemma 27 *Given a matching M in a multicycle G such that*

1. $s'(G \setminus M) = s'(G) - 1$,

2. *M has minimum size among all matchings satisfying condition 1,*

there exists a minimum sum edge coloring of G such that M is the set of edges colored with color $s'(G)$.

Proof. We distinguish three cases, a), b), and c), depending on the relative values of $\lceil m/k \rceil$ and Δ.

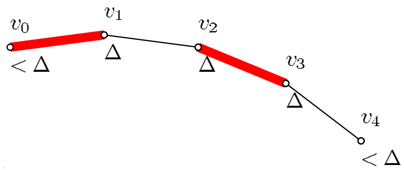
Case a) We first assume that $\lceil m/k \rceil > \Delta$ and k does not divide m, thus $m = \lfloor m/k \rfloor \cdot k + q$, with $q > 0$. In that case, M has size exactly q. To find a minimum sum coloring, we color the edges of M with color $\lceil m/k \rceil$. The remaining edges are colored using the "easy case" algorithm, which applies since $\lfloor m/k \rfloor \geq \Delta$ and the number of remaining edges is a multiple of k. This coloring must have minimum sum, because only one color class has not size k.

Case b) When $\Delta > \lceil m/k \rceil$, we have $s'(G) = \Delta$ from Theorem 40. We claim that in that case, M is a minimum matching that hits all vertices of degree Δ. To prove this, suppose otherwise. Then $(G \setminus M)$ has maximum degree Δ, and thus from Theorem 40, $s'(G \setminus M) = s'(G)$, contradicting condition 1. Now we have to ensure that there exists a minimum sum coloring such that M is the color class $s'(G) = \Delta$.

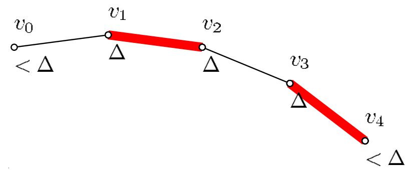
We consider a minimum sum coloring and the color class Δ in this coloring. This class, say M', must also be a matching hitting all vertices of degree Δ. We now describe a recoloring algorithm that, starting with this coloring, produces a coloring whose sum is not greater and whose color class Δ is exactly M. We define a *block* as a maximal sequence of adjacent vertices of degree Δ. The algorithm examines each block, and shifts the edges of M' if they do not match with those of M. Two cases can occur, depending on the parity of the block length.

The first case is when a block contains an odd number of vertices of degree Δ, say $v_1, v_2, \ldots, v_{2t+1}$ for some integer t. In that case, the only way in which M and M' can disagree is, without loss of generality, when M' contains edges of the form $v_0v_1, v_2v_3, \ldots, v_{2t}v_{2t+1}$, while M contains $v_1v_2, v_3v_4, \ldots, v_{2t+1}v_{2t+2}$ (see figure 4.3(a)-4.3(b)), where v_0 and v_{2t+2} are the predecessor of v_1 and the successor of v_{2t+1}, respectively. Since the degree of v_0 is, by definition of a block, strictly less than Δ, there must exist a color $\alpha \in C_{v_1} \setminus C_{v_0}$. Furthermore, since all vertices within the block have degree Δ, the color class for color α contains $t + 1$ edges of the form $v_{2i+1}v_{2i+2}$ for $0 \leq i \leq t$. Hence we can recolor the edges of M' of the form $v_{2i}v_{2i+1}$ for $0 \leq i \leq t$ with color α, and the $t + 1$ edges of color α within the block with color Δ. Note that at this point, the coloring might be not proper anymore, as two edges colored Δ might be incident to v_{2t+2}.

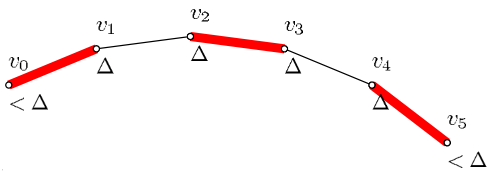
The other case is when a block contains an even number of vertices of degree Δ, say $v_1, v_2, \ldots, v_{2t}$ for some integer t. In that case, since M is minimum, it contains edges of the form $v_1v_2, v_3v_4, \ldots, v_{2t-1}v_{2t}$. The only way in which M' can disagree with M is by



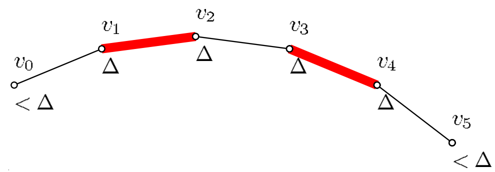
(a) odd case: edges of M'



(b) odd case: edges of M



(c) even case: edges of M'



(d) even case: edges of M

Figure 4.3: Illustration of the proof of Lemma 27.

containing edges $v_0v_1, v_2v_3, \ldots, v_{2t}v_{2t+1}$ (see figure 4.3(c)-4.3(d)). Like in the previous case, there must be a color $\alpha \notin C_{v_0}$, so we can recolor the edges of M' of the form $v_{2i}v_{2i+1}$ for $0 \leq i < t$ with color α, and the edges of color α within the block with color Δ. Note that the edge $v_{2t}v_{2t+1}$ of color Δ has not been recolored, thus $v_{2t-1}v_{2t}$ and $v_{2t}v_{2t+1}$ both have color Δ, and at this point the coloring is not proper anymore.

We proceed in this way for each block. Notice that the sum of the coloring is unaltered, and that the set of edges of color Δ is now a superset of M. Also, while G is not necessarily properly colored anymore, the graph $G \setminus M$ is properly colored with at most Δ colors. But since removing M decreases the strength, we know we can recolor $G \setminus M$ with $\Delta - 1$ colors without increasing the sum. Doing that and gluing back the edges of M colored with color Δ, we obtain a minimum sum coloring where only the edges of M have color Δ, as claimed.

Case c) Finally, in the case where $\lceil m/k \rceil = \Delta$, with $m = \lfloor m/k \rfloor \cdot k + q$, the matching M consists of at least q edges that together hit all vertices of degree Δ. If M has exactly size q, then case a) above applies, since we know that by removing M, we also decrease the maximum degree. Otherwise case b) applies. □

We have to make sure that the main step of the algorithm can be implemented efficiently.

Lemma 28 *Finding a matching M in a multicycle G such that $s'(G \setminus M) = s'(G) - 1$ and M has minimum size can be done in $O(n)$ time.*

Proof. The three cases of the previous proof must be checked. In the case where $\lceil m/k \rceil > \Delta$ and $m = \lfloor m/k \rfloor \cdot k + q$, we can pick any matching of size q, which can clearly be done in linear time. In the second case, when $\lceil m/k \rceil < \Delta$, we need to find a minimum matching hitting all vertices of degree Δ. This can be achieved in linear time as well by proceeding in a clockwise greedy fashion.

Finally, in the last case, we need to find a minimum set of at least q edges that together hit all vertices of degree Δ. This can also be achieved in $O(n)$ time as follows. We first find the minimum matching hitting all maximum degree vertices. If the resulting matching has size at least q, then we are done and back to the previous case. Otherwise, we need to include additional edges. For that purpose, we can proceed in the clockwise direction and iteratively extend each block in order to include the exact number of additional edges. This can take linear time as well if we took care to count the size of each block and of the gaps between them in the previous pass. □

Theorem 41 *Algorithm* MULTICYCLECOLOR *finds a minimum sum coloring of a multicycle on n vertices and with maximum degree Δ in time $O(\Delta n)$.*

Proof. The number of iterations of the algorithm is at most $s'(G) = \max\{\lceil m/k \rceil, \Delta\}$. Hence the running time is $O(\max\{\lceil m/k \rceil n, \Delta n\}) = O(\max\{m, \Delta n\}) = O(\Delta n)$. □

We deliberately ignored the situation in which after some iterations, the multicyle G_i does not contain a full cycle anymore, that is, one of the edge multiplicity m_i drops to 0. We are then left with a collection of disjoint *multipaths*, for which the minimum sum coloring problem becomes easier. This special case is described in the following section.

A linear time algorithm for even length multicycles

We turn to the special case $n = 2k$, that is, the number of vertices is even. We show that in that case, minimum sum colorings have a convenient property that can be exploited in a fast algorithm. This algorithm first colors a uniform multicycle contained in G such that the remaining edges of G form a (possibly unconnected) multipath. This multipath is then colored separately.

We begin this section by the following result on multipaths. We consider multipaths with vertices labelled $\{1, 2, \ldots, n\}$, such that edges are only between vertices of the form $i, i+1$.

Lemma 29 *There always exists a minimum sum edge coloring of a multipath H, such that its color classes E_i are maximum matchings in the graphs $H_i = H \setminus \cup_{j=1}^{i-1} E_j$; furthermore these matchings contain all the edges appearing in odd position from left to right in each connected component of H_i.*

Proof. Suppose H has been colored optimally. We want to transform such an edge coloring into another one that verifies the hypothesis of the theorem. Let i be the minimum positive integer for which E_i does not verify the hypothesis. Note that the color of every edge in H_i is at least i.

We can assume that H_i is connected, the following reasoning being applicable to each connected component. We first remark that E_i is a maximal matching, otherwise one edge can be recolored with color i, contradicting the optimality of the given edge coloring. Thus E_i can be partitioned in *blocks*, defined as maximal sequences of consecutive vertices $\{y_1, y_2, \ldots, y_{2t}\}$ such that one of the edges between y_{2j-1} and y_{2j} has color i, for $1 \leq j \leq t$. Two consecutive blocks are separated by a single vertex whose incident edges have colors strictly greater than i. We now show that if a block starts at an even vertex, it can be recolored without decreasing the color sum. We let y_0 be the vertex preceding y_1.

Recoloring: Let $\alpha_1 \neq i$ be any color appearing on the edges between y_0 and y_1. We recolor an edge of color α_1 with color i and color the edge between vertices y_1 and y_2 of color i with color α_1. Now, for each j, with $1 < j \leq t$, we recolor the edge $y_{2j-1}y_{2j}$ of color i with a color α_j appearing on the edges $y_{2j-2}y_{2j-1}$, and color the edge $y_{2j-2}y_{2j-1}$ of color α_j with color i. The color α_j is chosen such that $\alpha_j = \alpha_{j-1}$ if color α_{j-1} appears on edges $y_{2j-2}y_{2j-1}$, and it is any color appearing on edges $y_{2j-2}y_{2j-1}$ otherwise. At the end, we have two cases. Either y_{2t} is the last vertex of the path, and we are done, or there exist edges between y_{2t} and, say, y_{2t+1}. One of these edges may be of color α_t, and can be recolored with color i. Otherwise, any such edge can be recolored with color i. Since, by definition, y_{2t+1} was not incident to any edge with color i, this yields a proper coloring whose sum is not greater than the original one.

Now, it is clear that we can assume that every block starts at an odd vertex. This implies that there is only one block. Furthermore, this block must start with the first vertex of the path. Hence E_i is a maximum matching containing all the edges appearing in odd position. $\square$

From Lemma 29, we can deduce the following result, that settles the case of multipaths.

Theorem 42 *The greedy algorithm that iteratively picks a maximum matching formed by all edges appearing in odd position in each connected component of a multipath H, computes a minimum edge sum coloring of H in time $O(m)$.*

We now consider the case of even multicycles. We assume that the vertices in the multicycle G on $n = 2k$ vertices are labelled clockwise with integers $0, 1, \ldots, n-1$, and arithmetic operations are taken modulo n. For each $0 \leq i < n$, let m_i denote the number of parallel edges between two consecutive vertices i and $i+1$ in G. Let p be a positive integer. A multicycle G with $m = pn$ edges is called p-*uniform* if $m_i = p$ for every i such that $0 \leq i < n$.

Lemma 30 *Let G be a multicycle of even length and let $p = \min_i m_i$. Let f be any minimum sum edge coloring of G. Then, f can be transformed into another minimum sum edge coloring f' such that the first $2p$ color classes E_i induced by f', with $1 \leq i \leq 2p$, are such that $|E_i| = k$ and their union induces a p-uniform multicycle.*

Proof. Let G be a multicycle on n vertices, with $n = 2k$ for some integer $k > 1$. Let f be any minimum sum edge coloring of G. Clearly, as f is minimum, we have that $|E_1| \geq |E_2| \geq \cdots \geq |E_{\chi'}|$. Let us consider the following claim.

Claim 3 *The coloring f can be transformed into a minimum sum edge coloring f' having the property that the edges colored with colors 1 and 2 induce a subgraph of G isomorphic to a cycle.*

Notice that, by using Claim 3, the lemma follows directly by induction on p. So, in order to prove Claim 3, first notice that, by using a similar recoloring argument as in the proof of Lemma 29, we can deduce that $|E_1| = k$.

Now, without loss of generality, assume that f is such that there is an edge colored with color 1 between vertices $2j$ and $2j+1$ for each j with $0 \leq j < k$. Moreover, let $c \geq 2$ be the minimum color appearing on the edges between vertices $2j+1$ and $2j+2$, for all $0 \leq j < k$.

Suppose that there exists a maximal sequence $i_1, \ldots, i_{2t}$ of consecutive vertices in G, such that colors 1 and c belong to the set of colors assigned by f to the edges between vertices i_{2q-1} and i_{2q}, with $1 \leq q \leq t$. Then by using the same recoloring argument as in the proof of Lemma 29, we can move color c in order to transform such a sequence into a $(c, 1)$-path. Moreover, again by using the same recoloring argument as in the proof of Lemma 29, we can deduce that $|E_c| = k$.

So, if $c = 2$ we are done, otherwise, we can swap the colors 2 and c so that $|E_2| = k$ and $E_1 \cup E_2$ induce a cycle. □

Theorem 43 *There exists an $O(m)$-time algorithm for computing a minimum sum edge coloring of a multicycle G of even length with m edges.*

Proof. Let $n = 2k$ be the number of vertices in G and let $p = \min_i\{m_i\}$, for $0 \leq i < n$. For each $0 \leq j < k$, assign to p edges between vertices $2j$ and $2j+1$ the odd colors $1, 3, \ldots, 2p-1$ and assign to p edges between vertices $2j+1$ and $2j+2$ the even colors $2, 4, \ldots, 2p$.

The previous pn colored edges induce a subgraph of G isomorphic to a p-uniform multicycle. When removing this p-uniform multicycle from G, we obtain a multipath or a set of disjoint multipaths, the edges of which can be colored with colors in $\{2p+1, \ldots, s'(G)\}$, from Theorem 42.

Such a coloring can be computed in $O(m)$ time, and by Lemmas 30 and 29, it is a minimum sum edge coloring of G. □

4.2.2 Generalization

In the generalized optimal cost chromatic partition problem [49], each color has an integer cost, but this cost is not necessarily equal to the color itself. The cost of a vertex coloring is $\sum_{v \in V} c(f(v))$, where $c(i)$ is the cost of color i. For any set of costs, our proofs can be generalized to show that on one hand, the minimum number of colors needed in a minimum cost edge coloring of G is equal to $\chi'(G)$ when G is bipartite or a multicycle, and on the other hand that a minimum cost coloring can be computed in $O(\Delta n)$ time for multicycles.

In fact, our results can be generalized to an even broader class of edge coloring problems. Given an edge coloring $f : E \mapsto \mathbb{N}$, we define a cost $C(f)$ of the form:

$$C(f) = \sum_i c(i, |f^{-1}(i)|),$$

where $c : \mathbb{N} \times \mathbb{N} \mapsto \mathbb{R}$ is a real function of a color i and an integer k, and $f^{-1}(i)$ is the set of edges e such that $f(e) = i$. Hence the cost to minimize is a sum of the cost of each color class, itself defined as some function of the color and the size of the color class.

In the minimum sum coloring problem, the function c is defined by

$$c(i, k) = i \cdot k.$$

We further suppose that the functions $c(i, k)$ satisfy the following property:
Given two nonincreasing integer sequences $a_1 \geq a_2 \ldots \geq a_n$ and $b_1 \geq b_2 \ldots \geq b_n$ such that

$$\sum_{i=1}^{j} a_i \geq \sum_{i=1}^{j} b_i, \ \forall j = 1, \ldots, n,$$

we have

$$\sum_{i=1}^{n} c(i, a_i) \leq \sum_{i=1}^{n} c(i, b_i). \tag{4.1}$$

This property clearly holds in the minimum sum coloring problem. It formalizes the fact that when minimizing the cost $C(f)$, we are looking for a distribution of the color class sizes that is as nonuniform as possible. In particular, when an element (edge or vertex) in a color class i is recolored with a color $j < i$, whose class is larger, then the objective function decreases. This is the argument that we implicitly used in our proof of Theorem 40. It is also the argument that ensures the correctness of the algorithms.

Property (4.1) can also be shown to hold (see [27]) when the following two conditions are satisfied:

1. $c(i, k) = c(j, k) \ \forall i, j$, that is, when the cost of a class only depends on its size, in which case we will say that the functions are *separable*,

2. the functions $c(i, k) = c(k)$ are concave.

This is the case for instance in the minimum entropy edge coloring problem [12], for which $c(k) = -\frac{k}{m} \log \frac{k}{m}$. A number of other coloring problems falling in that class were recently studied by Fukunaga, Halldórsson, and Nagamochi [27].

For all minimum cost edge coloring problems whose objective function satisfies (4.1), all our results apply. In fact, the colorings that we compute are *robust* colorings, in the sense that they minimize every objective function satisfying the above property.

4.3 MSSC problem in block graphs

Assume that we are given a graph $G = (V, E)$ with a demand function $\omega : V \to Z^+$. We will denote $n = |V|$, $d(v)$ the degree of a vertex $v \in V$, $\Delta = \max_{v \in V} d(v)$ and $\omega_{\max} = \max_{v \in V} \omega(v)$. The neighborhood of a vertex v will be denoted by $N_G(v)$.

We can consider that the input of our problem is a graph $G = (V, E)$ with a demand function $\omega : V \to Z^+$, so the input size would be $|V| + |E| + \sum_{v \in V} \log(\omega(v))$. In the preemptive case it makes some sense to consider as the size of the problem $|V| + |E| + \sum_{v \in V} \omega(v)$, since it is the output size. Nevertheless, we will call (pseudo)polynomial time algorithms to those that are polynomial on $|V| + |E| + \sum_{v \in V} \omega(v)$.

It is known than the minimum number of colors that can be used in an optimum solution of the MSC problem on a graph G is bounded by $\Delta(G)+1$. Assume that we are given a graph $G = (V, E)$ with a demand function $\omega : V \to Z^+$. Denote by $C(G, \omega)$ (resp. $C'(G, \omega)$) be the minimum number of colors that can be used in an optimum solution of the non-preemptive (resp. preemptive) MSSC problem on G with demand function ω. It is easy to generalize the bound above and get $C'(G, \omega) \le \omega_{\max}(\Delta(G)+1)$. We now show the following lemma.

Lemma 31 *Let G be a graph and let ω be a demand function from the vertices of G to the set of positive integers. Then $C(G, \omega) \le 2\omega_{\max}(\Delta + 1)$.*

Proof. Let v be the vertex using the highest color. Since the goal is to minimize the total sum of colors, it uses at most the interval $[c + 1, c + \omega(v)]$, where c is the maximum color used by one of its neighbors. In the worst case, all of its neighbors use disjoint intervals, the smaller beginning at $\omega(v)$ and separated by intervals of size $\omega(v) - 1$. So $c \le \sum_{w \in N_G(v)}(\omega(w) + \omega(v) - 1) \le (2\omega_{\max} - 1)\Delta = 2\omega_{\max}\Delta - \Delta$, and thus $c + \omega(v) \le 2\omega_{\max}(\Delta + 1) - \Delta < 2\omega_{\max}(\Delta + 1)$. □

Let $P = V_1, \ldots, V_t$ be a partition of the vertices of a graph G with demand ω. We will call P-*good* to a coloring of (G, ω) where each V_i, $i = 1, \ldots, t$, is colored with sum $\Sigma(G[V_i], \omega)$. Clearly, a P-good coloring is optimum for the MSSC problem, and if G admits a P-good coloring, then every optimum coloring of G must be P-good.

4.3.1 Minimum sum set-coloring of trees

In this section we deal with the MSSC problem on trees. First, we show that the non-preemptive version of the MSSC problem on trees can be computed in (pseudo)polynomial time. Next, we show that the preemptive version of the MSSC problem on trees is NP-hard.

Non-preemptive case

The algorithm for solving the non-preemptive MSSC problem for trees is based on the idea of dynamic programming. For, we first choose an arbitrary vertex r of T as the root. For each vertex v of T, we denote by T_v the subtree of T rooted at vertex v. Let C be an upper bound for $C(T, \omega)$. We have a $n \times C$ table S such that, $S[v, j]$ represents the minimum sum for the subtree T_v when vertex v is assigned the interval $[j, j+\omega(v)-1]$, for every vertex $v \in T$ and every $1 \leq j \leq C$.

First, we define for each vertex v in T and each integer $1 \leq j \leq C$ the value $q_v(j)$ as follows:

$$q_v(j) = \begin{cases} \sum_{i=j}^{j+\omega(v)-1} i = j\omega(v) + \binom{\omega(v)}{2}, & \text{if } j+\omega(v)-1 \leq C, \\ \infty, & \text{otherwise.} \end{cases}$$

The algorithm computes the values of table S in a bottom-up way, from the leaves of the tree up to the root. If vertex v is a leaf then $S[v,j] = q_v(j)$ for all $1 \leq j \leq C$. Assume that vertex v is an internal vertex in T. Let $v_1, v_2, \ldots, v_k$ be the children vertices of internal vertex v. Assume that $S[v_i, s]$ is computed, for $1 \leq i \leq k$ and $1 \leq s \leq C$, and we want to compute the value of $S[v,j]$ (for some j, with $1 \leq j \leq C$). Now, for each children vertex v_i we compute the value $f_v(v_i, j)$ as follows:

$$f_v(v_i, j) = \min_{1 \leq t \leq C} \{S[v_i, t] : [t, t+\omega(v_i)-1] \cap [j, j+\omega(v)-1] = \emptyset\}.$$

The algorithm also can keep track the following information: $g_v(v_i, j) = \min\{t : S[v_i, t] = f_v(v_i, j)\}$, which will be used for reconstruct the coloring. Once the values $f_v(v_i, j)$ have been determined for each $1 \leq i \leq k$, we can compute the value of $S[v,j]$ as follows:

$$S[v,j] = q_v(j) + \sum_{1 \leq i \leq k} f_v(v_i, j).$$

Now, knowing how to compute the value of $S[v,j]$ from the computed values of children of v, the algorithm starts from the leaves of T, and fills in the table, from bottom to up, until it computes the value of $S[r,C]$, where r is the root of T. One can easily verify that the minimum value of $S[r,j]$, for $1 \leq j \leq C$, is the value of an optimal sum for the non-preemptive MSSC of (T, ω) and the root vertex r can be assigned the interval $[j, j+\omega(r)-1]$, where j is the minimum value m for which $S[r,m]$ is minimum. Finally, assuming that an internal vertex v has been assigned an interval $[j, j+\omega(v)-1]$, the interval assigned to each one of its children v_i will be the interval $[g_v(v_i, j), g_v(v_i, j) + \omega(v_i) - 1]$.

Notice that the complexity of computing $S[v,j]$ is $O(d(v)C)$, where $d(v)$ is the number of children of vertex v and C is the upper bound for the minimum number of colors needed in any optimal solution of the non-preemptive MSSC on (T, ω). In fact, in order to determine the value of $S[v,j]$, we need to compute the value $f_v(v_i, j)$ for each children v_i of v, each one of these values taking $O(C)$ steps. Therefore the overall complexity of the previous algorithm for computing an optimal solution for the non-preemptive MSSC problem on the entire tree is proportional to the number of edges, which, in a tree, gives rise to an $O(nC^2)$ algorithm. Thus, by Lemma 31, we have the following result.

Theorem 44 *The non-preemptive MSSC problem on trees can be solved in $O(n\Delta^2\omega_{\max}^2)$, that is, in (pseudo)polynomial time.*

Preemptive case

We will show that the preemptive MSSC problem on trees is in general NP-hard, even considering $\sum_{v\in V(T)} \omega(v)$ as the input size.

Theorem 45 *The preemptive MSSC problem on trees is NP-hard.*

Proof. The reduction is from 3-SAT. First we give some definitions and we introduce some families of special trees that will be used as gadgets in the NP-hardness proof.
For a, b positive integers, $a \leq b$, let $T_{[a,b]}$ be a tree with root r with demand $\omega(r) = b-a+1$ and, if $a > 1$, two children v_1, v_2 with demand $\omega(v_1) = \omega(v_2) = a - 1$ (see Figure 4.4).

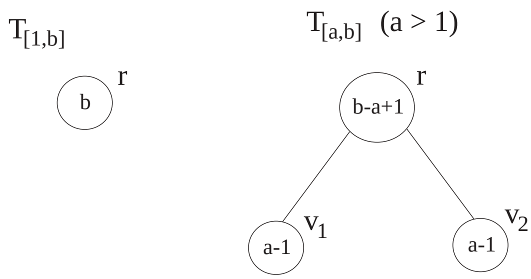


Figure 4.4: The tree $T_{[a,b]}$ for $1 \leq a \leq b$.

The trees $T_{[a,b]}$ admit P-good colorings for suitable partitions P : if $a = 1$ then the partition P is trivial, if $a > 1$ then the partition $P = \{r, v_1\}, \{v_2\}$ is such that $T_{[a,b]}$ admits a P-good coloring. Moreover, in every P-good coloring of $T_{[a,b]}$, vertex v_2 should receive colors $1, \ldots, a-1$ and therefore vertex r should receive colors $a, \ldots, b$ (and vertex v_1 colors $1, \ldots, a-1$ as well).
Let $\{a_1, \ldots, a_k\}$ be a set of positive integers, $a_1 < \cdots < a_k$, and let $S = \sum_{i=1}^{k} a_i - \binom{k+1}{2} + 1$. We will define the tree $T_{\{a_1,\ldots,a_k\}}$. The root r has demand $\omega(r) = 1$. The children of r are the following: a child v with demand $\omega(v) = k-1$; S children each of them being the root of $T_{[1,a_1-1]}$, when $a_1 > 1$; S children each of them being the root of $T_{[a_i+1,a_{i+1}-1]}$, when $a_{i+1} > a_i + 1$ for each $i = 1, \ldots, k-1$. Besides, vertex v has S children each of them being the root of $T_{[1,a_1-1]}$, when $a_1 > 1$; S children each of them being the root of $T_{[a_i+1,a_{i+1}-1]}$, when $a_{i+1} > a_i + 1$ for each $i = 1, \ldots, k-1$ (see Figure 4.5). We will analyze now the possible solutions to the MSSC problem on $T_{\{a_1,\ldots,a_k\}}$. If all the trees $T_{[a,b]}$ are colored in an optimum way, then r and v should receive colors $\{a_1, \ldots, a_k\}$, and the overall sum is $\sum_{i=1}^{k} a_i + D$ where D is the sum of $\Sigma(T_{[a,b]}, \omega)$ over all the trees $T_{[a,b]}$ involved in $T_{\{a_1,\ldots,a_k\}}$. On the other hand, suppose that r and v receive a set of colors $\{a'_1, \ldots, a'_k\}$ different from $\{a_1, \ldots, a_k\}$ in an optimum coloring of $T_{\{a_1,\ldots,a_k\}}$. Since D is locally optimum, then $\sum_{i=1}^{k} a'_i \leq \sum_{i=1}^{k} a_i$, so at least one of the colors not in $\{a_1, \ldots, a_k\}$ is less or equal than $a_k - 1$, and thus at least S trees $T_{[a,b]}$ are colored in a non-optimum way. Therefore the overall sum would be at least $\sum_{i=1}^{k} a'_i + D + S$ and since $\sum_{i=1}^{k} a'_i \geq \binom{k+1}{2}$, this contradicts the optimality of the coloring. Moreover, it is not difficult to see that for each $i = 1, \ldots, k$, there is an optimum coloring of $T_{\{a_1,\ldots,a_k\}}$ where r receives color a_i.

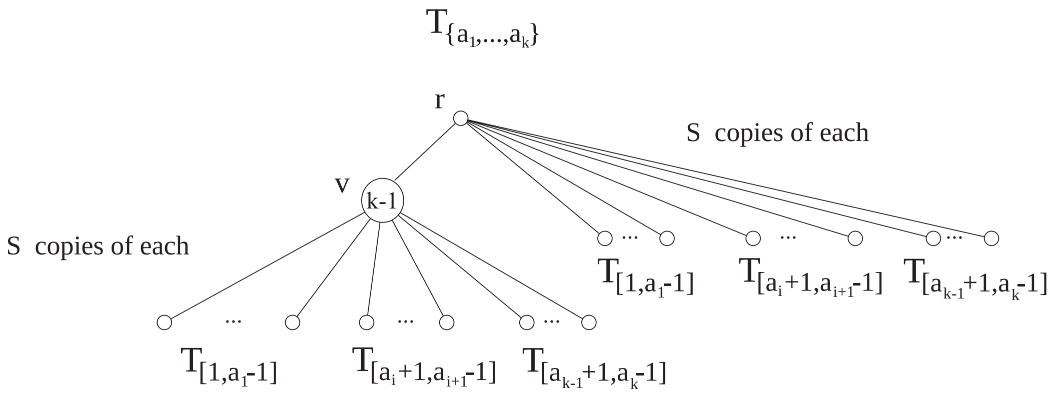


Figure 4.5: The tree $T_{\{a_1,\ldots,a_k\}}$.

Now, let $\mathcal{I}$ be an instance of 3-SAT, with n variables and m clauses. We will construct $T_{\mathcal{I}}$ as follows: it has a root r with demand $\omega(r) = n$; the root has $n + m + 1$ children $v_1, \ldots, v_n$, $w_1, \ldots, w_m$, all of them with demand 1, and z with demand $\omega(z) = n$; each v_i $(1 \leq i \leq n)$ is the root of a copy of $T_{\{2i-1,2i\}}$; each w_i $(1 \leq i \leq m)$ is the root of a copy of $T_{\{a_1^i,a_2^i,a_3^i\}}$, where a_1^i, a_2^i, a_3^i are the values corresponding to the three literals of the i-th clause of $\mathcal{I}$ in increasing order, assigning to variable k the value $2k-1$ and to its negation the value $2k$. Let $P = \{\{r, z\}, V(T_{\mathcal{I}}) \setminus \{r, z\}\}$ be a partition of the vertices of $T_{\mathcal{I}}$. We will show that $\mathcal{I}$ is satisfiable if and only if $(T_{\mathcal{I}}, \omega)$ admits a P-good coloring, that is, a coloring with sum $\binom{2n+1}{2} + \Sigma(T_{\mathcal{I}} \setminus \{r, z\}, \omega)$ (please note that this value can be computed in polynomial time based on the construction of $T_{\mathcal{I}}$ and the observations above). Suppose first that $\mathcal{I}$ is satisfiable and consider a truth assignment satisfying it. Then assign to z the values corresponding to true literals, to r the values corresponding to false literals, to each w_i the value of a literal satisfying its corresponding clause (and then extend this coloring to an optimum coloring of $T_{\{a_1^i,a_2^i,a_3^i\}}$), and to each v_i the value in $\{2i-1, 2i\}$ not used in r (and then extend this coloring to an optimum coloring of $T_{\{2i-1,2i\}}$). The coloring obtained is P-good. Conversely, suppose that $T_{\mathcal{I}}$ admits a P-good coloring. For $i = 1, \ldots, n$, since $T_{\{2i-1,2i\}}$ is colored in an optimum way, each v_i uses either color $2i-1$ or color $2i$. Moreover, since $\{r, z\}$ use the colors $\{1, \ldots, 2n\}$, r uses exactly one of $\{2i-1, 2i\}$ for each $i = 1, \ldots, n$, and z the other one. Let the variable i be true if $2i-1$ is used in z and false otherwise. For each $i = 1, \ldots, m$, since $T_{\{a_1^i,a_2^i,a_3^i\}}$ is colored in an optimum way, w_i uses one of the colors $\{a_1^i, a_2^i, a_3^i\}$ and then that color should not be used in r, so it should be used in z. If it is an odd color then the corresponding variable is true and appears in the i-th clause, otherwise the corresponding variable is false but its negation appears in the i-th clause. In both cases, the clause is satisfied and so $\mathcal{I}$ is satisfiable. □

However, if the maximum value of the demand function ω from vertices of a tree to the set of positive integers is bounded by a constant, then there is a polynomial-time algorithm for the preemptive MSSC problem on trees as it is shown in the next theorem.

Theorem 46 *Let $T = (V, E)$ be a tree and let $\omega : V \to Z^+$ be a demand function for T. Then the preemptive MSSC problem on T can be solved in $O(n(\Delta\omega_{\max})^{2\omega_{\max}})$ time. In* *particular, if $\omega_{\max}$ is bounded by a constant, it can be solved in polynomial time.*

Proof. Let n be the number of vertices in T and let Δ be the maximum degree of the vertices in T. Let m, k be positive integers, and let $[m]^k$ denote the set of all k-subsets of $[m]$, where $[m]$ denotes the set $\{1, \ldots, m\}$. Given an arbitrary finite set A of positive integers, we denote by $q(A)$ the value of the sum of the elements in A (i.e. $q(A) = \sum_{i \in A} i$). Let C be an upper bound for $C'(T, \omega)$, thus C is bounded by $O(\omega_{\max}\Delta)$. The algorithm is based on the idea of dynamic programming. Let r be an arbitrary vertex in T chosen as the root. For each vertex v of T, we denote by T_v the subtree of T rooted at vertex v. Now, for each vertex v in T, we construct an array S_v of length $\binom{C}{\omega(v)}$ such that $S_v[X]$ represents the minimum sum for the subtree T_v when vertex v is assigned the subset $X \in [C]^{\omega(v)}$. The algorithm computes the values of the arrays S_v in a bottom-up way, from the leaves of the tree up to the root as follows.

If v is a leaf then $S_v[X] = q(X)$, for each subset $X \in [C]^{\omega(v)}$. Now, assume that vertex v is an internal vertex in T. Let $v_1, v_2, \ldots, v_t$ be the children vertices of internal vertex v. Assume that $S_{v_i}[X]$ is computed, for all $1 \leq i \leq t$ and for all $X \in [C]^{\omega(v_i)}$, and we want to compute the value of $S_v[Y]$ for some fixed subset $Y \in [C]^{\omega(v)}$. First, for each children vertex v_i we compute the value $f_v(v_i, Y)$ as follows:

$$f_v(v_i, Y) = \min_{X \in [C]^{\omega(v_i)}} \{S_{v_i}[X] : X \cap Y = \emptyset\}.$$

Once the values $f_v(v_i, Y)$ have been computed for all $1 \leq i \leq t$, we can compute the value of $S_v[Y]$ as follows:

$$S_v[Y] = q(Y) + \sum_{i=1}^{t} f_v(v_i, Y).$$

Now, knowing how to compute the values of the array S_v from the computed values of children of v, the algorithm starts from the leaves of T, from bottom to up, until it computes the values of $S_r[Z]$ for all $Z \in [C]^{\omega(r)}$. It is clear that the minimum value of $S_r[Z]$ taken over all subsets $Z \in [C]^{\omega(r)}$ is the value of an optimal sum for the preemptive MSSC problem for (T, ω). Notice that additional information can be maintained in order to reconstruct the coloring without affecting the overall time complexity of the algorithm.

The length of each array S_v is bounded by $\binom{C}{\omega_{\max}}$. Therefore, the time complexity in order to compute the value of $S_v[Y]$ for a given vertex v and a fixed subset $Y \in [C]^{\omega(v)}$ is at most $d(v)\binom{C}{\omega_{\max}}$, where $d(v)$ denote the degree of vertex v in T. Thus, the time complexity needed to fulfill the array S_v is at most $d(v)\binom{C}{\omega_{\max}}^2$. Therefore, the overall complexity of the previous algorithm is proportional to the number of edges in T times $\binom{C}{\omega_{\max}}^2$, that is, $O(n(\Delta\omega_{\max})^{2\omega_{\max}})$. □

4.3.2 Minimum sum set-coloring of line graphs of trees

The MSSC problem on the line graph $L(G)$ of a graph G and demand function ω is equivalent to the Minimum Sum Edge Coloring (MSEC) of a multigraph $\widetilde{G}$, obtained from G by multiplying each edge e by $\omega(e)$. Therefore, in the sequel, we assume that we have as input to the MSEC problem a tree $T = (V, E)$ and a demand function ω from the edge-set E to the set of positive integers.

Non-preemptive case

In the following, we show that the non-preemptive MSSC problem on line graphs of trees is in general NP-hard, even considering $\sum_{e \in E(T)} \omega(e)$ as the input size. The reduction we use is based on the results given by Marx in [65] for a similar optimization problem. First we give some definitions and we introduce three families of special trees that will be used as gadgets in the NP-hardness proof. For any two positive integers i, j with $i < j$, let $[i, j]$ denote the consecutive interval $\{i, i+1, \ldots, j-1, j\}$ of integers. Denote by E_v the set of edges incident to vertex v. Let $l(v)$ be the minimum sum taken on E_v in any non-preemptive sum set edge-coloring. Clearly, $l(v) \geq \frac{|E_v|(|E_v|+1)}{2}$. Let $G = (A \cup B, E)$ be a bipartite multigraph. A lower bound for $\Sigma(L(G), \omega)$ is $l(A) = \sum_{v \in A} l(v)$. A non-preemptive edge-coloring Ψ will be called A-*good* if $\sum_{e \in E} \Psi(e) = l(A)$. Notice that every A-good non-preemptive edge-coloring is clearly an optimum coloring, and if there is an A-good non-preemptive edge-coloring, then every optimum coloring is A-good.

We define the tree T_i, for $i \geq 1$, as follows. The tree T_1 is an edge rv, where r is the root vertex and where $\omega(rv) = 1$. For $i > 1$, the tree T_i is a path on five vertices r, v, v_1, v_2, v_3, being r the root vertex and where $\omega(rv) = 1$, $\omega(vv_1) = \omega(v_2v_3) = i - 1$, and $\omega(v_1v_2) = i$ (see Figure 4.6). Vertices v and v_2 are in A (black vertices in the figure), the remaining ones are in B. Consider the coloring $\Psi(rv) = i$, $\Psi(vv_1) = \Psi(v_2v_3) = [1, i-1]$), $\Psi(v_1v_2) = [i, 2i-1]$. This is an A-good coloring, thus it is an optimum coloring and every optimum coloring is A-good. Therefore, if Φ is an optimum coloring for T_i then it must be an A-good coloring with vertices v and v_2 in A, and it is easy to see that this implies edge rv is assigned color i in every optimum coloring.

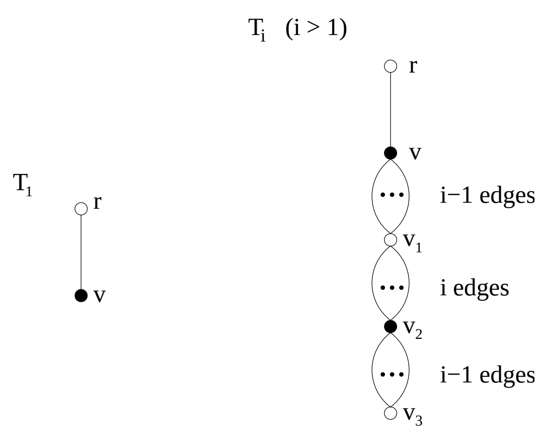


Figure 4.6: The tree T_i for $i \geq 1$.

A tree $T_{a,b,c}$ (for $a < b < c$) has root r having a single child v with $\omega(rv) = 1$; vertex v

has $c-1$ children $x, y, v_1, \ldots, v_{c-3}$ with $\omega(vx) = \omega(vy) = \omega(vv_1) = \cdots = \omega(vv_{c-3}) = 1$ (see Figure 4.7). Every vertex v_j is the root of a T_a, T_b and T_c tree, as defined in the previous paragraph. The black vertices in Figure 4.7 are in A. Notice that in every A-good (optimum) coloring of $T_{a,b,c}$ the edge rv is colored with color a, b or c, and there are three A-good colorings assigning a, b, and c to edge rv, respectively. The proof of this fact is exactly as appear in [65].

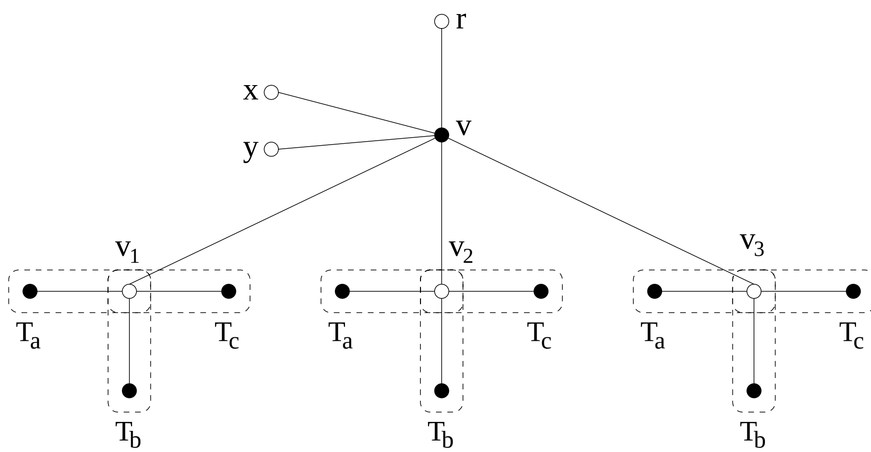


Figure 4.7: The tree $T_{a,b,c}$ with $c = 6$

Finally, we define tree $\hat{T}_i$ as follows. the vertex-set of $\hat{T}_i$ is composed by the vertices $r, x, v, v_1, v_2, v_3, v_4$ and v_5 where r is the root vertex, and the edge-set of $\hat{T}_i$ is the set $\{rv, vx, vv_1, v_1v_2, v_2v_3, v_3v_4, v_4v_5\}$, where $\omega(rv) = \omega(vx) = 2$, $\omega(vv_1) = \omega(v_2v_3) = \omega(v_4v_5) = i$, $\omega(v_1v_2) = 4$, and $\omega(v_3v_4) = i + 4$ (see Figure 4.8). We show that in every A-good (optimum) coloring of $\hat{T}_i$ the set of colors assigned to edges between vertices r and v is either $\{i+1, i+2\}$ or $\{i+3, i+4\}$. Let Ψ be an A-good coloring of $\hat{T}_i$. Notice that vertices v, v_2 and v_4 are in A. Thus, $\Psi(v_3v_4)$ can be equal to $[1, i+4]$ or equal to $[i+1, 2i+4]$. However, if $\Psi(v_3v_4) = [1, i+4]$ then $\Psi(v_2v_3)$ must be $[i+5, 2i+4]$, but as v_2 is in A, $i > 0$, and $\omega(v_1v_2) = 4$ then it contradicts that Ψ is an A-good coloring. Therefore, $\Psi(v_3, v_4) = [i+1, 2i+4]$ which implies that $\Psi(v_4v_5) = \Psi(v_2v_3) = \Psi(vv_1) = [1, i]$, $\Psi(v_1v_2) = [i+1, i+4]$. Moreover, as vertex v is in A and $\omega(rv) = \omega(vx) = 2$ then in an A-good non-preemptive coloring for $\hat{T}_i$, we have only two possibilities : $\Psi(rv) = \{i+1, i+2\}$ and $\Psi(vx) = \{i+3, i+4\}$, or $\Psi(rv) = \{i+3, i+4\}$ and $\Psi(vx) = \{i+1, i+2\}$.

Now, based in the three families of trees defined previously (i.e., the trees T_i, $T_{a,b,c}$ with $a < b < c$, and $\hat{T}_i$, resp.), we can prove the NP-hardness of the non-preemptive MSSC problem on line graphs of trees. The reduction is from 3-occurrence 3SAT, which is the restriction of 3SAT where every variable occurs at most three times. This problem is NP-complete even if every variable occurs at most twice positively and at most twice negatively (cf. [71]). Given a formula with n variables and m clauses, we construct a tree $T = (V, E)$ and a demand function $\omega : E \to Z^+$ such that T has an non-preemptive A-good edge-coloring if and only if the formula is satisfiable. Consider a variable x_k

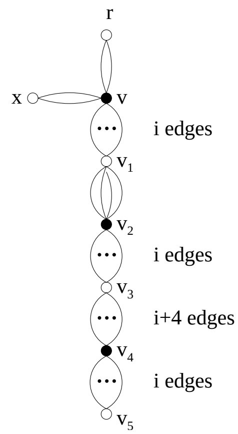


Figure 4.8: The tree $\hat{T_i}$ for $i \geq 1$

($0 \leq k < n$), which is the h-th literal of the i-th clause. Let $d_{i,h}$ be $4k+1$ if this is the first positive occurrence of x_k, $4k+2$ if this is the second positive occurrence, $4k+3$ if this is the first negated occurrence, and $4k+4$ if this is the second negated occurrence. The tree T has a vertex r which is the root of $n+m$ trees (assume that $r \notin A$). To each variable x_j corresponds a tree $\hat{T_{4j}}$, and to each clause i a tree $T_{d_{i,1},d_{i,2},d_{i,3}}$. This defines T and the demand function ω. The proof follows exactly as the one given by Marx (see Theorem 3.1 in [65]) for a similar optimization problem on trees. In fact, assume that a non-preemptive edge-coloring is A-good, then it is an A-good non-preemptive edge-coloring for all the $n+m$ subtrees (since $r \notin A$). Therefore, the root edge of $\hat{T_{4j}}$ corresponding to variable x_j uses either the set $\{4j+1, 4j+2\}$ or the set $\{4j+3, 4j+4\}$. Assign to the variable x_j the value "false" in the first case and "true" in the second case. It is a satisfying assignment: if the root edge of the tree corresponding to clause i uses a color from $\{4j+1, 4j+2, 4j+3, 4j+4\}$, the variable x_j satisfies clause i. Precisely, if it uses $4j+1$ or $4j+2$ (resp. $4j+3$ or $4j+4$), then x_j has the value "true" (resp. "false"), and by construction, x_j appears in clause i positively (resp. negatively). Conversely, given a satisfying assignment, we construct an A-good non-preemptive edge-coloring of the tree as follows. Take an A-good non-preemptive edge-coloring of the subtree $\hat{T_{4j}}$ corresponding to variable x_j such that its root edge uses the colors $\{4j+1, 4j+2\}$ (resp.$\{4j+3, 4j+4\}$) if x_j is "false" (resp. "true"). Since every clause is satisfied by some variable, we can choose an A-good non-preemptive edge-coloring for each subtree corresponding to a clause such that it does not conflict with any of the trees corresponding to the variables. Clearly, this will be an A-good non-preemptive edge-coloring of the tree. Therefore, as the previous reduction can be done in polynomial time, thus we have the following theorem.

Theorem 47 *The non-preemptive MSSC problem on line graphs of trees is NP-hard.*

In the following, we will show that under some constraints, there is a (pseudo)polynomial time complexity algorithm to solve the non-preemptive MSSC problem on line graphs of trees. Before, we need some preliminaries.

Let m, k be positive integers. Let $J \subset [m]$ be a subset of consecutive positive integers. Let $n_1, n_2, \ldots, n_{k+1}$ be positive integers (no necessarily different) such that $\sum_{i=1}^{k+1} n_i \leq m$

and such that $n_{k+1} = |J|$. Let $\mathbb{P}(m, J, n_1, \dots, n_k)$ be the set where each element is a k-set of intervals of consecutive integers $\{I_1, \dots, I_k\}$ pairwise disjoint contained in $[m] \setminus J$, such that $|I_i| = n_i$ for $1 \le i \le k$. Thus, we have the following result.

Lemma 32 *The cardinality of the set $\mathbb{P}(m, J, n_1, \dots, n_k)$ is bounded by m^k, and it can be computed in $O(k^2 m^k)$.*

Proof. Since the sets $\{I_1, \dots, I_k\}$ are intervals of consecutive integers, they are univocally defined by their starting point. So there are at most m^k possibilities for choosing the starting points of the k sets within the interval $[m]$ satisfying the constraints above. They can be generated by simple enumeration, and for each of them, the constraints satisfaction can be checked in $O(k^2)$ time. □

Theorem 48 *Let $T = (V, E)$ be a tree and let $\omega : E \to Z^+$ be a demand function defined on the edge-set of T. The non-preemptive MSSC problem for the line graph $L(T)$ of T can be solved in $O(n\Delta^{\Delta+3}\omega_{\max}^{\Delta+1})$. In particular, if Δ is bounded by a constant, then it can be solved in (pseudo)polynomial-time.*

Proof. The algorithm uses the dynamic programming method. Let n be the number of vertices of T. Without loss of generality, we choose as root of T a vertex r with degree equal to 1. As usual, we denote by T_v the subtree rooted at vertex v in T. Moreover, for each vertex v in T different from r, we denote by v' the father vertex of v. Finally, for each vertex v in T different from r, we denote by $T_{v'v}$ the subtree of T formed by T_v to which we join to v its father vertex v'. Given a subtree $T_{v'v}$, we say that v' is its root.

Assume now that C is an upper bound for $C(L(T), \omega)$. Since $\Delta(L(T)) \le 2\Delta(T)$, by Lemma 31, C is bounded by $O(\Delta\omega_{\max})$. We construct a $n \times C$ table S such that $S[v, j]$ represents the minimum sum for the subtree $T_{v'v}$ when edge $e = v'v$ is assigned the interval $[j, j + \omega(e) - 1]$, for every vertex $v \in T$ and every $1 \le j \le C$. First, we define for every vertex v ($v \ne r$) and every $1 \le j \le C$ the value $q_v(j)$ as follows:

$$q_v(j) = \begin{cases} j\omega(v'v) + \binom{\omega(v'v)}{2}, & \text{if } j + \omega(v'v) - 1 \le C, \\ \infty, & \text{otherwise.} \end{cases}$$

The algorithm computes the values of table S in a bottom-up way, from the leaves of T up to the root r. If vertex v is a leaf then $S[v, j] = q_v(j)$ for all $1 \le j \le C$. Otherwise, let v be an internal vertex in T with $v \ne r$. Let $v_1, v_2, \dots, v_k$ be the children vertices of v. Assume that $S[v_i, s]$ is computed, for all $1 \le i \le k$ and for all $1 \le s \le C$, and we want to compute the value of $S[v, j]$ for a fixed value j such that $j + \omega(v'v) - 1 \le C$ (otherwise, $S[v, j] = \infty$).

Consider now the set $\mathbb{P}_v(j) = \mathbb{P}(C, [j, j+\omega(v'v)-1], \omega(vv_1), \dots, \omega(vv_k))$. If $|\mathbb{P}_v(j)| = 0$ then $S[v, j] = \infty$. Otherwise, let X be an element of the set $\mathbb{P}_v(j)$. By definition, $X = \{I_1, \dots, I_k\}$, where for each i we have that $|I_i| = \omega(vv_i)$, I_i is an interval of consecutive integers, $I_i \cap [j, j + \omega(v'v) - 1] = \emptyset$ and $I_i \cap I_t = \emptyset$ whenever $i \ne t$. For each i, let $\alpha(X, i)$ be the minimum integer in the interval I_i of X.

Now, the value of $S[v,j]$ can be computed as follows:

$$S[v,j] = q_v(j) + \min_{X\in\mathbb{P}_v(j)}\{\sum_{i=1}^{k} S[v_i,\alpha(X,i)]\}.$$

Let z be the only children of root vertex r in T. Knowing how to compute the value of $S[v,j]$ from the computed values of children of v, the algorithm starts from the leaves filling the table, from bottom to up, until it computes the value of $S[z,C]$. It is easy to verify that the minimum value of $S[z,j]$, for $1 \leq j \leq C$, is the value of an optimal sum for the non-preemptive MSSC of $(L(T),\omega)$. Notice that additional information can be maintained in order to reconstruct the coloring without affecting the overall time complexity of the algorithm.

Now, in order to compute the value $S[v,j]$ for a given vertex v and a fixed integer j, the algorithm needs $O(C^{\Delta})$ time steps. In fact, by Lemma 32, the set $\mathbb{P}_v(j)$ has cardinality $O(C^{\Delta})$ and can be computed in $O(\Delta^2 C^{\Delta})$. Thus, the time complexity of compute the values $S[v,j]$, for all $1 \leq j \leq C$, is $O(\Delta^2 C^{\Delta+1})$. Finally, the overall time complexity of the previous algorithm on T is $O(n\Delta^2 C^{\Delta+1})$, thus $O(n\Delta^{\Delta+3}\omega_{\max}^{\Delta+1})$. □

4.4 Conclusions

In Section 4.1, we have studied the Minimum Sum Coloring (MSC) problem on P_4-sparse graphs. We have introduced the concept of maximal sequence associated with an optimal solution of the MSC problem of any graph. Next, based in such maximal sequences, we have shown that there is a large sub-family of P_4-sparse graphs for which the MSC problem can be solved in polynomial-time. An interesting open problem is the complexity of the MSC problem on P_4-sparse graphs.

In Section 4.2, we have studied the MSEC problem on multicycles and we have given a closed-form expression for the chromatic edge strength for such familly of multigraphs, thereby extending a theorem due to Berge. It is shown that the minimum sum can be achieved with a number of colors equal to the chromatic index. We also propose simple algorithms for finding a minimum sum edge coloring of a multicycle. Finally, these results are generalized to a large family of minimum cost coloring problems.

Finally, in Section 4.3, we have defined the Minimum Sum Set Coloring (MSSC) problem which consists in assign a set of $\omega(v)$ positive integers to each vertex v of a graph so that the intersection of sets assigned to adjacent vertices be empty and the sum of the assigned set of numbers to each vertex of the graph is minimum. Clearly, when $\omega(v) = 1$ for each vertex v of the graph, the MSSC problem becomes the MSC problem. We have shown that the MSSC problem on trees is polynomial-time solvable in the *non-preemptive* case (i.e. the set of integers assigned to each vertex is a consecutive interval) but NP-hard in the *preemptive* case. Finally, we have shown that the *non-preemptive* case of the MSSC problem is NP-hard for line graphs of trees.

Chapter 5

Conclusions and Perspectives

In this manuscript, I have shown the recent progress on some variations of the famous classical problem of graph Theory : (i) the b-coloring problem, (ii) coloring properties on direct product graphs, and (iii) the Minimum Sum coloring problem. For each one of these problems, I have given some conclusions and some open problems.

In this section, I propose some research perspectives for each one of the coloring problems discussed in this manuscript.

b-coloring

- The computational complexity of the b-chromatic problem for chordal graphs is not known. A related problem consists in constructing polynomial-time approximation algorithms for computing the b-chromatic number of chordal graphs, or for subclasses of b-continuous graphs. In fact, even for subfamilies of chordal graphs like interval graphs, the complexity of this problem is unknown.

- An interesting research perspective on this topic is the study of exact and parametrized algorithms for some classes of graphs. A graduate thesis on this perspective, which I co-supervised with Flavia Bonomo, is actually in execution by a Ph.D. student at the Computer Science Department of the University of Buenos Aires, Argentina.

Direct products of graphs

One of the most famous open problem on this topic is the Hedetniemi conjecture : $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$. This problem is open even if G and H are both finite undirected vertex-transitive graphs. In Chapter 3, we have obtained a positive answer to Hedetniemi conjecture for some families of vertex-transitive graphs : Kneser graphs, Circular Graphs and powers of cycles. Notice that the last two subfamilies of graphs belong to the family of circulant graphs, that is, Cayley graphs of cyclic groups.

- It will be very interesting to study the chromatic number of the direct product of finite many undirected circulant graphs. Clearly, if Hedetniemi conjecture holds in this case, it means that it holds for Cayley graphs of finite Abelian groups.

- Another way to attack the Hedetniemi conjecture on vertex-transitive graphs can be the study of a more strong conjecture, proposed by Tardif, on such family of graphs : $\alpha_k(G \times H) = \max\{\alpha_k(G).|H|, \alpha_k(H).|G|\}$, where $\alpha_k(G)$ is the maximal size of an induced k-colorable subgraph of G.

- Concerning the idomatic partitions of direct product of finite many complete graphs, it will be interesting to know if there are other relations between properties of the direct product of some groups associated to such Cayley graphs, as the one found in [78].

Minimum Sum Coloring

Concerning this problem, two research perspectives may be suggested :

- As mentioned in Section 4.2 of Chapter 4, the line graph of a multicycle is a proper circular arc graph. However, not all proper circular arc graphs are line graphs of multicycles. Therefore, it will be interesting to extend our results to this family of graphs. In fact, it is known that the MSC problem is NP-hard for general interval graphs, but polynomial-time solvable for proper interval graphs. The former result implies that this problem is also NP-hard for general circular arc graphs. The computational complexity of this problem for proper circular arc graphs is an open question.

- The results presented in Section 4.1 of Chapter 4 are the first ones for the MSC problem on the family of P_4-sparse graphs. However, neither the computational complexity nor approximation algorithms are known for such a family of graphs.

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