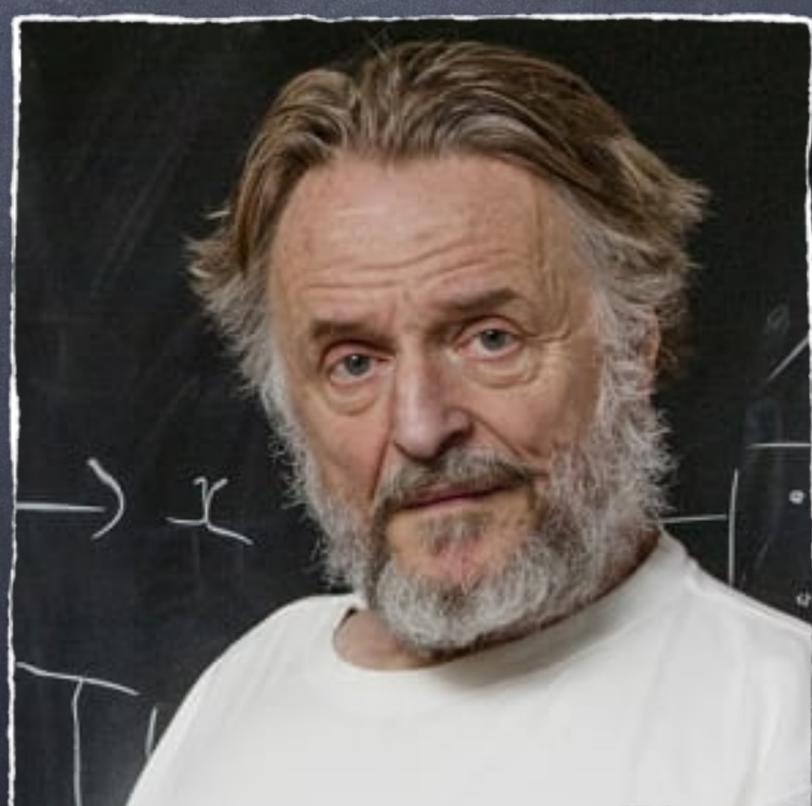
Alice au Pays des Frises et des Pavages

François Bergeron, LACIM

Frises



Conway

Coxeter

Frises

2 Triangulated Polygons and Frieze Patterns

	0		0		0		0		0		0		0		0		0		0		0		0	
1		1		1		1		1		1		1		1		1		1		1		1		1
	1		2		2		3		1		2		4		1		2		2		3		1	
3		1		3		5		2		1		7		3		1		3		5		2		1
	2		1		7		3		1		3		5		2		1		7		3		1	
3		1		2		4		1		2		2		3		1		2		4		1		2
	1		1		1		1		1		1		1		1		1		1		1		1	
0		0		0		0		0		0		0		0		0		0		0		0		0

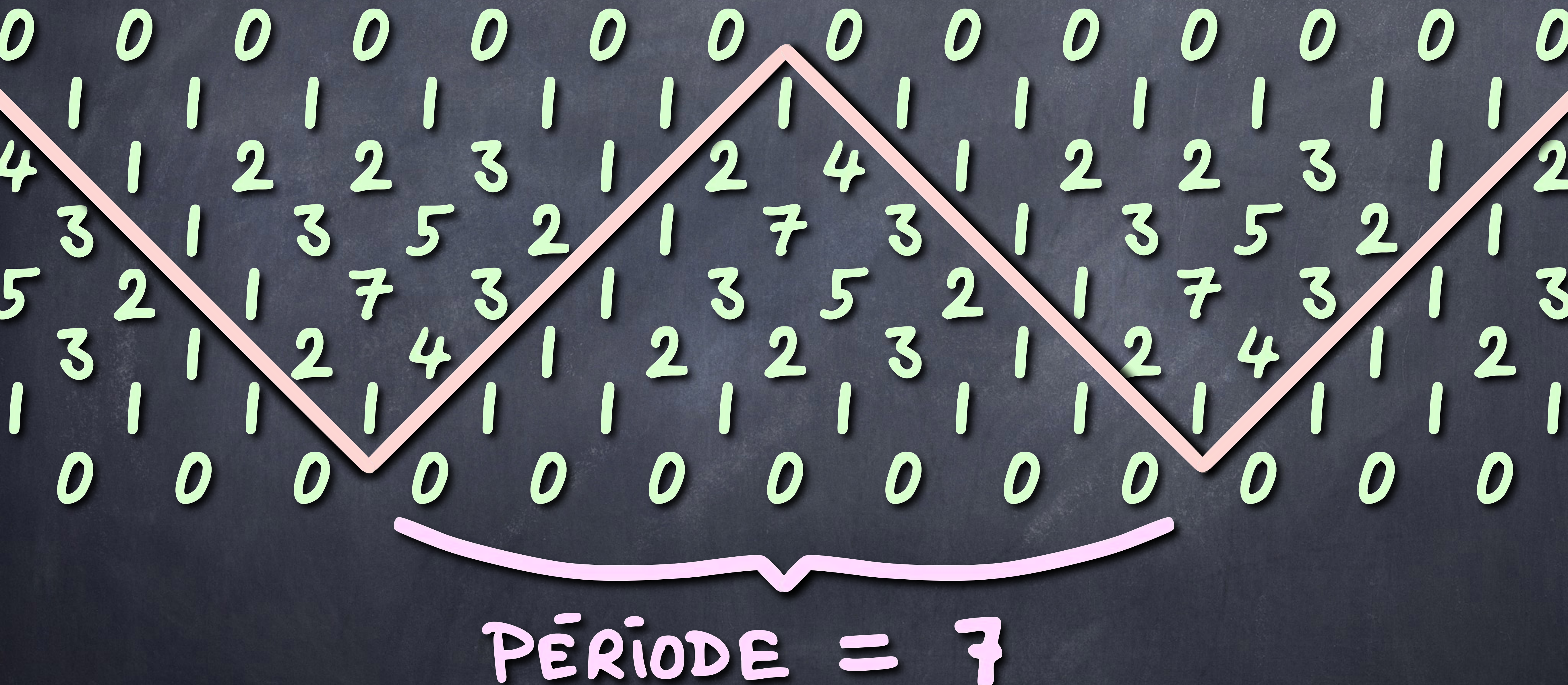
Table 1.

The first author once asked an audience of a hundred students of mathematics to look at the pattern in Table 1 and find the simple rule connecting each number with its neighbors and allowing the pattern to be extended indefinitely to the right and left. After an embarrassingly long time it was necessary to break the suspense by explaining that any four numbers forming a diamond, such as

$$\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$$

satisfy the relation $ad - bc = 1$, which may also be written $c = (ad - 1)/b$; this is called the **unimodular rule.** Later, to test the effect of a brilliant brain, the same pattern was shown to Paul Erdős: he needed only a few seconds!

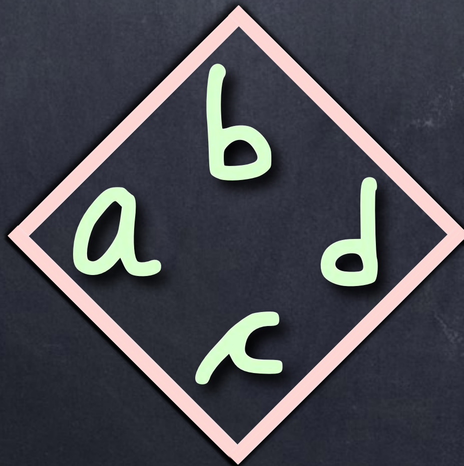
Frises



François Bergeron, LACIM

Frises

```
0   0   0   0   0   0   0   0   0   0   0   0   0   0
  1   1   1   1   1   1   1   1   1   1   1   1   1
4   1   2   2   3   1   2   4   1   2   2   3   1   2
  3   1   3   5   2   1   7   3   1   3   5   2   1
5   2   1   7   3   1   3   5   2   1   7   3   1   3
  3   1   2   4   1   2   2   3   1   2   4   1   2
1   1   1   1   1   1   1   1   1   1   1   1   1   1
  0   0   0   0   0   0   0   0   0   0   0   0   0
```



$\rightsquigarrow \quad ad - bc = 1$

François Bergeron, LaCIM

LARGEUR

0 0 0 0 0 0 0 0 0 0 0
1 1 1 1 1 1 1 1 1 1
2 3 1 2 4 1 2 2 3 1 2
5 2 1 7 3 1 3 5 2 1
7 3 1 3 5 2 1 7 3 1 3
4 1 2 2 3 1 2 4 1 2
1 1 1 1 1 1 1 1 1 1 1
0 0 0 0 0 0 0 0 0 0

} = 4

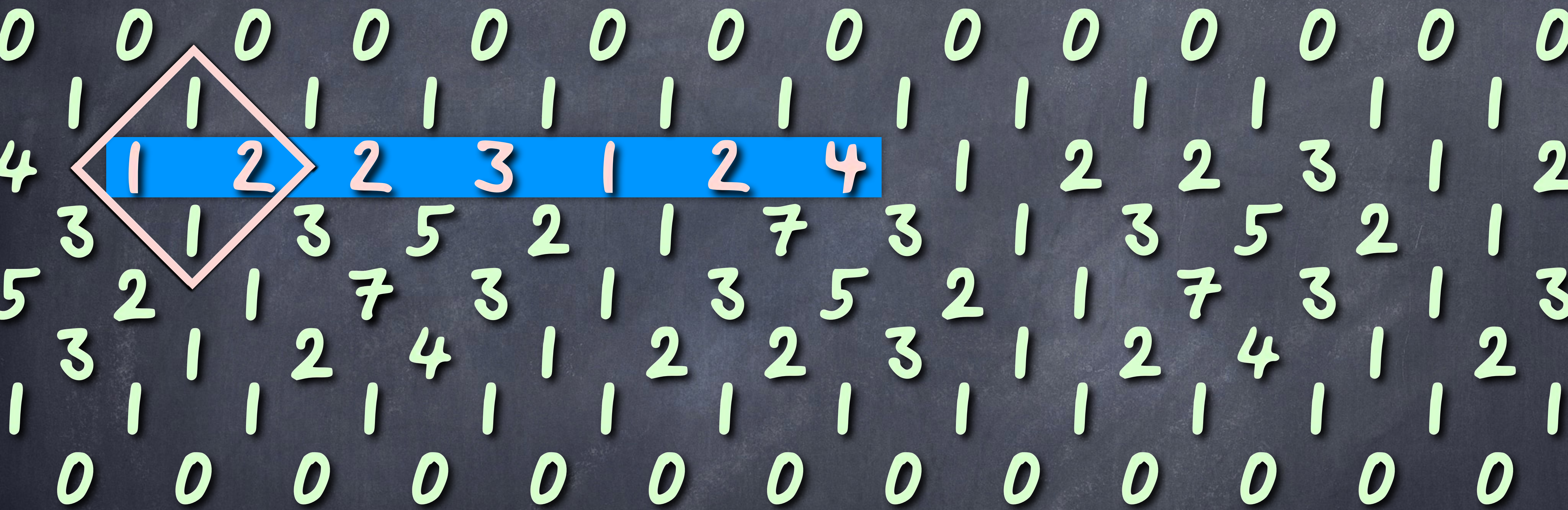
François Bergeron, LACIM

THÉORÈME (CONWAY-COXETER)

LES FRISES D'ENTIERS
DE LARGEUR n SONT
PÉRIODIQUES,
DE PÉRIODE $n+3$.
ET, ELLES SONT
CARACTÉRISEES PAR ...

FRANÇOIS BERGERON, LACIM

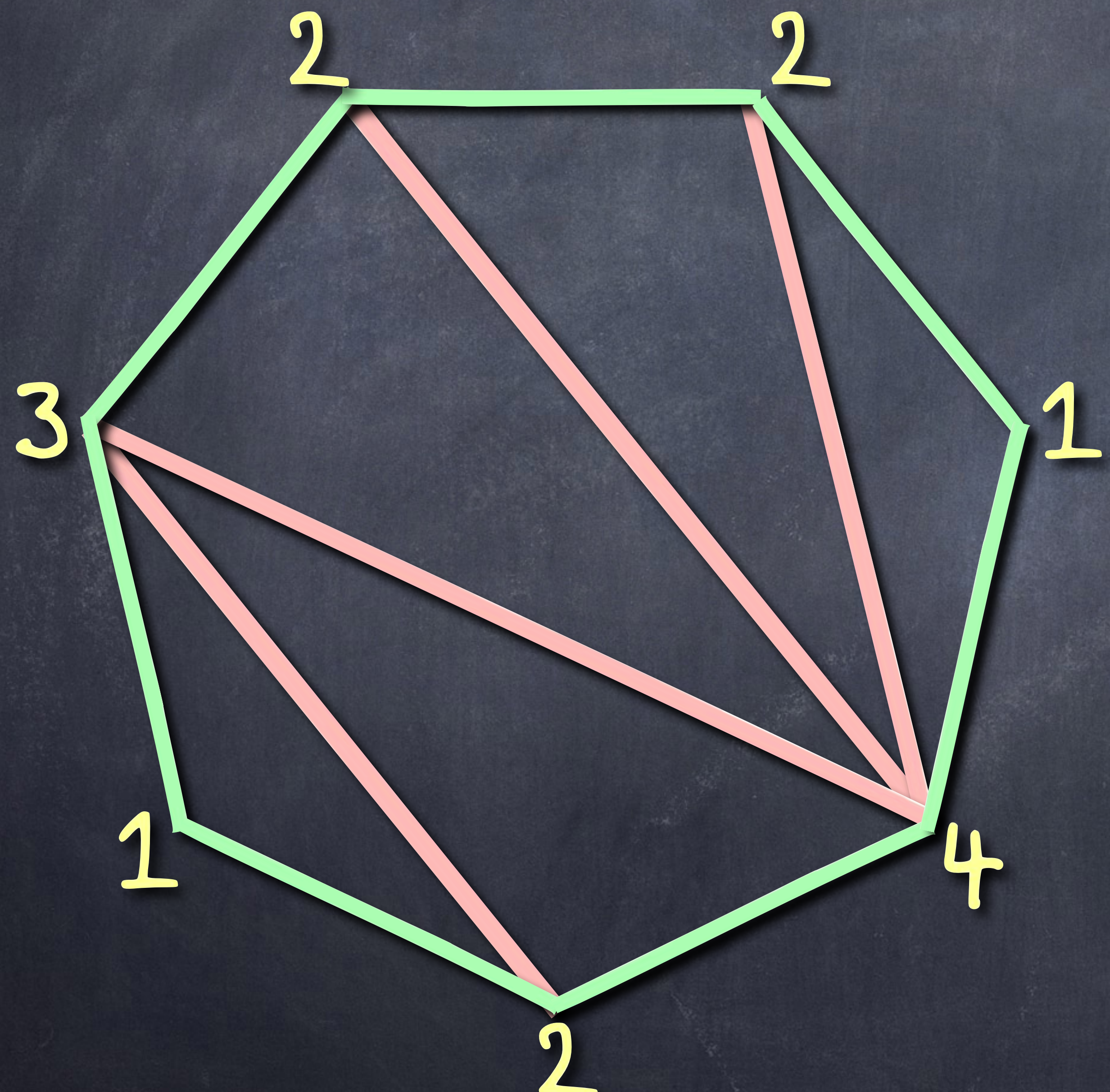
QUIDDITÉ



FRANÇOIS BERGERON, LACIM

QUIDDITÉ

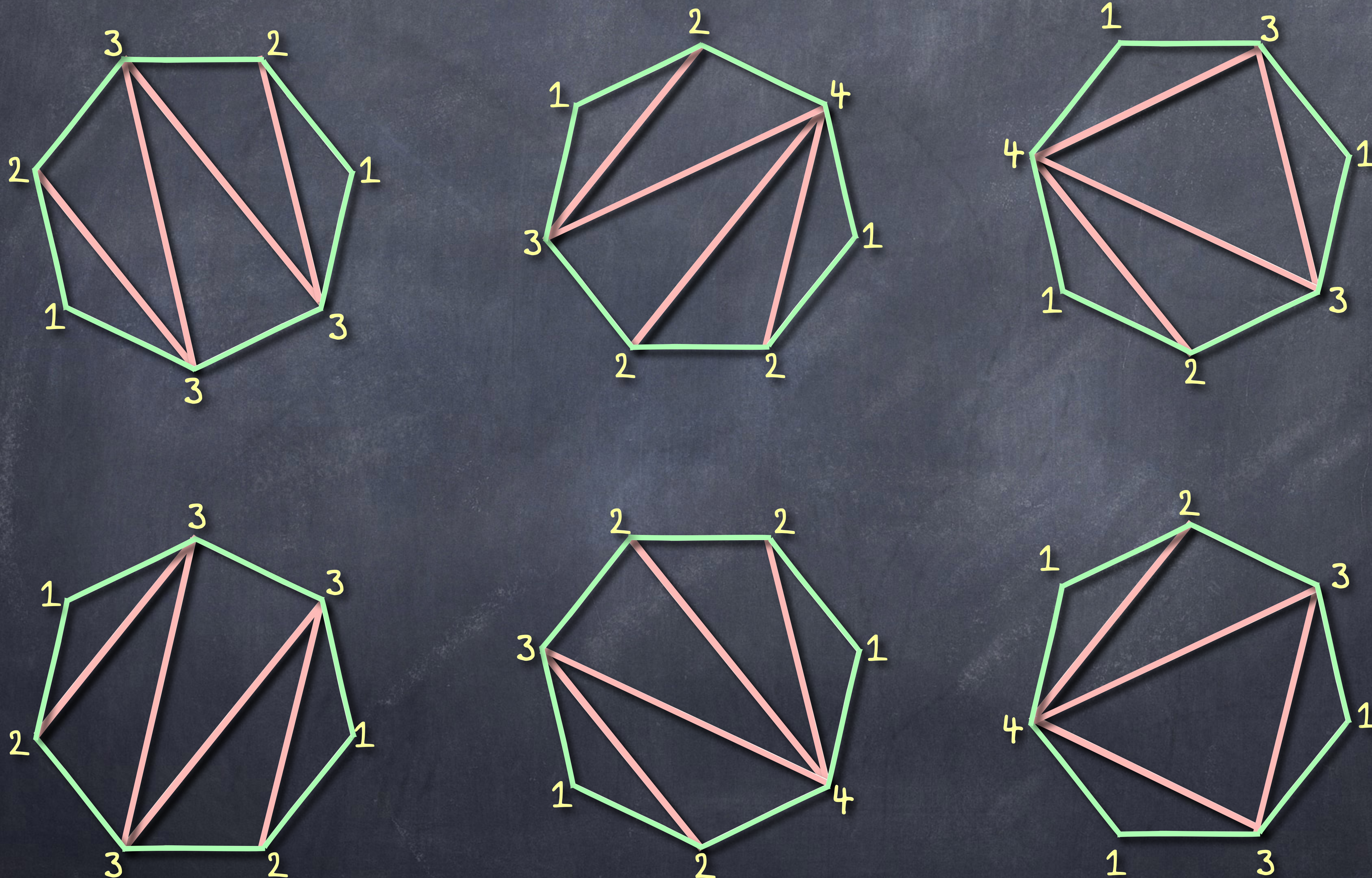
1	2	2	3	1	2	4



TRIANGULATION

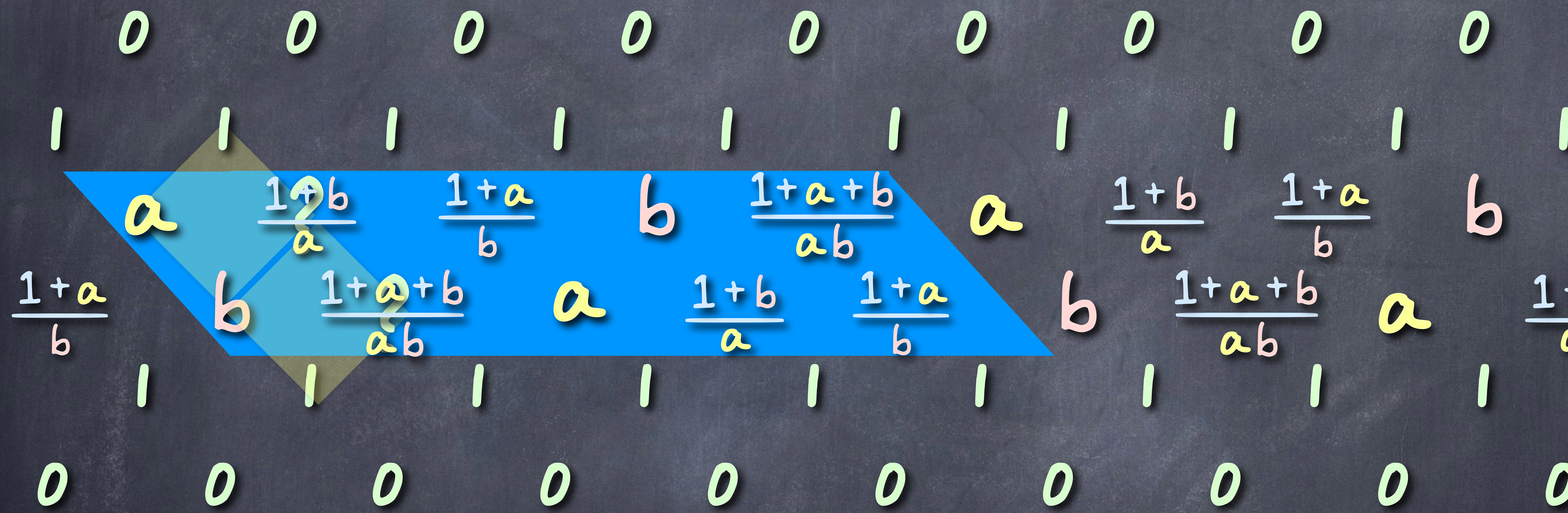
FRANÇOIS BERGERON, LACIM

Triangulations



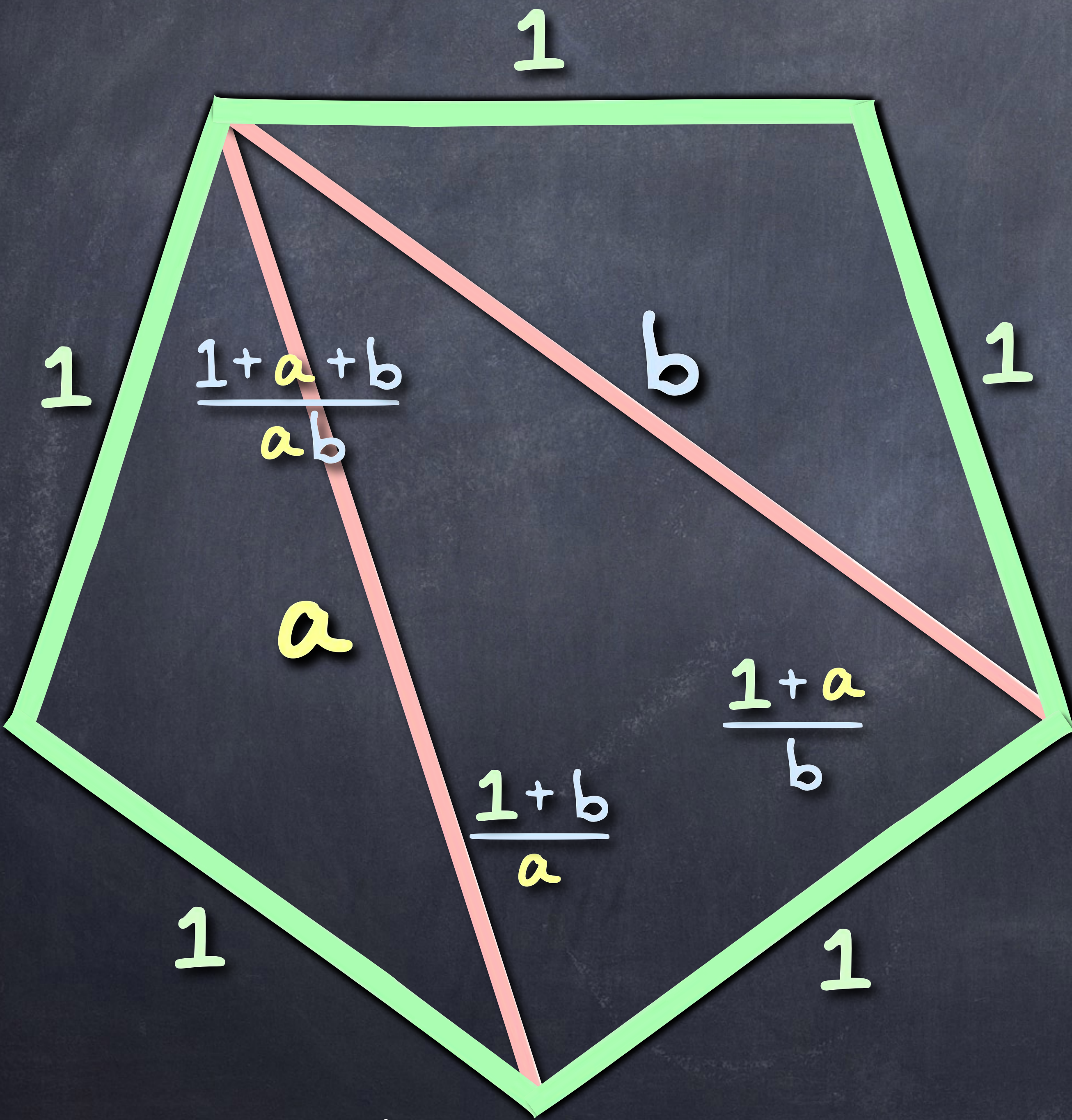
François Bergeron, LaCIM

Périodicité



François Bergeron, LACIM

Algèbres Amassées




Fomin


Zelevinsky

François Bergeron, LACIM

- F. Chapoton, Enumerative properties of generalized associahedra, Sém. Lothar. Combin. 51 (2004/05), Art. B51b, 16 pp.
- P. Di Francesco et R. Kedem, Q-systems as cluster algebras. II. Cartan matrix of finite type and the polynomial property, Lett. Math. Phys. 89 (2009), no. 3, 183–216.
- C. Geiß, B. Leclerc, et J. Schröer, Cluster structures on quantum coordinate rings, arXive:1104.0531.
- D. Hernandez et B. Leclerc, Cluster algebras and quantum affine algebras, arXiv:0903.1452.
- B. Keller, Cluster algebras, quiver representations and triangulated categories, arXive:0807.1960.
- M. Kontsevich et Y. Soibelman, Stability structures, Donaldson-Thomas invariants and cluster transformations, arXiv:0811.2435.

Version matricielle

0
4 1 0
5 3 1 1 0
1 3 2 1 2 1 0
0 1 1 1 3 2 1 0
0 1 2 7 5 3 1 0
0 1 4 3 2 1 1 0
0 1 1 1 1 2 1 0
0 1 2 3 7 4 1 0
0 1 2 5 3 1 1 0
0 1 3 2 1 2 1 0
0 1 1 1 3 2 1 0
0 1 2 7 5 3 1 0
0 1 4 3 2 1 1 0
0 1 1 1 1 2
0 1 2 3
0 1

François Bergeron, LACIM

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 1$$

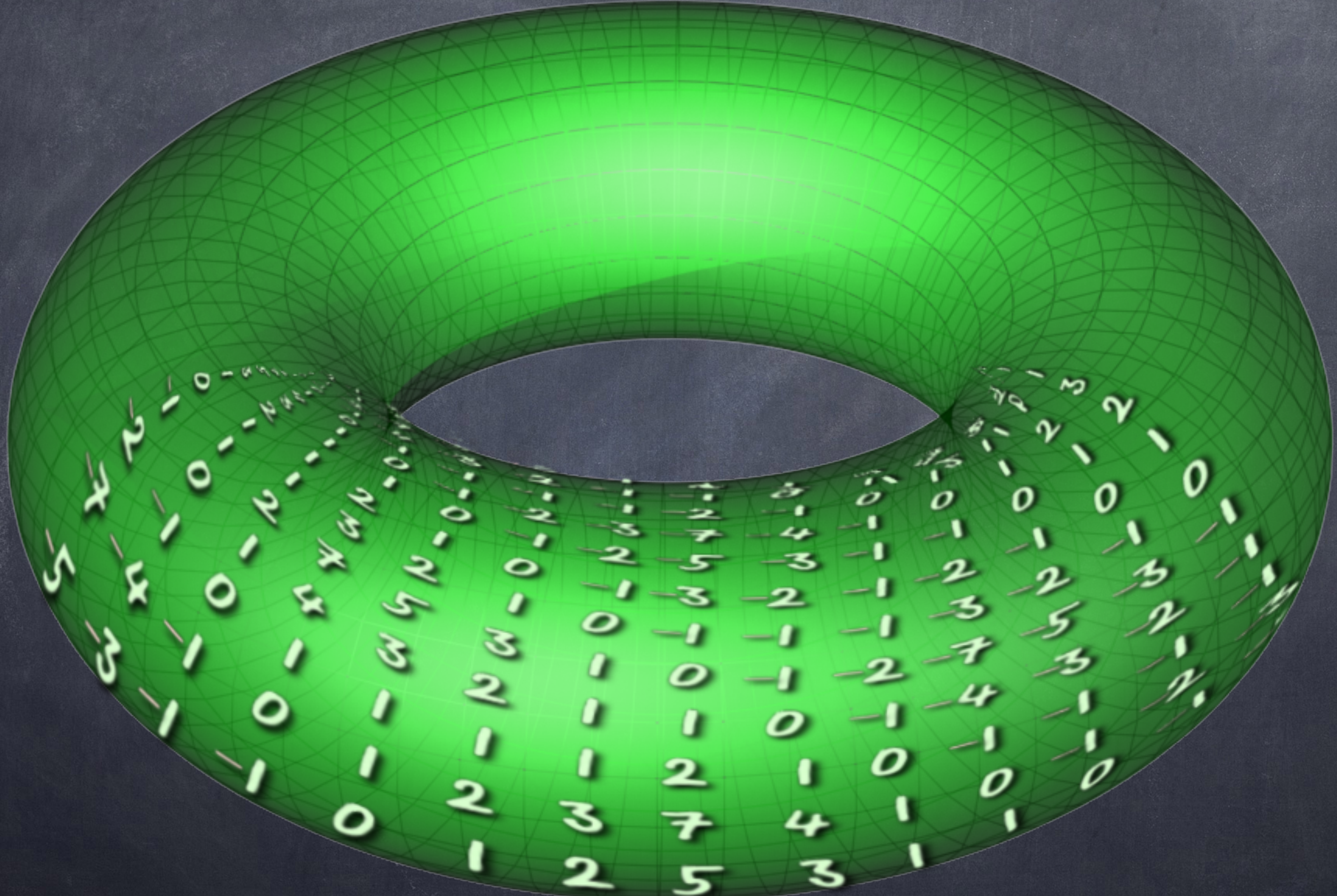
3 3 1 1 0
1 3 2 1 2 1 0
0 1 1 1 3 2 1 0
0 1 2 7 5 3 1 0
0 1 4 3 2 1 1 0
0 1 1 1 1 2 1 0
0 1 2 3 7 4 1 0
0 1 2 5 3 1 1 0
0 1 3 2 1 2 1 0
0 1 1 1 3 2 1 0
0 1 2 7 5 3 1 0
0 1 4 3 2 1 1 0
0 1 1 1 1 2

François Bergeron, LaCIM

⋱	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋰
⋯	1	3	2	1	2	1	0	−1	−3	−2	−1	−2	−1	0	1	3	⋯
⋯	0	1	1	1	3	2	1	0	−1	−1	−1	−3	−2	−1	0	1	⋯
⋯	−1	0	1	2	7	5	3	1	0	−1	−2	−7	−5	−3	−1	0	⋯
⋯	−1	−1	0	1	(4)	3	2	1	1	0	−1	(−4)	−3	−2	−1	−1	⋯
⋯	−1	−2	−1	0	1	1	1	1	2	1	0	−1	−1	−1	−1	−2	⋯
⋯	−3	−7	−4	−1	0	1	2	3	7	4	1	0	−1	−2	−3	−7	⋯
⋯	−2	−5	−3	−1	−1	0	1	2	5	3	1	1	0	−1	−2	−5	⋯
⋯	−1	−3	−2	−1	−2	−1	0	1	3	2	1	2	1	0	−1	−3	⋯
⋯	0	−1	−1	−1	−3	−2	−1	0	1	1	1	3	2	1	0	−1	⋯
⋯	1	0	−1	−2	−7	−5	−3	−1	0	1	2	7	5	3	1	0	⋯
⋯	1	1	0	−1	(−4)	−3	−2	−1	−1	0	1	(4)	3	2	1	1	⋯
⋯	1	2	1	0	−1	−1	−1	−1	−2	−1	0	1	1	1	1	2	⋯
⋰	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

FRANÇOIS BERGERON, LACIM

Frise Torique



François Bergeron, LACIM

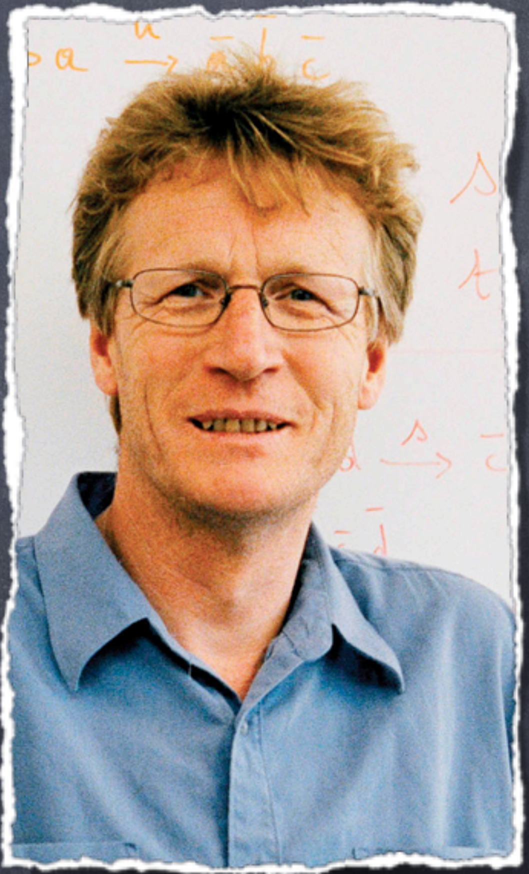
RANG = 2

⋱	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋰
⋯	1	3	2	1	2	1	0	−1	−3	−2	−1	−2	−1	0	1	3	⋯
⋯	0	1	1	1	3	2	1	0	−1	−1	−1	−3	−2	−1	0	1	⋯
⋯	−1	0	1	2	7	5	3	1	0	−1	−2	−7	−5	−3	−1	0	⋯
⋯	−1	−1	0	1	4	3	2	1	1	0	−1	−4	−3	−2	−1	−1	⋯
⋯	−1	−2	−1	0	1	1	1	1	2	1	0	−1	−1	−1	−1	−2	⋯
⋯	−3	−7	−4	−1	0	1	2	3	7	4	1	0	−1	−2	−3	−7	⋯
⋯	−2	−5	−3	−1	−1	0	1	2	5	3	1	1	0	−1	−2	−5	⋯
⋯	−1	−3	−2	−1	−2	−1	0	1	3	2	1	2	1	0	−1	−3	⋯
⋯	0	−1	−1	−1	−3	−2	−1	0	1	1	1	3	2	1	0	−1	⋯
⋯	1	0	−1	−2	−7	−5	−3	−1	0	1	2	7	5	3	1	0	⋯
⋯	1	1	0	−1	−4	−3	−2	−1	−1	0	1	4	3	2	1	1	⋯
⋯	1	2	1	0	−1	−1	−1	−1	−2	−1	0	1	1	1	1	2	⋯
⋰	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

François Bergeron, LaCIM

Pavages SL_k

En collaboration avec



Christophe Reutenauer

PAVAGES SL_k

$$\mathcal{P} : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}$$

$$\mathcal{P} = (p_{ij})$$

$$\mathcal{P}^{\langle k \rangle}_{(a,b)} := \det \left(p_{(a,b)+(i,j)} \right)_{0 \leq i,j \leq k-1}$$

$$\mathcal{P}^{\langle k \rangle}_{(a,b)} = 1, \quad \text{POUR TOUT } (a,b) \in \mathbb{Z} \times \mathbb{Z}$$

FRANÇOIS BERGERON, LACIM

Pavages SL_2

$$\rho : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{N}^+$$

⋱	⋮	⋮	⋮	⋮	⋮		⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋰
⋯	887	567	247	174	101	28	11	5	4	3	2	1	1	⋯
⋯	158	101	44	31	18	5	2	1	1	1	1	1	2	⋯
⋯	61	39	17	12	7	2	1	1	2	3	4	5	11	⋯
⋯	25	16	7	5	3	1	1	2	5	8	11	14	31	⋯
	14	9	4	3	2	1	2	5	13	21	29	37	82	
⋯	3	2	1	1	1	1	3	8	21	34	47	60	133	⋯
⋯	1	1	1	2	3	4	13	35	92	149	206	263	583	⋯
⋯	1	2	3	7	11	15	49	132	347	562	777	992	2199	⋯
⋰	⋮	⋮	⋮	⋮	⋮		⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 1$$

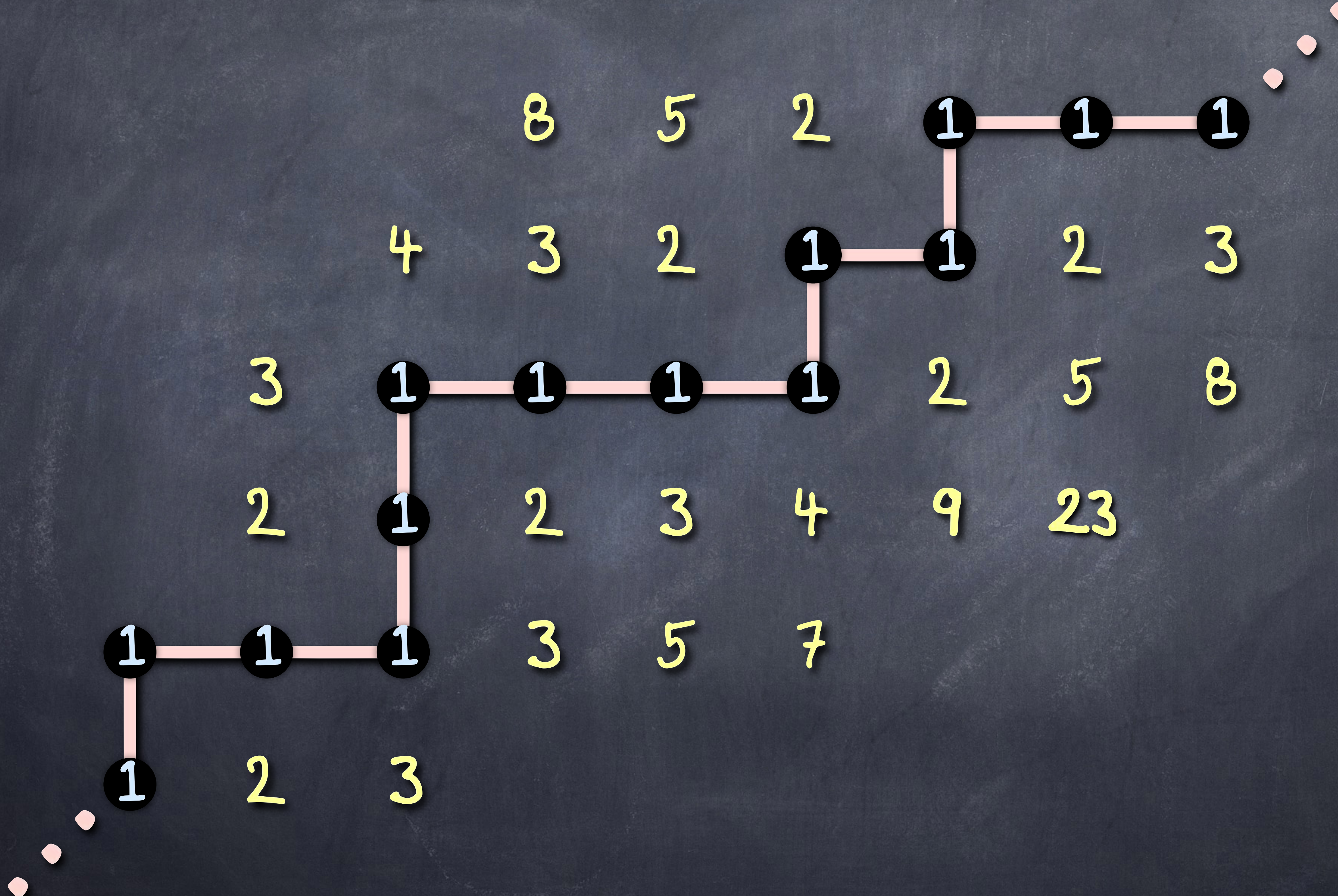
François Bergeron, LaCIM

Pavages SL_3

$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$⋰$
$\cdots$	8997	1782	353	70	14	3	1	1	$\cdots$
$\cdots$	1782	353	70	14	3	1	1	2	$\cdots$
$\cdots$	353	70	14	3	1	1	2	5	$\cdots$
$\cdots$	70	14	3	1	1	2	5	14	$\cdots$
$\cdots$	14	3	1	1	2	5	14	42	$\cdots$
$\cdots$	3	1	1	2	5	14	42	131	$\cdots$
$\cdots$	1	1	2	5	14	42	131	417	$\cdots$
$⋰$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

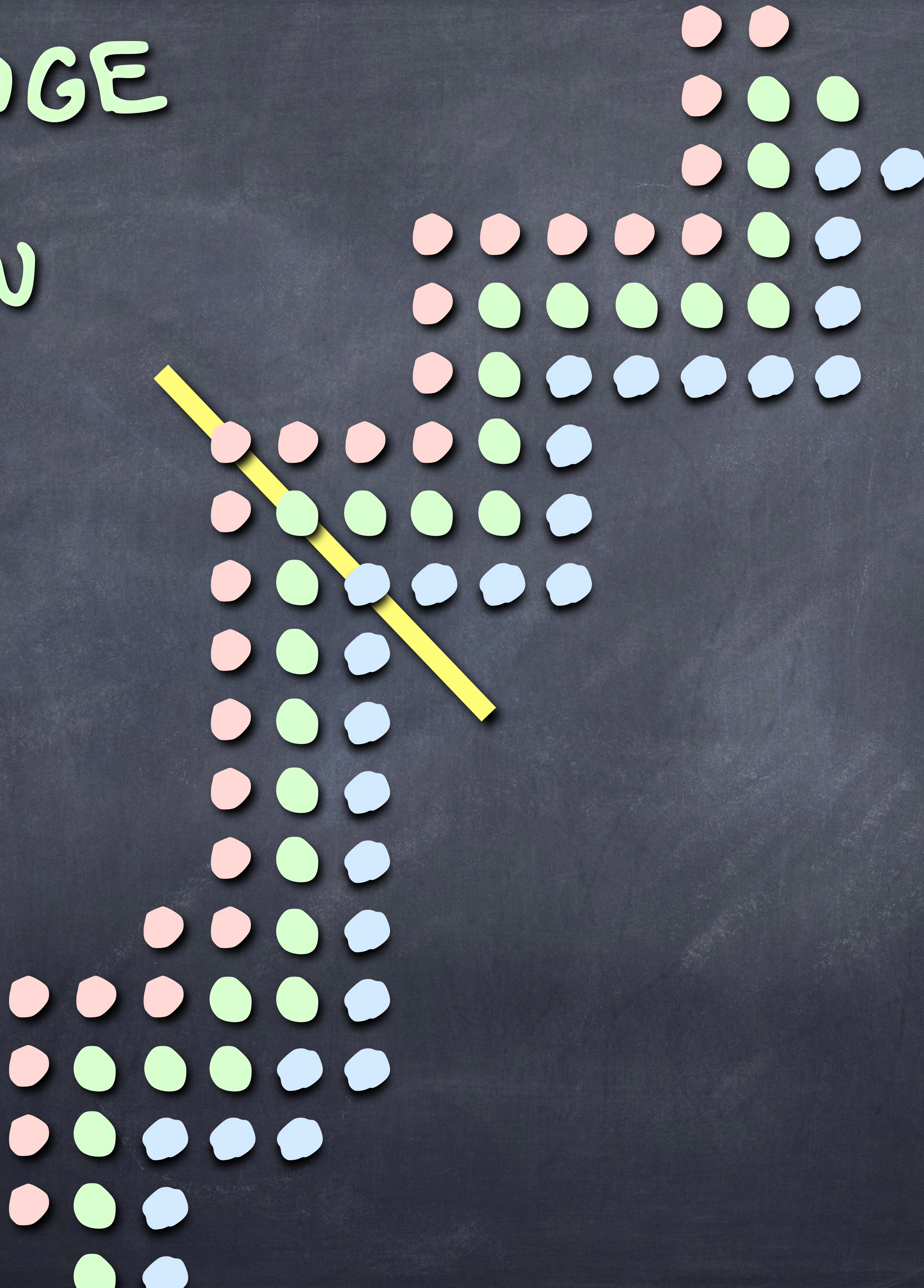
François Bergeron, LACIM

Pavages SL_2 associé à un chemin Nord-Est



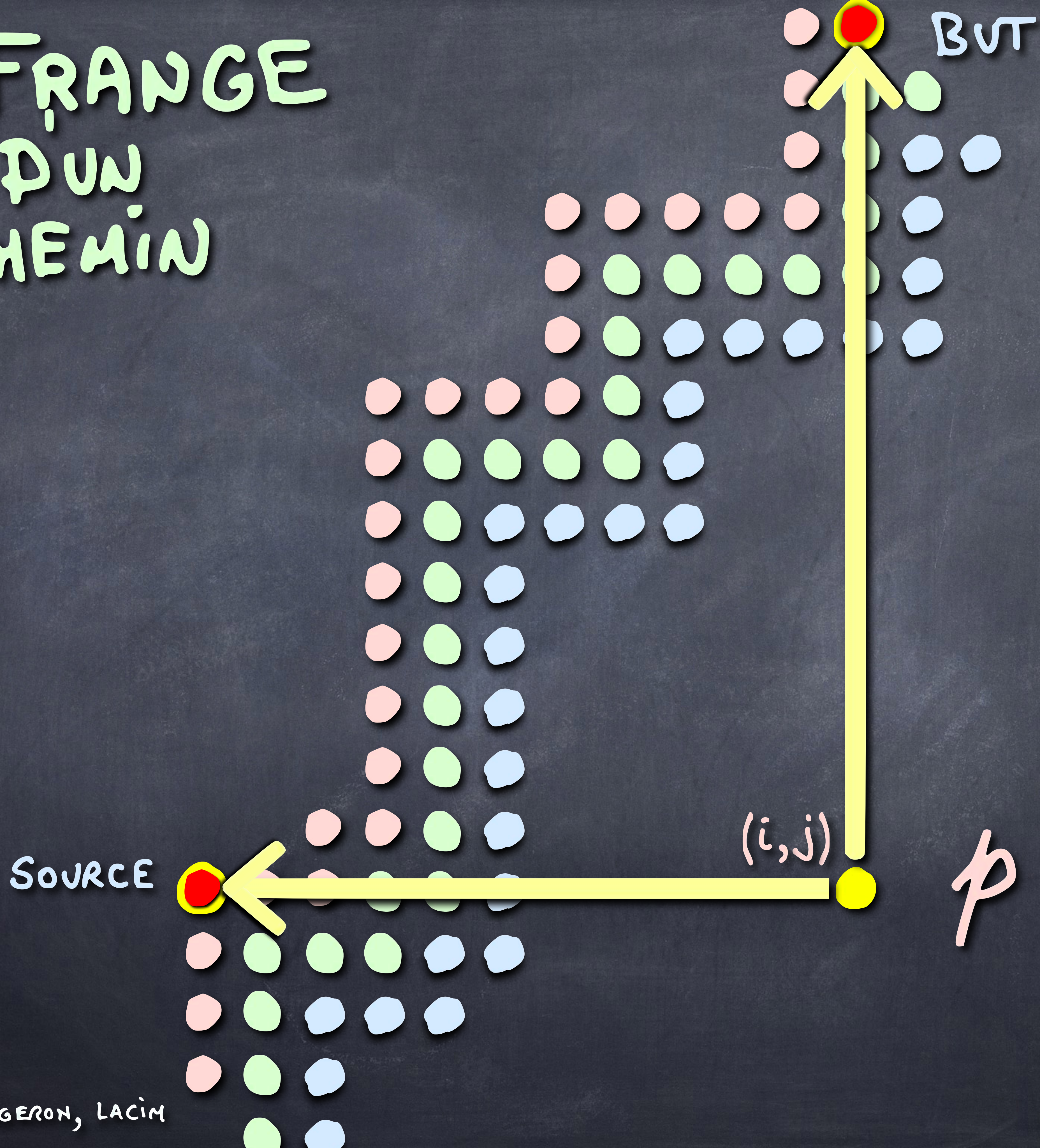
François Bergeron, LACIM

3-FRANGE D'UN CHEMIN



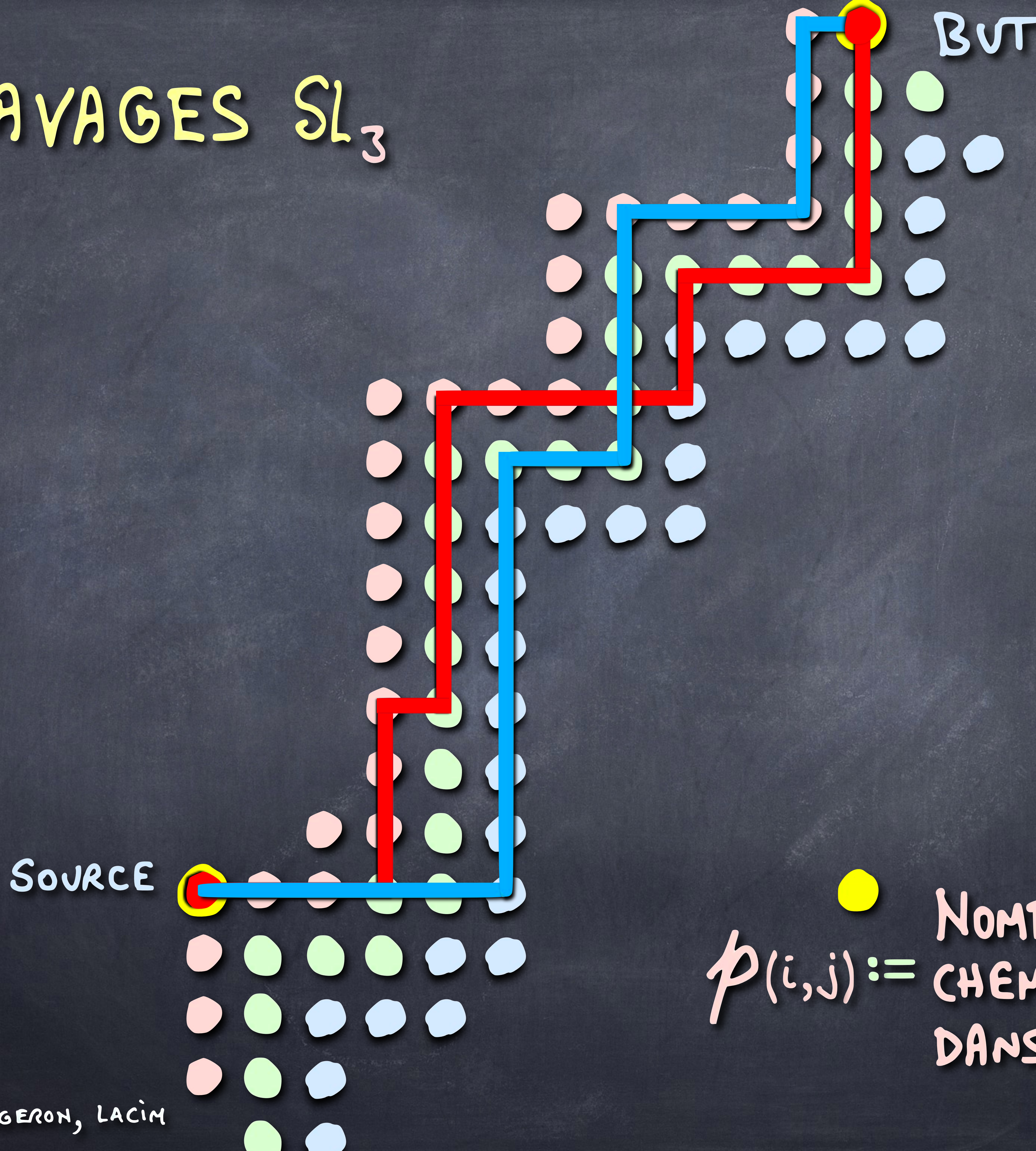
FRANÇOIS BERGERON, LACIM

3 - FRANGE D'UN CHEMIN



FRANÇOIS BERGERON, LACIM

Pavages SL_3



$\varphi(i,j) :=$ Nombre de chemins N-E dans la frange

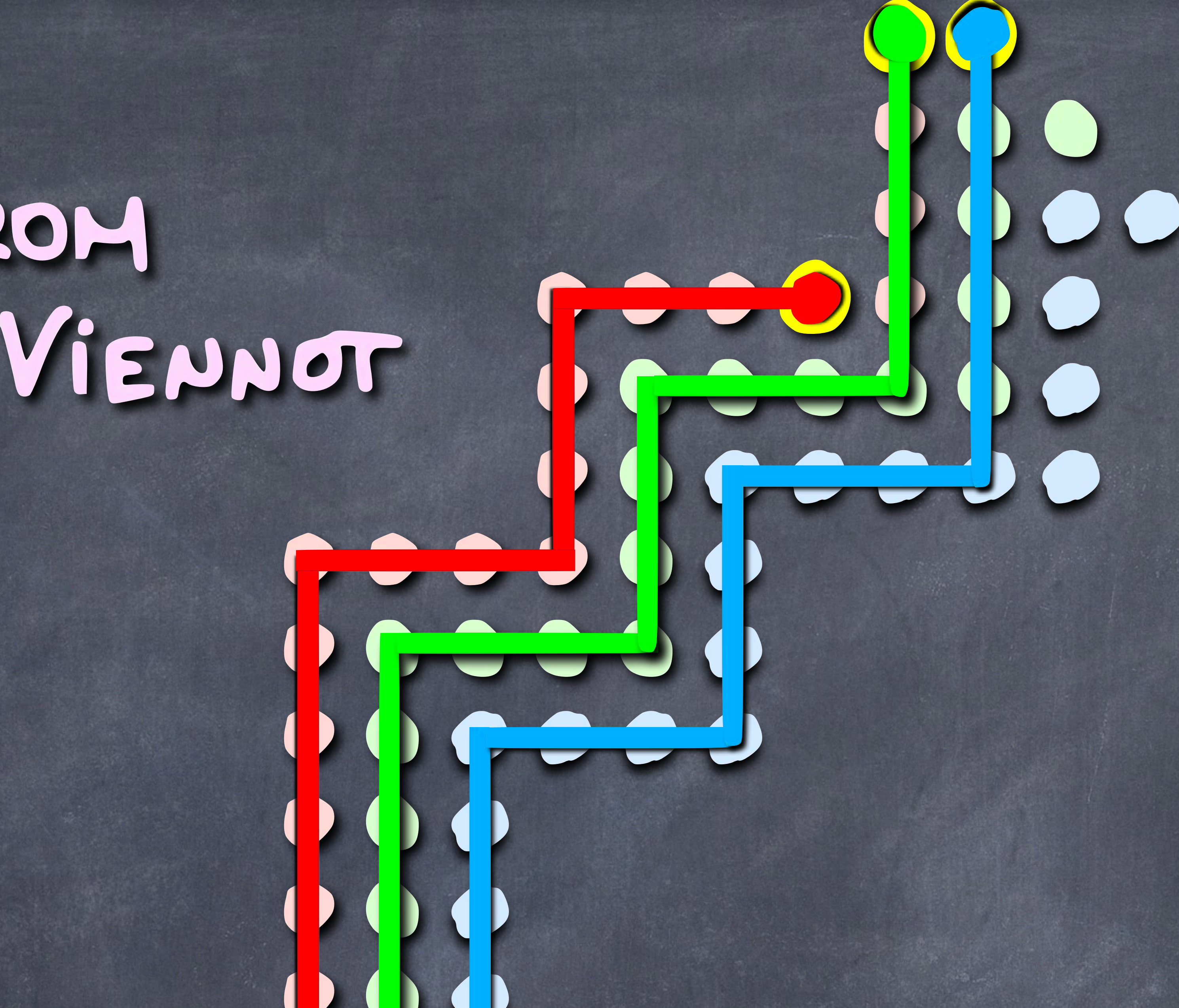
François Bergeron, LaCIM

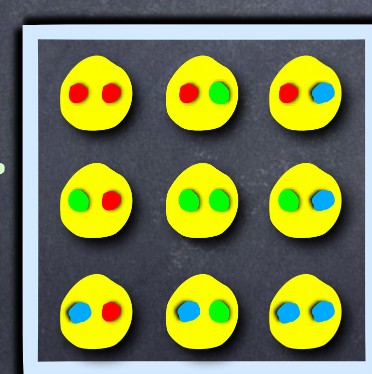
THÉORÈME (B. - REUTENAUER)

POUR TOUT CHEMIN N.-E.
IL EXISTE UN UNIQUE PAVAGE-SL_k
(DOCILE) QUI EST DONNÉ PAR
L'ENUMÉRATION DE CHEMINS
DANS LA k-FRANGE DU CHEMIN

FRANÇOIS BERGERON, LACIM

Lindstrom Gessel-Viennot

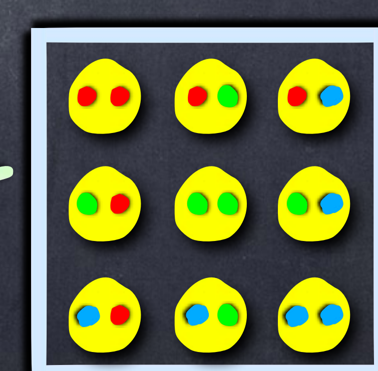


det  = 1

$p(i,j) :=$ Nombre de chemins N-E dans la frange

François Bergeron, LACIM

Lindstrom Gessel-Viennot

det 

François Bergeron, LACIM

PAVAGE-SL_k DOCILE

FRANÇOIS BERGERON, LACIM

UN PAVAGE-SL_k ρ EST DIT **DOCILE** SSI $\text{RANG}(\rho) = k$. S'INON, IL EST DIT **SAUVAGE**.

FRANÇOIS BERGERON, LACIM

PAVAGE-SL_2 SAUVAGE

⋱	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋰
⋯	1	a	-1	b	1	c	-1	d	1	e	-1	⋯
⋯	0	1	0	-1	0	1	0	-1	0	1	0	⋯
⋯	-1	f	1	g	-1	h	1	i	-1	j	1	⋯
⋯	0	-1	0	1	0	-1	0	1	0	-1	0	⋯
⋯	1	k	-1	l	1	m	-1	n	1	p	-1	⋯
⋯	0	1	0	-1	0	1	0	-1	0	1	0	⋯
⋰	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

FRANÇOIS BERGERON, LACIM

UN PAVAGE-SL_k $\wp$ EST DIT <u>0-LIBRE</u>

SSI

$$\wp_{(a,b)}^{\langle k-1\rangle} \neq 0$$

POUR TOUT $(a,b) \in \mathbb{Z} \times \mathbb{Z}$.

FRANÇOIS BERGERON, LACIM

Proposition (B. – Reutenauer)

> **Tout pavage 0-libre est docile**

Preuve [Loi de condensation]

$$\det(e) \times \det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \det\begin{pmatrix} \det\begin{pmatrix} a & b \\ d & e \end{pmatrix} & \det\begin{pmatrix} b & c \\ e & f \end{pmatrix} \\ \det\begin{pmatrix} d & e \\ g & h \end{pmatrix} & \det\begin{pmatrix} e & f \\ h & i \end{pmatrix} \end{pmatrix}$$

Charles Lutwidge Dodgson

Loi de condensation

$$\det\left(\det(e) \times \det\begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix}\right) = \det\begin{pmatrix} \det\begin{pmatrix} a & b \\ d & e \end{pmatrix} & \det\begin{pmatrix} b & c \\ e & f \end{pmatrix} \\ \det\begin{pmatrix} e & f \\ h & i \end{pmatrix} & \det\begin{pmatrix} e & f \\ h & i \end{pmatrix} \end{pmatrix}$$

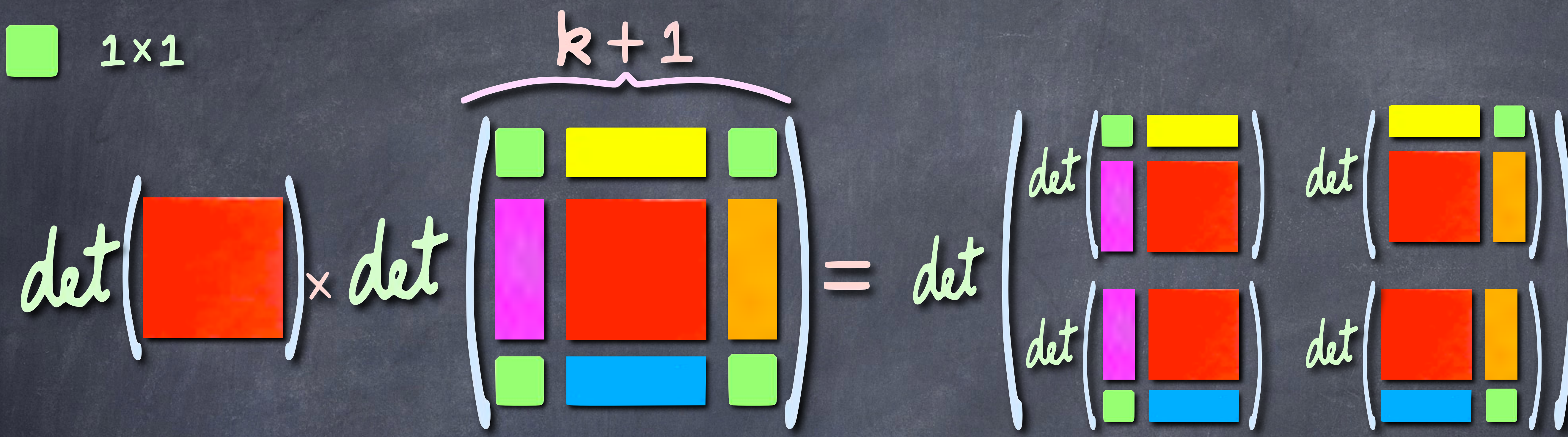
Identité de Desnanot–Jacobi

Jacobi

Dodgson

François Bergeron, LACIM

Loi de condensation



$$\mathcal{P}^{\langle k-1\rangle}_{(a+1,b+1)} \times \mathcal{P}^{\langle k+1\rangle}_{(a,b)} = \det\begin{pmatrix} \mathcal{P}^{\langle k\rangle}_{(a,b+1)} & \mathcal{P}^{\langle k\rangle}_{(a+1,b+1)} \\ \mathcal{P}^{\langle k\rangle}_{(a,b)} & \mathcal{P}^{\langle k\rangle}_{(a+1,b)} \end{pmatrix}$$

$\mathcal{P}^{\langle k-1\rangle}_{(a+1,b+1)} \neq 0$

$= 0$ ... QED

François Bergeron, LACIM

• Pavages SL_k (associé à un chemin)

$$\rho : \mathbb{Z}\times\mathbb{Z} \longrightarrow \mathbb{N}[\ldots, a, a^{-1}, \ldots]$$

$$\begin{array}{ccccc}
\ddots & \vdots & \vdots & \vdots & \iddots \\
\cdots & \frac{1}{v}+\frac{bd}{c}+\frac{uw}{cv} & d & e & \cdots \\
\cdots & b & c & \frac{u}{d}+\frac{ce}{d} & \cdots \\
\cdots & a & \frac{w}{b}+\frac{ac}{b} & \frac{v}{c}+\frac{ace}{bd}+\frac{au}{bd}+\frac{ew}{bd}+\frac{uw}{bcd} & \cdots \\
\iddots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

François Bergeron, LACIM

- FRISES SL_k (VIA COORDONNÉES DE PLÜCKER)
- T-SYSTÈMES (PHYSIQUE)
- MATRICES À SIGNES ALTERNANTS
- RÉCURRENCE OCTAÉDRALE (YANG-BAXTER)

FRANÇOIS BERGERON, LACIM

Fin

(avec C. Reutenauer)
Sl_k-tilings of the plane,
Illinois J. of Mathematics,
Volume 54, Number 1 (2010), 263-300.
(arXiv :0812.3566)