Independence and coloring properties of direct products of some vertex-transitive graphs *

Mario Valencia-Pabon† Juan Vera‡

Abstract

Let $\alpha(G)$ and $\chi(G)$ denote the independence number and chromatic number of a graph G respectively. Let $G \times H$ be the direct product graph of graphs G and H. We show that if G and H are circular graphs, Kneser graphs, or powers of cycles, then $\alpha(G \times H) = \max\{\alpha(G)|V(H)|, \alpha(H)|V(G)|\}$ and $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$.

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1 Introduction

In this paper, we study the independence and chromatic numbers of finite direct products graphs of circular graphs, Kneser graphs and powers of cycles. In the case of circular and Kneser graphs, this is done via classical homomorphisms. For the direct product graph of powers of cycles, we first analyze its independence number and then we use such a result to compute its chromatic number.

The *direct product* $G \times H$ of two graphs G and H is defined by $V(G \times H) = V(G) \times V(H)$, and where two vertices $(u_1, u_2), (v_1, v_2)$ are joined by an edge in $E(G \times H)$ if $\{u_1, v_1\} \in E(G)$ and $\{u_2, v_2\} \in E(H)$. This product is commutative and associative in a natural way (see reference [10] for a detailed description on product graphs). A coloring of $G \times H$ can be easily derived from a coloring of any of its factors, hence $\chi(G \times H) \leq \min\{\chi(G), \chi(H)\}$. One of the outstanding problems in graph theory is a formula concerning the chromatic number of the direct product of any two graphs G and H, called the *Hedetniemi conjecture* [8] (see also [6, 7] and ref.), which states $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$. The inherent difficulty of Hedetniemi's conjecture lies in finding lower bounds for $\chi(G \times H)$. In this paper we prove the Hedetniemi's conjecture to be true in some classes of vertex-transitive graphs.

On the other hand, if I is an independent set of one factor, the pre-image of I under the projection is an independent set of the product. Then, $\alpha(G \times H) \geq \max\{\alpha(G)|H|, \alpha(H)|G|\}$. In this case it is known that the equality does not hold in general. In fact, Jha and Klavžar show in [11] that for any graph G with at least one edge and for any $j \in \mathbb{N}$ there is a graph H such that $\alpha(G \times H) > \max\{\alpha(G).|V(H)|, \alpha(H).|V(G)|\} + j$. In [16], Tardif asks whether

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†Departamento de Matemáticas, Universidad de los Andes, Cra. 1 No. 18A - 70, Bogotá, Colombia (mvalenci@uniandes.edu.co).

‡Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh PA 15213-3890, USA (jvera@andrew.cmu.edu).

$\alpha_k(G \times H) = \max\{\alpha_k(G)|H|, \alpha_k(H)|G|\}$ always holds for vertex-transitive graphs, where $\alpha_k(G)$ is the maximal size of an induced k-colourable subgraph of G. In this paper, we analyze this problem for some vertex-transitive graphs when $k = 1$.

In other related work, Larose and Tardif investigate in [12] the relationship between projectivity and the structure of maximal independent sets of finite direct products of several copies of the same graph G, being G a circular graph, a Kneser graph or a truncated simplices.

Independence and chromatic properties of circular graphs and Kneser graphs are analyzed using graph homomorphism. An edge-preserving map from $\phi : V(G) \to V(H)$ is called a *homomorphism* from G to H and it is denoted by $\phi : G \to H$. We say that G and H are *homomorphically equivalent* if there exist $\phi : G \to H$ and $\psi : H \to G$. Notice that if there is $\phi : G \to H$ then $\chi(G) \leq \chi(H)$. In particular if G and H are homomorphically equivalent then $\chi(G) = \chi(H)$. The following result is direct.

Lemma 1 *Let G be a graph and let H be an induced subgraph of G. Then, $G \times H$ and H are homomorphically equivalent and therefore, $\chi(G \times H) = \chi(H)$.*

In the context of vertex transitive graphs The "No-Homomorphism" lemma of Albertson and Collins is useful to get bounds on the size of independent sets.

Lemma 2 *(Albertson-Collins [2]) Let G, H be graphs such that H is vertex-transitive and there is a homomorphism $\phi : G \to H$. Then,*

$$\frac{\alpha(G)}{|V(G)|} \geq \frac{\alpha(H)}{|V(H)|}.$$

The chromatic number of a graph G and its independence number are closely related via the inequality

$$\chi(G) \geq \lceil |V(G)|/\alpha(G) \rceil.$$

Let K_n denotes the complete graph on n vertices. By using this relation, Lemma 2, and Lemma 1, we can deduce the following well known result.

Corollary 1 *Let $k \geq 2$ be an integer and let $n_1, n_2, \ldots, n_k$ be positive integers. Then,*

$$\alpha\left(\prod_i K_{n_i}\right) = \max_i \left\{ \left(\prod_j n_j\right)/n_i \right\} \text{ and } \chi\left(\prod_i K_{n_i}\right) = \min_i \{n_i\},$$

where $1 \leq i, j \leq k$.

2 Circular graphs

Let m, n be integers such that $m \geq 2n > 0$. The *circular graph* C_n^m is the Cayley graph for the cyclic group $\mathbb{Z}_m$ with connector set $\{n, n+1, n+2, \ldots, m-n\}$. These graphs play an important role in the definition of the star chromatic number defined by Vince in [17]. The following result can be easily deduced.

Lemma 3 *Let m, n be integers with $m \geq 2n > 0$. Then, $\alpha(C_n^m) = n$ and $\chi(C_n^m) = \lceil \frac{m}{n} \rceil$.*

Concerning homomorphisms between circular graphs, Bondy and Hell show in [3] the following result.

Lemma 4 *(Bondy-Hell [3]) Let m, n, k be positive integers such that $m \geq 2n$. Then, C_n^m and C_{kn}^{km} are homomorphically equivalent.*

Lemma 5 *Let r, m be positive integers and let $n_1, n_2, \ldots, n_r$ be positive integers such that $n_1 \leq n_2 \leq \ldots \leq n_r$ and $m \geq 2n_i$, for each $i \in [r]$. Then, $C_{n_r}^m$ is a subgraph of the graph $C_{n_1}^m \times C_{n_2}^m \times \ldots \times C_{n_r}^m$.*

Proof Let $\phi : C_{n_r}^m \to \prod_i C_{n_i}^m$ be the map defined by $x \mapsto (x, x, \ldots, x)$ for all $x \in V(C_{n_r}^m)$. It is easy to deduce that this map is an injective graph homomorphism. $\square$

By Lemma 5 and Lemma 1 we have the following result.

Corollary 2 *Let r, m be positive integers and let $n_1, n_2, \ldots, n_r$ be positive integers such that $m \geq 2n_i$, for each $i \in [r]$. Then, $\chi\left(\prod_i C_{n_i}^m\right) = \min_i\{\chi(C_{n_i}^m)\} = \min_i\left\{\left\lceil \frac{m}{n_i} \right\rceil\right\}$.*

Let $m_1, m_2, \ldots, m_r$ be positive integers, with $r \geq 1$. We denote by $[m_1, m_2, \ldots, m_r]$ the least common multiple of $m_1, m_2, \ldots, m_r$.

Theorem 1 *Let r be a positive integer, and let $m_1, m_2, \ldots, m_r, n_1, n_2, \ldots, n_r$ be positive integers such that $m_i \geq 2n_i$, for each $i \in [r]$. Then, $\chi\left(\prod_i C_{n_i}^{m_i}\right) = \min_i\{\chi(C_{n_i}^{m_i})\}$.*

Proof Let $m = [m_1, m_2, \ldots, m_r]$ and $k_i = m/m_i$ for each $i \in [r]$. By Lemma 4, for each i, we have $C_{n_ik_i}^m$ homomorphically equivalent to $C_{n_i}^{m_i}$. Therefore $\prod_i C_{n_ik_i}^m$ is homomorphically equivalent to $\prod_i C_{n_i}^{m_i}$. By Corollary 2, we have

$$\chi\left(\prod_i C_{n_i}^{m_i}\right) = \chi\left(\prod_i C_{n_ik_i}^m\right) = \min_i\left\{\chi(C_{n_ik_i}^m)\right\} = \min_i\left\{\left\lceil \frac{m_i}{n_i} \right\rceil\right\} = \min_i\left\{\chi(C_{n_i}^{m_i})\right\}.$$

$\square$

Lemma 6 *Let r, m be positive integers and let $n_1, n_2, \ldots, n_r$ be positive integers such that $m \geq 2n_i$, for each $i \in [r]$. Then, $\alpha\left(\prod_i C_{n_i}^m\right) = m^{r-1}\max_i\{\alpha(C_{n_i}^m)\} = m^{r-1}\max_i\{n_i\}$.*

Proof W.l.o.g. we can assume that $n_1 \leq n_2 \leq \ldots \leq n_r$. By Lemma 5, the graph $C_{n_r}^m$ is a subgraph of the graph $C_{n_1}^m \times C_{n_2}^m \times \ldots \times C_{n_r}^m$ and thus, there is a natural homomorphism (i.e. the inclusion map) from $C_{n_r}^m$ to $\prod_i C_{n_i}^m$. Moreover, as $\prod_i C_{n_i}^m$ is vertex-transitive, by Lemma 2 we have $\alpha(C_{n_r}^m)/m \geq \alpha\left(\prod_i C_{n_i}^m\right)/m^r$. Therefore,

$$\alpha\left(\prod_i C_{n_i}^m\right) \leq m^{r-1}\alpha(C_{n_r}^m) = m^{r-1}n_r = m^{r-1}\max_i\{n_i\}. \quad \square$$

Theorem 2 *Let r be a positive integer, and let $m_1, m_2, \ldots, m_r, n_1, n_2, \ldots, n_r$ be positive integers such that $m_i \geq 2n_i$, for each $i \in [r]$. Let $M = m_1m_2\ldots m_r$. Then, $\alpha\left(\prod_i C_{n_i}^{m_i}\right) = \max_i\{\alpha(C_{n_i}^{m_i})M/m_i\} = \max_i\{n_iM/m_i\}$.*

Proof Let $m = [m_1, m_2, \ldots, m_r]$ and let $k_i = m/m_i$ for each $i \in [r]$. By Lemma 4, $\prod_i C_{n_ik_i}^m$ is homomorphically equivalent to $\prod_i C_{n_i}^{m_i}$. Moreover, as $\prod_i C_{n_ik_i}^m$ and $\prod_i C_{n_i}^{m_i}$ are vertex-transitive, by Lemma 2, we have $\alpha(\prod_i C_{n_ik_i}^m)/m^r = \alpha(\prod_i C_{n_i}^{m_i})/M$. Now, by Lemma 6, we have $\alpha(\prod_i C_{n_ik_i}^m) = m^{r-1}\max_i\{n_ik_i\}$. W.l.o.g. we can assume that $n_1k_1 \leq n_2k_2 \leq \ldots \leq n_rk_r$. Therefore, $\alpha(\prod_i C_{n_i}^{m_i}) = n_rk_rM/m = m_1m_2\ldots m_{r-1}n_r = \max_i\{n_iM/m_i\} = \max_i\{\alpha(C_{n_i}^{m_i})M/m_i\}$. $\square$

3 Kneser graphs

Let m, n be positive integers such that $m \geq 2n$. The *Kneser graph* K_n^m is the graph whose vertices are the n-subsets of $\{0, 1, \ldots, m-1\}$, where two vertices are adjacent if they are disjoint. In a celebrated paper, Lovász shows the following result.

Theorem 3 *(Lovász [13]) The chromatic number of K_n^m is $m - 2n + 2$.*

The independence number of Kneser graphs is related to the following classical inequality.

Theorem 4 *(Erdös-Ko-Rado, [5]) Let m, n be positive integers such that $n < m/2$, and $\mathbb{F}$ a family of pairwise intersecting n-subsets of $[m]$. Then $|\mathbb{F}| \leq \binom{m-1}{n-1}$.*

Theorem 4 implies that the sets $I_k = \{A \in V(K_n^m) : k \in A\}$ are independent sets of maximal cardinality in K_n^m, for $k = 0, 1, \ldots, m-1$. Hilton-Milner [9], show that those are the only independent sets of maximal cardinality in K_n^m.

Concerning homomorphisms between Kneser graphs, Stahl shows the following useful result.

Theorem 5 *(Stahl [15]) Let m, n be integers such that $n > 1$ and $m \geq 2n$. Then, there is an homomorphism from K_n^m to K_{n-1}^{m-2}.*

Lemma 7 *Let n, r be positive integers and let $m_1 \leq m_2 \leq \ldots \leq m_r$ be positive integers such that $m_i \geq 2n$, for $i \in [r]$. Then, $K_n^{m_1}$ is a subgraph of the graph $K_n^{m_1} \times K_n^{m_2} \times \ldots \times K_n^{m_r}$.*

Proof Let $\Phi : K_n^{m_1} \to \prod_i K_n^{m_i}$ be the map defined by $\Phi(A) = (A, A, \ldots, A)$ for all $A \in V(K_n^{m_1})$. It is clear that this map is an injective homomorphism. $\square$

By Lemma 7, Lemma 1 and Theorem 3 we can deduce the following result.

Corollary 3 *Let n, r be positive integers and let $m_1, m_2, \ldots, m_r$ be positive integers such that $m_i \geq 2n$, for $i \in [r]$. Then, $\chi(\prod_i K_n^{m_i}) = \min_i\{\chi(K_n^{m_i})\} = \min_i\{m_i\} - 2n + 2$.*

Lemma 8 *Let r be a positive integer, and let $m_1, m_2, \ldots, m_r, n_1, n_2, \ldots, n_r$ be positive integers such that $m_i \geq 2n_i$, for $i \in [r]$, and assume that $n_1 \leq n_2 \leq \ldots \leq n_r$, with $n_r > 1$. Then, there is a graph homomorphism $\Phi : \prod_i K_{n_r}^{m_i + 2(n_r - n_i)} \to \prod_i K_{n_i}^{m_i}$.*

Proof By Theorem 5, for each $i \in [r]$, there is a graph homomorphism $\phi_i : K_{n_r}^{m_i+2(n_r-n_i)} \to K_{n_i}^{m_i}$. Therefore, there is a graph homomorphism $\Phi : \prod_i K_{n_r}^{m_i+2(n_r-n_i)} \to \prod_i K_{n_i}^{m_i}$. $\square$

Theorem 6 *Let r be a positive integer, and let $m_1, m_2, \ldots, m_r, n_1, n_2, \ldots, n_r$ be positive integers such that $m_i \geq 2n_i$, for $i \in [r]$. Then, $\chi\left(\prod_i K_{n_i}^{m_i}\right) = \min_i\{\chi(K_{n_i}^{m_i})\}$.*

Proof W.l.o.g. we can assume that $n_1 \leq n_2 \leq \ldots \leq n_r$, and assume that $n_r > 1$. Then, by Lemma 8, there is a graph homomorphism $\Phi : \prod_i K_{n_r}^{m_i+2(n_r-n_i)} \to \prod_i K_{n_i}^{m_i}$, which implies that $\chi(\prod_i K_{n_i}^{m_i}) \geq \chi\left(\prod_i K_{n_r}^{m_i+2(n_r-n_i)}\right)$. By Corollary 3 we have $\chi\left(\prod_i K_{n_r}^{m_i+2(n_r-n_i)}\right) = \min_i\{m_i + 2(n_r - n_i) - 2n_r + 2\} = \min_i\{m_i - 2n_i + 2\} = \min_i\{\chi(K_{n_i}^{m_i})\}$. $\square$

Let m, n be positive integers such that $m \geq 2n$. The circular graph C_n^m is a subgraph of the Kneser graph K_n^m. More precisely the map $\phi : C_n^m \rightarrow K_n^m$ defined by $\phi(u) = \{u, u+1, \ldots, u+n-1\}$ (arithmetic operations are taken modulo m) is an injective graph homomorphism. Notice that the Erdös-Ko-Rado inequality (Theorem 4) can be easily deduced by using the fact that C_n^m is a subgraph of K_n^m, and then, using the No-Homomorphism-Lemma (Lemma 2). In the same way, we can deduce the independence number of the direct product of Kneser graphs, which is a particular case of a more general result of Ahlswede, Aydinian, and Khachatrian [1] in extremal set theory.

Theorem 7 *Let r be a positive integer, and let $m_1, m_2, \ldots, m_r, n_1, n_2, \ldots, n_r$ be positive integers such that $m_i \geq 2n_i$, for $i \in [r]$. Let $N = \prod_i \binom{m_i}{n_i}$. Then,*

$$\alpha\left(\prod_i K_{n_i}^{m_i}\right) = \max_i \left\{\alpha(K_{n_i}^{m_i})N/\binom{m_i}{n_i}\right\}.$$

Proof We know that for each $i \in [r]$, we have that $C_{n_i}^{m_i}$ is a subgraph of $K_{n_i}^{m_i}$. Therefore, there is a homomorphism from $\prod_i C_{n_i}^{m_i}$ to $\prod_i K_{n_i}^{m_i}$. Let $M = \prod_i m_i$. By Lemma 2, we have $\alpha(\prod_i C_{n_i}^{m_i})/M \geq \alpha(\prod_i K_{n_i}^{m_i})/N$. Moreover, by Theorem 2, $\alpha\left(\prod_i C_{n_i}^{m_i}\right) = \max_i\{n_i M/m_i\}$. Thus, $\alpha(\prod_i K_{n_i}^{m_i}) \leq N \max_i\{n_i/m_i\} = \max_i\{\binom{m_i-1}{n_i-1}N/\binom{m_i}{n_i}\} = \max_i\{\alpha(K_{n_i}^{m_i})N/\binom{m_i}{n_i}\}$, which proves this theorem. $\square$

4 Powers of cycles

For positive integers n and a such that $n \geq 2a$, we denote by $C(n,a)$ the graph with vertex set $\{0, 1, \ldots, n-1\}$ and edge set $\{ij : i - j \equiv \pm k \mod n, 1 \leq k \leq a\}$; the graph $C(n,a)$ is the a-th power of the n-cycle $C(n,1)$. Notice that graph $C(n,a)$ is the complement graph of the circular graph C_{a+1}^n. Prowse and Woodall analyze in [14] a restricted coloring problem (the list-coloring problem) on powers of cycles. In particular, they show the following result.

Theorem 8 *(Prowse-Woodall [14]) Let n, a be positive integers such that $a \leq n/2$ and $n = q(a+1) + r$, where $q \geq 1$ and $0 \leq r \leq a$. Then, $\alpha(C(n,a)) = \lfloor \frac{n}{a+1} \rfloor = q$ and $\chi(C(n,a)) = \lceil \frac{n}{\alpha(C(n,a))} \rceil = a + 1 + \lceil \frac{r}{q} \rceil$.*

Let $V_1, V_2, \ldots, V_j$ be a vertex decomposition (i.e. a partition of the vertex set V) of the graph G. Then, it is easy to deduce that $\alpha(G) \leq \sum_i \alpha(G[V_i])$, where, for $1 \leq i \leq j$, $G[V_i]$ denotes the subgraph of G induced by V_i.

Lemma 9 *Let m, n, a be positive integers such that $a \leq n/2$, and let $\alpha = \alpha(C(n,a))$. Then,*

$$\alpha(K_m \times C(n,a)) = \max\{n, m\alpha\}.$$

Proof Let $n = q(a+1) + r$, with $q \geq 1$ and $0 \leq r \leq a$. By Theorem 8 we have that $\alpha = q$, and thus we need to prove that $\alpha(K_m \times C(n,a)) \leq \max\{n, mq\}$. Let I be a maximal independent set of $K_m \times C(n,a)$. We can assume that $|I| > n$. Otherwise, the lemma trivially holds. Thus, there exists $j \in \{0, \ldots, n-1\}$ such that there are at least two vertices in I with the second coordinate equal to j. As $C(n,a)$ is vertex transitive, we can assume that $j = 0$. As I is an independent set, there is no vertex in I having as second coordinate an integer i, such that $0 < i \leq a$ or such that $n - a \leq i \leq n - 1$. Thus, as $0 \leq r \leq a$, we can assume that the remaining vertices of I form an independent set in the induced subgraph $K_m \times C(n,a)[\{a+1, a+2, \ldots, n-r-1\}]$. This induced subgraph admits a vertex decomposition into $q-1$ subgraphs all of them isomorphic to $K_m \times K_{a+1}$. Therefore, by using Corollary 1, we have that $|I| \leq m + (q-1)\alpha(K_m \times K_{a+1}) = m + (q-1)\max\{m, a+1\}$. If $m \geq a+1$ then $|I| \leq mq$. Otherwise, $|I| \leq (a+1)q \leq n$. $\square$

Theorem 9 *For $i = 1, 2$, let n_i, a_i be positive integers such that $n_i \geq 2a_i$, and let $\alpha_i = \alpha(C(n_i, a_i))$. Then,*

$$\alpha(C(n_1, a_1) \times C(n_2, a_2)) = \max\{\alpha_1 n_2, \alpha_2 n_1\}.$$

Proof For $i = 1, 2$, arithmetic operations on the vertex set of $C(n_i, a_i)$ will be taken modulo n_i. Let $n_i = q_i(a_i+1) + r_i$, with $q_i \geq 1$ and $0 \leq r_i \leq a_i$. By Theorem 8, $\alpha_i = q_i$, for $i = 1, 2$. Let I be a maximal independence set in the graph $C(n_1, a_1) \times C(n_2, a_2)$. We should prove that $|I| \leq \max\{q_1 n_2, q_2 n_1\}$. We define $I_1 = \{x \in I : (x_1 - 1, x_2) \in I \text{ or } (x_1 + 1, x_2) \in I\}$ and $I_2 = I \setminus I_1$. For $x \in I$ we define $S_x = \{(x_1, x_2 + i) : i = 0, \ldots, a_2\}$ if $x \in I_1$ and $S_x = \{(x_1 + i, x_2) : i = 0, \ldots, a_1\}$ if $x \in I_2$.

Claim 1 *Let $x, y \in I$. If $x \neq y$ then $S_x \cap S_y = \emptyset$.*

Let $x, y \in I$ be such that $x \neq y$. First we show $y \notin S_x$ and $x \notin S_y$. W.l.o.g. assume $y \in S_x$. If $x \in I_1$, then $x_1 = y_1$ and $0 < y_2 - x_2 \leq a_2$. By the maximality of I, $\{(x_1, x_2 + i) : i = 1, \ldots, y_2 - x_2\} \subset I$, contradicting $x \in I_1$. By a similar argument $x \notin I_2$. Now, assume $S_x \cap S_y \neq \emptyset$. Note that if $x, y \in I_1$ or $x, y \in I_2$, then $x \in S_y$ or $y \in S_x$. Therefore, $x \in I_1$ if and only if $y \in I_2$. W.l.o.g. assume $x \in I_1$ and $y \in I_2$. Let $z \in S_x \cap S_y$. Then $z_1 = x_1$ and $z_2 = y_2$. Thus, $0 \leq y_2 - x_2 \leq a_2$ and $0 \leq x_1 - y_1 \leq a_1$, contradicting $x, y \in I$, proving this Claim.

Now, w.l.o.g. assume that $a_1 \leq a_2$ and $|I| > n_2 q_1$; and let $A = \cup_{x \in I} S_x$. By Claim 1, we have $|A| = |I_1|(a_2+1) + |I_2|(a_1+1) \geq |I|(a_1+1) > n_2 q_1(a_1+1)$. Then there is $0 \leq j < n_2$ such that $A_j = \{0 \leq x < n_1 : (x, j) \in A\}$ has size larger than $q_1(a_1+1)$. Given $x \in A_j$ let $\hat{x}$ be defined as the only point in I such that $(x, j) \in S_{\hat{x}}$. Also, for $i = 1, 2$, let $B_i = \{x \in A_j : \hat{x} \in I_i\}$ and let $B'_2 = \{x \in A_j : (x, j) = \hat{x} \in I_2\}$. By Claim 1, we have B'_2 is an independence set in $C(n_1, a_1)$ and $|A_j| = (a_1+1)|B'_2| + |B_1| \leq (a_1+1)q_1 + |B_1|$. Therefore, B_1 is nonempty. As $C(n_1, a_1) \times C(n_2, a_2)$ is vertex-transitive we can assume $A_j = \{x^i : i = 1, \ldots, |A_j|\}$ ordered such that $x^{i+1} > x^i$ for all i and $x^1 = 0 \in B_1$. Notice that as $|A_j| > q_1(a_1+1)$, then $x^{i+1} - x^i \leq a_1$ for all i. Now we want to prove B_2 is empty. For this assume $B'_2 \neq \emptyset$ and let $k = \min_i \left\{x^i \in B'_2\right\} = \min_i \left\{x^i \in B_2\right\}$. Then $x^{k-1} \in B_1$. Now, $\hat{x}^{k-1}, \hat{x}^k \in I$, but $\hat{x}^k_1 - \hat{x}^{k-1}_1 = x^k - x^{k-1} \leq a_1$ and $\hat{x}^{k-1}_2 - \hat{x}^k_2 = j - \hat{x}^{k-1}_2 \leq a_2$. Then $\hat{x}^{k-1}_2 = j$ and by maximality of I we get $x^k = x^{k-1} + 1$, but this contradicts $\hat{x}^k \in I_2$. Therefore B_2 is empty.

Finally, by a similar argument to the one above, for every $1 \leq i \leq |A_j|$ we have $\hat{x}^i_2 = \hat{x}^{i+1}_2$ and $x^{i+1} = x^i + 1$. Therefore there is $0 \leq j' < n_2$ such that $[0, n_1 - 1] \times \{j'\} \subseteq I$. W.l.o.g assume $j' = 0$. The vertices in $I \setminus [0, n_1 - 1] \times \{0\}$ belong to the induced subgraph $C(n_1, a_1) \times C(n_2, a_2)[\{a_2+1, a_2+2, \ldots, n_2 - r_2 - 1\}]$, which admits a vertex decomposition into $q_2 - 1$ subgraphs all of them isomorphic to $C(n_1, a_1) \times K_{a_2+1}$. Therefore, by Lemma 9, we have that $|I| \leq n_1 + \alpha(C(n_1, a_1) \times K_{a_2+1})(q_2 - 1) = n_1 + (q_2 - 1)\max\{n_1, (a_2+1)q_1\} \leq \max\{q_2 n_1, q_1 n_2\}$. $\square$

Theorem 10 *For $i = 1, 2$, let n_i, a_i be positive integers such that $n_i \geq 2a_i$, and let $\alpha_i = \alpha(C(n_i, a_i))$. Then,*

$$\chi(C(n_1, a_1) \times C(n_2, a_2)) = \min \{\chi(C(n_1, a_1)), \chi(C(n_2, a_2))\} = \min \left\{ \left\lceil \frac{n_1}{\alpha_1} \right\rceil, \left\lceil \frac{n_2}{\alpha_2} \right\rceil \right\}.$$

Proof For $i = 1, 2$, let $n_i = q_i(a_i + 1) + r_i$, with $q_i \geq 1$ and $0 \leq r_i \leq a_i$. By Theorem 8 we have that $\chi(C(n_i, a_i)) = \lceil \frac{n_i}{\alpha_i} \rceil$, where $\alpha_i = q_i$. Moreover, by Theorem 9, we have that $\alpha(C(n_1, a_1) \times C(n_2, a_2)) = \max\{n_1\alpha_2, n_2\alpha_1\}$. So, we have that $\chi(C(n_1, a_1) \times C(n_2, a_2)) \geq \lceil \frac{n_1 n_2}{\max\{n_1\alpha_2, n_2\alpha_1\}} \rceil$. Thus, if $n_1\alpha_2 \geq n_2\alpha_1$ then $\chi(C(n_1, a_1) \times C(n_2, a_2)) \geq \lceil \frac{n_1 n_2}{n_1 \alpha_2} \rceil = \lceil \frac{n_2}{\alpha_2} \rceil = \chi(C(n_2, a_2))$. Otherwise, $\chi(C(n_1, a_1) \times C(n_2, a_2)) \geq \lceil \frac{n_1 n_2}{n_2 \alpha_1} \rceil = \lceil \frac{n_1}{\alpha_1} \rceil = \chi(C(n_1, a_1))$. Therefore, $\chi(C(n_1, a_1) \times C(n_2, a_2)) \geq \min\{\chi(C(n_1, a_1)), \chi(C(n_2, a_2))\}$. $\square$

Let F and G be graphs. The *map graph* F^G has the set of functions from $V(G)$ to $V(F)$ as its vertices; two such functions f and h are adjacent in F^G if and only if whenever u and v are adjacent in G, the vertices $f(u)$ and $h(v)$ are adjacent in F. Notice that a vertex in F^G has a loop on it if and only if the corresponding function is a graph homomorphism. In order to simplify the study of Hedetniemi's conjecture, El-Zahar and Sauer show in [4] the following result (see also [6]).

Theorem 11 *(El-Zahar, Sauer [4]) Suppose $\chi(G) > n$. Then K_n^G is n-colourable if and only if $\chi(G \times H) > n$ for all graphs H such that $\chi(H) > n$.*

A consequence of Theorem 11 is the following lemma, that follows by induction.

Lemma 10 *Let $\mathbb{F}$ be a non empty family of graphs such that for any two graphs $G, H \in \mathbb{F}$ (not necessarily different) we have that $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$. Let $G_1, G_2, \ldots, G_k$ be a collection of graphs in $\mathbb{F}$. Then, $\chi(\prod G_i) = \min\{\chi(G_i)\}$.*

We have not be able to generalize Theorem 9 for any finite product of powers of cycles graphs, and so it remains as an open problem. However, by using Lemma 10 we can generalize Theorem 10 as follows.

Theorem 12 *Let r be a positive integer, and let $n_1, n_2, \ldots, n_r, a_1, a_2, \ldots, a_r$ be positive integers such that $n_i \geq 2a_i$, for each $i \in [r]$. Then, $\chi\left(\prod_i C(n_i, a_i)\right) = \min_i\{\chi(C(n_i, a_i)\}$.*

Another interesting open problem is the structure of the independent sets of finite direct products of vertex-transitive graphs such as the ones studied in this paper.

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