On universal differential equations

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Outline

INTRODUCTION

Picard-Vessiot theory of ordinary differential equations

$(\mathbf{k}, \partial)$ differential ring. $\mathrm{Const}(\mathbf{k}) = \{c \in \mathbf{k} | \partial c = 0\}$ is supposed to be a field.

$$(ODE) \quad (a_n\partial^n + a_{n-1}\partial^{n-1} + \ldots + a_0)y = 0, \quad a_0, \ldots, a_{n-1}, a_n \in \mathbf{k}.$$

a_n^{-1} is supposed to exist.

Definition 1

1. Let $y_1, \ldots, y_n$ be $\mathrm{Const}(\mathbf{k})$-linearly independent solutions of (ODE). Then $\{y_1, \ldots, y_n\}$ is called a fundamental set of solutions of (ODE) and it generates a $\mathrm{Const}(\mathbf{k})$-vector subspace of dimension at most n.
2. If[1] $M = \mathbf{k}\{y_1, \ldots, y_n\}$ and $\mathrm{Const}(M) = \mathrm{Const}(\mathbf{k})$ then M is called a Picard-Vessiot extension related to (ODE)
3. Let $\mathbf{k} \subset \mathbb{K}_1$ and $\mathbf{k} \subset \mathbb{K}_2$ be differential rings. An isomorphism of rings $\sigma : \mathbb{K}_1 \to \mathbb{K}_2$ is a differential $\mathbf{k}$-isomorphism if
$$\forall a \in \mathbb{K}_1, \quad \partial(\sigma(a)) = \sigma(\partial a) \text{ and, if } a \in \mathbf{k},\ \sigma(a) = a.$$
If $\mathbb{K}_1 = \mathbb{K}_2 = \mathbb{K}$, the differential galois group of $\mathbb{K}$ over $\mathbf{k}$ is by
$$\mathrm{Gal}_{\mathbf{k}}(\mathbb{K}) = \{\sigma | \sigma \text{ is a differential } \mathbf{k}\text{-automorphism of } \mathbb{K}\}.$$

[1] Let R_1, R_2 be differential rings s.t. $R_1 \subset R_2$. Let S be a subset of R_2. $R_1\{S\}$ denotes the smallest differential subring of R_2 containing R_1. $R_1\{S\}$ is the ring (over R_1) generated by S and their derivatives of all orders.

Linear differential equations and Dyson series

Let $a_0, \ldots, a_n \in \mathbb{C}(z)$, $\quad a_n(z)\partial^n y(z) + \ldots + a_1(z)\partial y(z) + a_0(z)y(z) = 0$.

$$(ED) \quad \begin{cases} \partial q(z) &= A(z)q(z), & A(z) \in \mathcal{M}_{n,n}(\mathbb{C}(z)), \\ q(z_0) &= \eta, & \lambda \in \mathcal{M}_{1,n}(\mathbb{C}), \\ y(z) &= \lambda q(z), & \eta \in \mathcal{M}_{n,1}(\mathbb{C}). \end{cases}$$

By successive Picard iterations, with the initial point $q(z_0) = \eta$, we get[2]
$y(z) = \lambda U(z_0; z)\eta$, where $U(z_0; z)$ is the following functional expansion

$$U(z_0; z) = \sum_{k \geq 0} \int_{z_0}^{z} A(z_1)dz_1 \int_{z_0}^{z_1} A(z_2)dz_2 \ldots \int_{z_0}^{z_{k-1}} A(z_k)dz_k, \text{(Dyson series)}$$

and $(z_0, z_1 \ldots, z_k, z)$ is a subdivision of the path of integration $z_0 \rightsquigarrow z$.
In order to find the matrix $\Omega(z_0; z)$ s.t.

$$U(z_0; z) = \exp[\Omega(z_0; z)] = \top \exp \int_{z_0}^{z} A(s)ds, \qquad \text{(Feynman's notation)}$$

Magnus computed $\Omega(z_0; z)$ as limit of the following Lie-integral-functionals

$$\begin{aligned} \Omega_1(z_0; z) &= \int_{z_0}^{z} A(z)ds, \\ \Omega_k(z_0; z) &= \int_{z_0}^{z} [A(z) + [A(z), \Omega_{k-1}(z_0; s)]/2 \\ &\quad + [[A(z), \Omega_{k-1}(z_0; s)], \Omega_{k-1}(z_0; s)]/12 + \ldots)ds. \end{aligned}$$

[2] Subject to convergence.

Fuchsian linear differential equations

$\sigma := \{s_i\}_{i=0,..,m}$: set of simple poles of (ED), $m \geq 1$.

Let $\mathcal{H}(\Omega)$ be the ring of holomorphic functions over $\Omega := \widetilde{\mathbb{C} \setminus \sigma}$.

$$A(z) = \sum_{i=0}^{m} M_i u_i(z), \quad \text{where} \quad \begin{cases} M_i & \in \mathcal{M}_{n,n}(\mathbb{C}), \\ u_i(z) = 1/(z - s_i) & \in \mathbb{C}(z). \end{cases}$$

$$(ED) \quad \begin{cases} \partial q(z) &=& \left(\sum_{i=0}^{m} M_i u_i(z)\right) q(z), \\ q(z_0) &=& \eta, \\ y(z) &=& \lambda q(z), \end{cases}$$

Let X^* be the set of words over $X = \{x_0, \ldots, x_m\}$ and

$$\alpha_{z_0}^{z} \otimes \mathcal{M} : \mathbb{C}\langle X \rangle \otimes \mathbb{C}\langle X \rangle \to \mathcal{M}_{n,n}(\mathcal{H}(\Omega))$$

($z_0 \rightsquigarrow z$ is the path of integration previously introduced) s.t.

$$\mathcal{M}(1_{X^*}) = \mathtt{Id}_n \quad \text{and} \quad \mathcal{M}(x_{i_1} \cdots x_{i_k}) = M_{i_1} \ldots M_{i_k},$$

$$\alpha_{z_0}^{z}(1_{X^*}) = 1_{\mathcal{H}(\Omega)} \quad \text{and} \quad \alpha_{z_0}^{z}(x_{i_1} \cdots x_{i_k}) = \int_{z_0}^{z} \frac{dz_1}{z_1 - s_{i_1}} \ldots \int_{z_0}^{z_{k-1}} \frac{dz_k}{z_k - s_{i_k}}.$$

Then[3] $y(z) = \lambda U(z_0; z) \eta$ with

$$U(z_0; z) = \sum_{w \in X^*} \mathcal{M}(w) \alpha_{z_0}^{z}(w) = (\mathcal{M} \otimes \alpha_{z_0}) \sum_{w \in X^*} w \otimes w.$$

[3] Subject to convergence.

Examples of linear dynamical systems

Example 2 (Hypergeometric equation)

Let t_0, t_1, t_2 be parameters and

$$z(1-z)\ddot{y}(z) + [t_2 - (t_0 + t_1 + 1)z]\dot{y}(z) - t_0 t_1 y(z) = 0.$$

Let $q_1(z) = -y(z)$ and $q_2(z) = (1-z)\dot{y}(z)$. Hence, one has

$$y(z) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} q_1(z) \\ q_2(z) \end{pmatrix}$$

and

$$\begin{aligned} \begin{pmatrix} \dot{q}_1(z) \\ \dot{q}_2(z) \end{pmatrix} &= \left(\frac{M_0}{z} + \frac{M_1}{1-z} \right) \begin{pmatrix} q_1(z) \\ q_2(z) \end{pmatrix} \\ &= (u_0(z) M_0 + u_1(z) M_1) \begin{pmatrix} q_1(z) \\ q_2(z) \end{pmatrix}, \end{aligned}$$

where $u_0(z) = z^{-1}, u_1(z) = (1-z)^{-1}$ and

$$M_0 = -\begin{pmatrix} 0 & 0 \\ t_0 t_1 & t_2 \end{pmatrix} \quad \text{and} \quad M_1 = -\begin{pmatrix} 0 & 1 \\ 0 & t_2 - t_0 - t_1 \end{pmatrix}.$$

Fuchsian nonlinear differential equations

$$(NED)\quad \begin{cases} \partial q(z) &= \left(\sum_{i=0}^{m} T_i(q)u_i(z)\right)(q),\\ q(z_0) &= q_0,\\ y(z) &= f(q(z)),\end{cases}$$

where

- $u_i(z) = (s_i - z)^{-1}$ and s_i is a simple poles of (NDE),
- the state $q = (q_1, \ldots, q_n)$ belongs the complex analytic manifold Q of dimension n and q_0 is the initial state,
- the observation $f \in \mathcal{O}$, with $\mathcal{O}$ the ring of analytic functions over Q,
- for $i = 0..1$, $T_i = (T_i^1(q)\partial/\partial q_1 + \cdots + T_i^m(q)\partial/\partial q_m)$ is an analytic vector field over Q, with $T_i^j(q) \in \mathcal{O}$, for $j = 1, \ldots, n$.

With X and $\alpha_{z_0}^z$ given as previously, let the morphism τ be defined by $\tau(1_{X^*}) = \mathtt{Id}$ and $\tau(x_{i_1} \cdots x_{i_k}) = T_{i_1} \ldots T_{i_k}$. Then[4] $y(z) = \mathcal{T} \circ f_{|_{q_0}}$ with

$$\mathcal{T} = \sum_{w \in X^*} \tau(w)\alpha_{z_0}^z(w) = (\tau \otimes \alpha_{z_0}) \sum_{w \in X^*} w \otimes w.$$

[4] Subject to convergence.

Examples of nonlinear dynamical systems (1/2)

Example 3 (Harmonic oscillator)

Let k_1, k_2 be parameters and $\partial^2 y(z) + k_1 y(z) + k_2 y^2(z) = u_1(z)$ which can be represented by the following state equations (with $n = 1$)

$$
\begin{aligned}
y(z) &= q(z),\\
\partial q(z) &= A_0(q)u_0(z) + A_1(q)u_1(z),\\
\text{where}\quad A_0 &= -(k_1 q + k_2 q^2)\frac{\partial}{\partial q} \quad\text{and}\quad A_1 = \frac{\partial}{\partial q}.
\end{aligned}
$$

Example 4 (Duffing equation)

Let a, b, c be parameters and $\partial^2 y(z) + a\partial y(z) + by(z) + cy^3(z) = u_1(z)$ which can be represented by the following state equations (with $n = 2$)

$$
\begin{aligned}
y(z) &= q_1(z),\\
\begin{pmatrix}\partial q_1(z)\\ \partial q_2(z)\end{pmatrix} &= \begin{pmatrix} q_2 \\ -(aq_2 + b^2 q_1 + cq_1^3)\end{pmatrix} u_0(z) + \begin{pmatrix}0\\1\end{pmatrix} u_1(z)\\
&= A_0(q)u_0(z) + A_1(q)u_1(z),\\
\text{where}\quad A_0 &= -(aq_2 + b^2 q_1 + cq_1^3)\frac{\partial}{\partial q_2} + q_2\frac{\partial}{\partial q_1} \quad\text{and}\quad A_1 = \frac{\partial}{\partial q_2}.
\end{aligned}
$$

Examples of nonlinear dynamical systems (2/2)

Example 5 (Van der Pol oscillator)

Let γ, g be parameters and

$$\partial^2 x(z) - \gamma[1 + x(z)^2]\partial x(z) + x(z) = g\cos(\omega z)$$

which can be tranformed into (with C is some constant of integration)

$$\partial x(z) = \gamma[1 + x(z)^2/3]x(z) - \int_{z_0}^{z} x(s)ds + \frac{g}{\omega}\sin(\omega z) + C.$$

Supposing $x = \partial y$ and $u_1(z) = g\sin(\omega z)/\omega + C$, it leads then to

$$\partial^2 y(z) = \gamma[\partial y(z) + (\partial y(z))^3/3] + y(z) + u_1(z)$$

which can be represented by the following state equations (with $n = 2$)

$$\begin{aligned} y(z) &= q_1(z), \\ \begin{pmatrix} \partial q_1(z) \\ \partial q_2(z) \end{pmatrix} &= \begin{pmatrix} q_2 \\ \gamma(q_2 + q_2^3/3) + q_1 \end{pmatrix} u_0(z) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_1(z) \\ &= A_0(q)u_0(z) + A_1(q)u_1(z), \end{aligned}$$

$$\text{where} \quad A_0 = [\gamma(q_2 + q_2^3/3) + q_1]\frac{\partial}{\partial q_2} + q_2\frac{\partial}{\partial q_1} \quad \text{and} \quad A_1 = \frac{\partial}{\partial q_2}.$$

DUAL LAWS AND REPRESENTATIVE SERIES

Dual law in bialgebra

Startting with a $\mathbf{k}-\mathbf{AAU}$ ($\mathbf{k}$ is a ring) $\mathcal{A}$. Dualizing $\mu : \mathcal{A}\otimes_{\mathbf{k}}\mathcal{A}\to\mathcal{A}$, we get the transpose ${}^t\mu : \mathcal{A}^\vee \to (\mathcal{A}\otimes_{\mathbf{k}}\mathcal{A})^\vee$ so that we do not get a co-multiplication in general.

- Remark that when $\mathbf{k}$ is a field, the following arrow is into (due to the fact that $\mathcal{A}^\vee\otimes_{\mathbf{k}}\mathcal{A}^\vee$ is torsionfree)
$$\Phi : \mathcal{A}^\vee\otimes_{\mathbf{k}}\mathcal{A}^\vee \to (\mathcal{A}\otimes_{\mathbf{k}}\mathcal{A})^\vee.$$
- One restricts the codomain of ${}^t\mu$ to $\mathcal{A}^\vee\otimes_{\mathbf{k}}\mathcal{A}^\vee$ and then the domain to $({}^t\mu)^{-1}\Phi(\mathcal{A}^\vee\otimes_{\mathbf{k}}\mathcal{A}^\vee) =: \mathcal{A}^\circ$.

$$\begin{array}{ccc}
\mathcal{A}^\vee & \xrightarrow{\quad {}^t\mu \quad} & (\mathcal{A}\otimes_{\mathbf{k}}\mathcal{A})^\vee \\
{\scriptstyle can}\uparrow & & \uparrow{\scriptstyle \Phi} \\
\mathcal{A}^\circ & \xrightarrow{\quad \Delta_\mu \quad} & \mathcal{A}^\vee\otimes_{\mathbf{k}}\mathcal{A}^\vee \\
{\scriptstyle can}\uparrow & & \uparrow{\scriptstyle j\otimes j} \\
\mathcal{A}^{\circ\circ} & \xrightarrow{\quad \Delta_\mu \quad} & \mathcal{A}^\circ\otimes_{\mathbf{k}}\mathcal{A}^\circ
\end{array}$$

The descent can stop at first step for a field $\mathbf{k}$ and then $\mathcal{A}^{\circ\circ} = \mathcal{A}^\circ$. The coalgebra $(\mathcal{A}^\circ, \Delta_\mu)$ is called the Sweedler's dual of $(\mathcal{A}, \mu)$.

Case of algebras noncommutative series

- Denoting the (ordered) alphabets $Y := \{y_k\}_{k\geq 1}$ (with $y_1 \succ y_2 \succ \ldots$) or $X := \{x_0, x_1\}$ (with $x_1 \succ x_0$) by $\mathcal{X}$, we use the correspondence among words of the free monoid $(\mathcal{X}^*, \mathrm{conc}, 1_{\mathcal{X}^*})$:
$$(s_1, \ldots, s_r) \in \mathbb{N}_+^r \leftrightarrow y_{s_1} \ldots y_{s_r} \in Y^* \underset{\pi_Y}{\overset{\pi_X}{\rightleftharpoons}} x_0^{s_1-1} x_1 \ldots x_0^{s_r-1} x_1 \in X^* x_1.$$

- Let $\mathcal{L}yn\mathcal{X}$ denote the set of Lyndon words generated by $\mathcal{X}$.

- Let $(\mathcal{L}ie_A\langle\langle\mathcal{X}\rangle\rangle, [.])$ and $(A\langle\langle\mathcal{X}\rangle\rangle, \mathrm{conc})$ (resp. $\mathcal{L}ie_A\langle\mathcal{X}\rangle, [.])$ and $(A\langle\mathcal{X}\rangle, \mathrm{conc})$) denote the algebras of (Lie) series (resp. polynomials) with coefficients in the ring A, over $\mathcal{X}$.

- $\mathcal{H}_{⧢}(\mathcal{X}) := (A\langle\mathcal{X}\rangle, \mathrm{conc}, 1_{\mathcal{X}^*}, \Delta_{⧢}, \mathrm{e})$ and $\mathcal{H}_{\text{⧢}\!\!\!\!\!\!-}(Y) := (A\langle Y\rangle, \mathrm{conc}, 1_{Y^*}, \Delta_{\text{⧢}\!\!\!\!\!\!-}, \mathrm{e})$ with[5]
$$\forall x \in \mathcal{X}, \quad \Delta_{⧢} x = x \otimes 1_{\mathcal{X}^*} + 1_{\mathcal{X}^*} \otimes x,$$
$$\forall y_i \in Y, \quad \Delta_{\text{⧢}\!\!\!\!\!\!-} y_i = y_i \otimes 1_{Y^*} + 1_{Y^*} \otimes y_i + \sum_{k+l=i} y_k \otimes y_l.$$

- The dual law associated to conc is defined by
$$\forall w \in \mathcal{X}^*, \quad \Delta_{\mathrm{conc}}(w) = \sum_{u,v \in \mathcal{X}^*, uv=w} u \otimes v.$$

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[5] Or equivalently, for $x, y \in \mathcal{X}, y_i, y_j \in Y$ and $u, v \in \mathcal{X}^*$ (resp. Y^*), $u ⧢ 1_{\mathcal{X}^*} = 1_{\mathcal{X}^*} ⧢ u = u$ and $xu ⧢ yv = x(u ⧢ yv) + y(xu ⧢ v)$, $u \text{⧢}\!\!\!\!\!\!- 1_{Y^*} = 1_{Y^*} \text{⧢}\!\!\!\!\!\!- u = u$ and $x_i u \text{⧢}\!\!\!\!\!\!- y_j v = y_i(u \text{⧢}\!\!\!\!\!\!- y_j v) + y_j(y_i u \text{⧢}\!\!\!\!\!\!- v) + y_{i+j}(u \text{⧢}\!\!\!\!\!\!- v)$.

Dualizable laws in conc-shuffle bialgebras (1/2)

We can exploit the basis of words as follows

1. Any bilinear law (shuffle, stuffle or any) $\mu : A\langle\mathcal{X}\rangle \otimes_A A\langle\mathcal{X}\rangle \to A\langle\mathcal{X}\rangle$ can be decribed through its structure constants wrt to the basis of words, *i.e.* for $u, v, w \in \mathcal{X}^*$, $\Gamma_{u,v}^{w} := \langle \mu(u \otimes v) | w \rangle$ so that
$$\mu(u \otimes v) = \sum_{w \in \mathcal{X}^*} \Gamma_{u,v}^{w} w.$$
2. In the case when $\Gamma_{u,v}^{w}$ is locally finite in w, we say that the given law is dualizable, the arrow ${}^t\mu$ restricts nicely to $A\langle\mathcal{X}\rangle \hookrightarrow A\langle\langle\mathcal{X}\rangle\rangle$ and one can define on the polynomials a comultiplication by
$$\Delta_\mu(w) := \sum_{u,v \in \mathcal{X}^*} \Gamma_{u,v}^{w} u \otimes v.$$
3. When the law μ is dualizable, we have

$$\begin{array}{ccc} A\langle\langle\mathcal{X}\rangle\rangle & \xrightarrow{\ {}^t\mu\ } & A\langle\langle\mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle \\ {\scriptstyle can}\uparrow & & \uparrow{\scriptstyle \Phi|_{A\langle\mathcal{X}\rangle \otimes_A A\langle\mathcal{X}\rangle}} \\ A\langle\mathcal{X}\rangle & \xrightarrow{\ \Delta_\mu\ } & A\langle\mathcal{X}\rangle \otimes_A A\langle\mathcal{X}\rangle \end{array}$$

The arrow Δ_μ is unique to be able to close the rectangle and $\Delta_\mu(P)$ is defined as above.

Dualizable laws in conc-shuffle bialgebras (2/2)

4. Proof that the arrow $A\langle\mathcal{X}\rangle \otimes_A A\langle\mathcal{X}\rangle \longrightarrow A\langle\langle\mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle$ is into:
   Let $T = \sum_{i=1}^{n} P_i \otimes_A Q_i$ such that $\Phi(T) = 0$. Rewriting T as a finitely supported sum $T = \sum_{u,v \in \mathcal{X}^*} c_{u,v}\, u \otimes v$ (this is indeed the iso between $A\langle\mathcal{X}\rangle \otimes_A A\langle\mathcal{X}\rangle$ and $A[\mathcal{X}^* \times \mathcal{X}^*]$), $\Phi(T)$ is by definition of Φ the double series (here a polynomial) s.t. $\langle\Phi(T)|u \otimes v\rangle = c_{u,v}$. If $\Phi(T) = 0$, then for all $(u, v) \in \mathcal{X}^* \times \mathcal{X}^*$, $c_{u,v} = 0$ entailing $T = 0$.

We extend by linearity and infinite sums, for $S \in A\langle\langle Y\rangle\rangle$ (resp. $A\langle\langle\mathcal{X}\rangle\rangle$), by

$$\begin{aligned}
\Delta_{\uplus} S &= \sum_{w \in Y^*} \langle S|w\rangle \Delta_{\uplus} w &\in A\langle\langle Y^* \otimes Y^*\rangle\rangle,\\
\Delta_{\texttt{conc}} S &= \sum_{w \in \mathcal{X}^*} \langle S|w\rangle \Delta_{\texttt{conc}} w &\in A\langle\langle \mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle,\\
\Delta_{\sqcup\!\sqcup} S &= \sum_{w \in \mathcal{X}^*} \langle S|w\rangle \Delta_{\sqcup\!\sqcup} w &\in A\langle\langle \mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle.
\end{aligned}$$

$\underline{A\langle\langle\mathcal{X}\rangle\rangle \otimes A\langle\langle\mathcal{X}\rangle\rangle \text{ embeds injectively}}$ in[6] $A\langle\langle\mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle \cong [A\langle\langle\mathcal{X}\rangle\rangle]\langle\langle\mathcal{X}\rangle\rangle$.

[6] $A\langle\langle\mathcal{X}\rangle\rangle \otimes A\langle\langle\mathcal{X}\rangle\rangle$ contains the elements of the form $\sum_{i \in I \text{ finite}} G_i \otimes D_i$, for $(G_i, D_i) \in A\langle\langle\mathcal{X}\rangle\rangle \times A\langle\langle\mathcal{X}\rangle\rangle$. But since elements of $M \otimes N$ are finite combination of $m_i \otimes n_i, m_i \in M, n_i \in N$ then $\sum_{i \geq 0} u^i \otimes v^i$ belongs to $A\langle\langle\mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle$ and does not belong to $A\langle\langle\mathcal{X}\rangle\rangle \otimes A\langle\langle\mathcal{X}\rangle\rangle$, for $u, v \in \mathcal{X}^{\geq 1}$.

Extended Ree's theorem

Let $S \in A\langle\langle Y \rangle\rangle$ (resp. $A\langle\langle \mathcal{X} \rangle\rangle$), A is a commutative ring containing $\mathbb{Q}$. The series S is said to be

1. a $⧢$ (resp. conc, $\sqcup\!\sqcup$)-character iff, for any $w, v \in Y^*$ (resp. $\mathcal{X}^*$), $\langle S|w\rangle\langle S|v\rangle = \langle S|w ⧢ v\rangle$ (resp. $\langle S|wv\rangle, \langle S|w \sqcup\!\sqcup v\rangle$) and $\langle S|1\rangle = 1$.
2. an infinitesimal $⧢$ (resp. conc, $\sqcup\!\sqcup$)-character iff, for any $w, v \in Y^*$ (resp. $\mathcal{X}^*$), $\langle S|w ⧢ v\rangle = \langle S|w\rangle\langle v|1_{Y^*}\rangle + \langle w|1_{Y^*}\rangle\langle S|v\rangle$ (resp. $\langle S|wv\rangle = \langle S|w\rangle\langle v|1_{\mathcal{X}^*}\rangle + \langle w|1_{\mathcal{X}^*}\rangle\langle S|v\rangle$, $\langle S|w \sqcup\!\sqcup v\rangle = \langle S|w\rangle\langle v|1_{\mathcal{X}^*}\rangle + \langle w|1_{\mathcal{X}^*}\rangle\langle S|v\rangle$).
3. a group-like series iff $\langle S|1_{\mathcal{X}^*}\rangle = 1$ and $\Delta_{⧢} S = \Phi(S \otimes S)$ (resp. $\Delta_{\mathrm{conc}} S = \Phi(S \otimes S), \Delta_{⧢} S = \Phi(S \otimes S)$).
4. a primitive series iff $\Delta_{⧢} S = 1_{Y^*} \otimes S + S \otimes 1_{Y^*}$ (resp. $\Delta_{\mathrm{conc}} S = 1_{\mathcal{X}^*} \otimes S + S \otimes 1_{\mathcal{X}^*}, \Delta_{\sqcup\!\sqcup} S = 1_{\mathcal{X}^*} \otimes S + S \otimes 1_{\mathcal{X}^*}$).

Then the following assertions are equivalent

1. S is a $⧢$ (resp. conc and $\sqcup\!\sqcup$)-character.
2. $\log S$ an infinitesimal $⧢$ (resp. conc and $\sqcup\!\sqcup$)-character.
3. S is group-like, for $\Delta_{⧢}$ (resp. Δ_{conc} and $\Delta_{\sqcup\!\sqcup}$).
4. $\log S$ is primitive, for $\Delta_{⧢}$ (resp. Δ_{conc} and $\Delta_{\sqcup\!\sqcup}$).

Extension by continuity (infinite sums)

Now, suppose that the ring A (containing $\mathbb{Q}$) is a field $\mathbf{k}$. Then

$$\Delta_{\sqcup\!\sqcup} : \mathbf{k}\langle\mathcal{X}\rangle \to \mathbf{k}\langle\mathcal{X}\rangle \otimes \mathbf{k}\langle\mathcal{X}\rangle \text{ and } \Delta_{\uplus} : \mathbf{k}\langle Y\rangle \to \mathbf{k}\langle Y\rangle \otimes \mathbf{k}\langle Y\rangle$$

are graded for the multidegree. Then $\Delta_{\uplus}$ is graded for the length. Their extension to the completions (*i.e.* $\mathbf{k}\langle\langle\mathcal{X}\rangle\rangle$ and $\mathbf{k}\langle\langle\mathcal{X}^* \otimes \mathcal{X}^*\rangle\rangle$) are continuous and then, when exist, commute with infinite sums. Hence[7,8],

$$\forall c \in \mathbf{k}, \quad \Delta_{\sqcup\!\sqcup}(cx)^* = \sum_{n\geq 0} c^n \Delta_{\sqcup\!\sqcup} x^n = \sum_{n\geq 0} c^n \sum_{j=0}^{n} \binom{n}{j} x^j \otimes x^{n-j}.$$

For $c \in \mathbb{N}_{\geq 2}$ which is neither a field nor a ring (containing $\mathbb{Q}$), we also get

$$(cx)^* = (c-1)^{-1} \sum_{a,b\in\mathbb{N}_{\geq 1}, a+b=c} (ax)^* \sqcup\!\sqcup (bx)^* \quad \in \mathbb{N}_{\geq 2}\langle\langle\mathcal{X}\rangle\rangle,$$

$$\Delta_{\sqcup\!\sqcup}(cx)^* \neq (c-1)^{-1} \sum_{a,b\in\mathbb{N}_{\geq 1}, a+b=c} (ax)^* \otimes (bx)^* \quad \in \mathbb{Q}\langle\langle\mathcal{X}\rangle\rangle \otimes \mathbb{Q}\langle\langle\mathcal{X}\rangle\rangle,$$

because

$$\langle \mathrm{LHS} | x \otimes 1_{\mathcal{X}^*}\rangle = c \quad \text{and} \quad \langle \mathrm{RHS} | x \otimes 1_{\mathcal{X}^*}\rangle = (c-1)^{-1} \sum_{a=1}^{c-1} a = \frac{c}{2}.$$

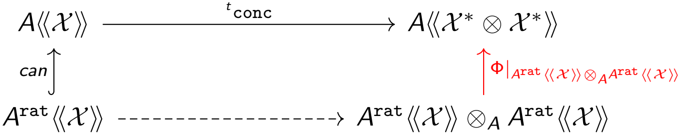
For $c \in \mathbb{Z}$ (or even $\mathbb{Q}, \mathbb{R}, \mathbb{C}$), the such decomposition is not finite.

[7]For $S \in A\langle\langle\mathcal{X}\rangle\rangle$ s.t. $\langle S|1_{\mathcal{X}^*}\rangle = 0$, $S^* = \sum_{n\geq 0} S^n$ is called Kleene star of S.

[8]$\Delta_{\sqcup\!\sqcup} x^n = (\Delta_{\sqcup\!\sqcup} x)^n = (1_{\mathcal{X}^*} \otimes x + x \otimes 1_{\mathcal{X}^*})^n = \sum_{j=0}^{n} \binom{n}{j} x^j \otimes x^{n-j}$.

Case of rational series and of Δ_{conc}

$A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$ denotes the algebraic closure by[9] $\{\mathrm{conc},+,*\}$ of $\widehat{A.\mathcal{X}}$ in $A\langle\langle\mathcal{X}\rangle\rangle$.



The dashed arrow may not exist in general, but for any $R\in A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$ admitting (λ,μ,η) as linear representation of dimension n, we can get

$${}^t\mathrm{conc}(R)=\Phi(\sum_{i=1}^n G_i\otimes D_i).$$

Indeed, since $\langle R|xy\rangle=\lambda\mu(xy)\eta=\lambda\mu(x)\mu(y)\eta$ $(x,y\in\mathcal{X})$ then, letting e_i is the vector such that ${}^te_i=\begin{pmatrix}0&\ldots&0&1&0&\ldots&0\end{pmatrix}$, one has

$$\langle R|xy\rangle=\sum_{i=1}^n\lambda\mu(x)e_i{}^te_i\mu(y)\eta=\sum_{i=1}^n\langle G_i|x\rangle\langle D_i|y\rangle=\sum_{i=1}^n\langle G_i\otimes D_i|x\otimes y\rangle.$$

G_i (resp. D_i) admits then (λ,μ,e_i) (resp. $({}^te_i,\mu,\eta)$) as linear representation. If $A=\mathbf{k}$ being a field then, due to the injectivity of Φ, all expressions of the type $\sum_{i=1}^n G_i\otimes D_i$, of course, coincide. Hence, the dashed arrow (a restriction of Δ_{conc}) in the above diagram is well-defined.

[9] $A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$ is closed under $\sqcup\!\sqcup$. $A^{\mathrm{rat}}\langle\langle Y\rangle\rangle$ is also closed under $\uplus$.

Representative series and Sweedler's dual

Theorem 6 (representative series)

Let $S \in A\langle\langle\mathcal{X}\rangle\rangle$. The following assertions are equivalent

1. *The series S belongs to $A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$.*
2. *There exists a linear representation (ν, μ, η) (of rank n) for S with $\nu \in M_{1,n}(A), \eta \in M_{n,1}(A)$ and a morphism of monoids $\mu : \mathcal{X}^* \to M_{n,n}(A)$ s.t. $S = \sum_{w\in\mathcal{X}^*}(\nu\mu(w)\eta)w$.*
3. *The shifts[10] $\{S \triangleleft w\}_{w\in\mathcal{X}^*}$ (resp. $\{w \triangleright S\}_{w\in\mathcal{X}^*}$) lie within a finitely generated shift-invariant A-module.*

Moreover, if A is a field $\mathbf{k}$, the previous assertions are equivalent to

4. *There exists $(G_i, D_i)_{i\in F\, finite}$ s.t. $\Delta_{\mathrm{conc}}(S) = \sum_{i\in F\, finite} G_i \otimes D_i$.*

Hence, $\mathcal{H}^{\circ}_{\sqcup\!\sqcup}(\mathcal{X}) = (\mathbf{k}^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle, \sqcup\!\sqcup, 1_{\mathcal{X}^*}, \Delta_{\mathrm{conc}}, \mathrm{e})$ and $\mathcal{H}^{\circ}_{\uplus}(Y) = (\mathbf{k}^{\mathrm{rat}}\langle\langle Y\rangle\rangle, \uplus, 1_{\mathcal{X}^*}, \Delta_{\mathrm{conc}}, \mathrm{e})$.

Now, let $A_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle$ (resp. $A^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle$) be the set of exchangeable[11] series (resp. series admitting a linear representation with commuting matrices).

[10] The *left* (resp. *right*) shift of S by P is $P \triangleright S$ (resp. $S \triangleleft P$) defined by, for $w \in \mathcal{X}^*$, $\langle P \triangleright S|w\rangle = \langle S|wP\rangle$ (resp. $\langle S \triangleleft P|w\rangle = \langle S|Pw\rangle$).

[11] *i.e.* if $S \in A_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle$ then $(\forall u, v \in \mathcal{X}^*)((\forall x \in \mathcal{X})(|u|_x = |v|_x) \Rightarrow \langle S|u\rangle = \langle S|v\rangle)$.

Kleene stars of the plane and conc-characters

Theorem 7 (rational exchangeable series)

1. *If the $\mathbb{Q}$-algebra A is a field* $\mathbf{k}$ *then for any* $S \in \mathbf{k}\langle\langle\mathcal{X}\rangle\rangle$,
   $\Delta_{\texttt{conc}}(S) = S \otimes S, \langle S|1_{\mathcal{X}^*}\rangle = 1 \iff S = (\sum_{x\in\mathcal{X}} c_x x)^*$ *with* $c_x \in \mathbf{k}$.
   Thus, any solution of $\nabla S = MS$ *(or* $\nabla S = SM$*) and* $\langle S|1_{\mathcal{X}^*}\rangle = 1$ *is group-like, for* $\Delta_{\texttt{conc}}$, *where* $\nabla S = S - 1_{\mathcal{X}^*}$ *and* $M = (\sum_{x\in\mathcal{X}} c_x x)$.
2. $A^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle \subset A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle \cap A_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle$. *If A is a field then the equality holds and* $A^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle = A^{\mathrm{rat}}\langle\langle x_0\rangle\rangle ⧢ A^{\mathrm{rat}}\langle\langle x_1\rangle\rangle$ *and, for the algebra of series over subalphabets* $A^{\mathrm{rat}}_{\mathrm{fin}}\langle\langle Y\rangle\rangle := \cup_{F\subset_{finite} Y} A^{\mathrm{rat}}\langle\langle F\rangle\rangle$, *we get*[12]
   $A^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle Y\rangle\rangle \cap A^{\mathrm{rat}}_{\mathrm{fin}}\langle\langle Y\rangle\rangle = \cup_{k\geq 0} A^{\mathrm{rat}}\langle\langle y_1\rangle\rangle ⧢ \ldots ⧢ A^{\mathrm{rat}}\langle\langle y_k\rangle\rangle \subsetneq A^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle Y\rangle\rangle$.
3. $\forall x \in \mathcal{X}, A^{\mathrm{rat}}\langle\langle x\rangle\rangle = \{P(1-xQ)^{-1}\}_{P,Q\in A[x]}$. *If* $\mathbf{k}$ *is an algebraically closed field then* $\mathbf{k}^{\mathrm{rat}}\langle\langle x\rangle\rangle = \mathrm{span}_{\mathbf{k}}\{(ax)^* ⧢ \mathbf{k}\langle x\rangle | a \in K\}$.
4. *If A is a $\mathbb{Q}$-algebra without zero divisors,* $\{x^*\}_{x\in\mathcal{X}}$ *(resp.* $\{y^*\}_{y\in Y}$*) are* conc*-character and algebraically independent over* $(A\langle\mathcal{X}\rangle, ⧢)$ *(resp.* $(A\langle Y\rangle, ⧢)$*) within* $(A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle, ⧢)$ *(resp.* $(A^{\mathrm{rat}}\langle\langle Y\rangle\rangle, ⧢)$*).*

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[12] The following identity lives in $A^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle Y\rangle\rangle$ but not in $A^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle Y\rangle\rangle \cap A^{\mathrm{rat}}_{\mathrm{fin}}\langle\langle Y\rangle\rangle$,
$(y_1 + \ldots)^* = \lim_{k\to+\infty}(y_1 + \ldots + y_k)^* = \lim_{k\to+\infty} y_1^* ⧢ \ldots ⧢ y_k^* = ⧢_{k\geq 1} y_k$.

CONTINUITY OVER CHEN SERIES

Iterated integrals

Let Ω be a simply connected domain admitting $1_{\mathcal{H}(\Omega)}$ as neutral element.
Let $\mathcal{A} := \mathcal{H}(\Omega)$ and let $\mathcal{C}_0$ be a differential subring of $\mathcal{A}$ ($\partial(\mathcal{C}_0) \subset \mathcal{C}_0$) which is an integral domain containing $\mathbb{C}$.
$\mathbb{C}\{\{(g_i)_{i\in I}\}\}$ denotes the differential subalgebra of $\mathcal{A}$ generated by $(g_i)_{i\in I}$, *i.e.* the $\mathbb{C}$-algebra generated by g_i's and their derivatives
$\{u_x\}_{x\in\mathcal{X}}$: elements in $\mathcal{C}_0 \cap \mathcal{A}^{-1}$ in correspondence with $\{\theta_x\}_{x\in\mathcal{X}}$ ($\theta_x = u_x^{-1}\partial$).
The iterated integral associated to $x_{i_1}\ldots x_{i_k} \in \mathcal{X}^*$, over the differential forms $\omega_i(z) = u_{x_i}(z)dz$, and along a path $z_0 \rightsquigarrow z$ on Ω, is defined by

$$\begin{aligned}
\alpha_{z_0}^z(1_{\mathcal{X}^*}) &= 1_\Omega,\\
\alpha_{z_0}^z(x_{i_1}\ldots x_{i_k}) &= \int_{z_0}^{z}\omega_{i_1}(z_1)\ldots\int_{z_0}^{z_{k-1}}\omega_{i_k}(z_k).\\
\partial\alpha_{z_0}^z(x_{i_1}\ldots x_{i_k}) &= u_{x_{i_1}}(z)\int_{z_0}^{z}\omega_{i_2}(z_2)\ldots\int_{z_0}^{z_{k-1}}\omega_{i_k}(z_k).
\end{aligned}$$

$$\begin{aligned}
\mathrm{span}_{\mathbb{C}}\{\partial^l\alpha_{z_0}^z(w)\}_{w\in\mathcal{X}^*,l\geq 0} &\subset \mathrm{span}_{\mathbb{C}\{\{(u_x)_{x\in\mathcal{X}}\}\}}\{\alpha_{z_0}^z(w)\}_{w\in\mathcal{X}^*}\\
&\subset \mathrm{span}_{\mathbb{C}\{\{(u_x^{\pm1})_{x\in\mathcal{X}}\}\}}\{\alpha_{z_0}^z(w)\}_{w\in\mathcal{X}^*}\\
&\cong \mathbb{C}\{\{(u_x^{\pm1})_{x\in\mathcal{X}}\}\}\otimes_{\mathbb{C}}\mathrm{span}_{\mathbb{C}}\{\alpha_{z_0}^z(w)\}_{w\in\mathcal{X}^*}?
\end{aligned}$$

These iterated integrals satisfy the following Chen's lemma

$$\forall u,v\in\mathcal{X}^*,\quad \alpha_{z_0}^z(u ⧢ v) = \alpha_{z_0}^z(u)\alpha_{z_0}^z(v).$$

Chen series

The ring A is supposed contain $\mathbb{Q}$. On $\mathcal{H}_{⧢}(\mathcal{X})$ and $\mathcal{H}_{\text{⧢}\!\!\!-}(Y)$, we also get

$$\mathcal{D}_{\mathcal{X}} := \sum_{w\in\mathcal{X}^*} w\otimes w = \sum_{w\in\mathcal{X}^*} S_w\otimes P_w = \prod_{l\in\mathcal{L}yn\mathcal{X}}^{\searrow} e^{S_l\otimes P_l},$$

$$\mathcal{D}_{Y} := \sum_{w\in Y^*} w\otimes w = \sum_{w\in Y^*} \Sigma_w\otimes \Pi_w = \prod_{l\in\mathcal{L}yn Y}^{\searrow} e^{\Sigma_l\otimes \Pi_l},$$

where $\{P_l\}_{l\in\mathcal{L}yn\mathcal{X}}$ (resp. $\{\Pi_l\}_{l\in\mathcal{L}yn Y}$) is a basis of Lie algebra of primitive elements and $\{S_l\}_{l\in\mathcal{L}yn\mathcal{X}}$ (resp. $\{\Sigma_l\}_{l\in\mathcal{L}yn Y}$) is a transcendence basis of $(A\langle\mathcal{X}\rangle, ⧢, 1_{\mathcal{X}^*})$ (resp. $(A\langle Y\rangle, \text{⧢}\!\!\!-, 1_{Y^*})$).

The Chen series of $\{\omega_i\}_{i\geq 1}$ and along $z_0 \rightsquigarrow z$ is defined as follows

$$C_{z_0\rightsquigarrow z} := \sum_{w\in\mathcal{X}^*} \alpha_{z_0}^z(w)w = (\alpha_{z_0}^z\otimes \mathtt{Id})\mathcal{D}_{\mathcal{X}} = \prod_{l\in\mathcal{L}yn\mathcal{X}}^{\searrow} e^{\alpha_{z_0}^z(S_l)P_l} \in \mathcal{H}(\Omega)\langle\langle\mathcal{X}\rangle\rangle.$$

Theorem 8

If $R\in\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$ with minimal representation of dimension n then[13] *$y(z_0,z)=\alpha_{z_0}^z(R)=\langle R\| C_{z_0\rightsquigarrow z}\rangle$ and there exists $l=0,..,n-1$ s.t. $\{\partial^k y\}_{0\leq k\leq l}$ is $\mathcal{C}_0$-linearly independent and $a_l,\ldots,a_1,a_0\in\mathcal{C}_0$ s.t.*

$$(a_l\partial^l + a_{l-1}\partial^{l-1} + \ldots + a_1\partial + a_0)y = 0.$$

[13] Subject to convergence.

Continuity, indiscernability and growth condition

For $i = 0, 2$, let $(\mathbf{k}_i, \|.\|_i)$ be a semi-normed space and $g_i \in \mathbb{Z}$.

Definition 9

1. Let $\mathcal{C}l$ be a class of $\mathbf{k}_1\langle\langle X\rangle\rangle$ and $S \in \mathbf{k}_2\langle\langle X\rangle\rangle$.

   a) S is said to be *continuous* over $\mathcal{C}l$ if, for any $\Phi \in \mathcal{C}l$, the sum $\sum_{w\in X^*} \|\langle S|w\rangle\|_2 \|\langle \Phi|w\rangle\|_1$ is convergent.
   We will denote $\langle S \parallel \Phi\rangle$ the sum $\sum_{w\in X^*}\langle S|w\rangle\langle\Phi|w\rangle$.
   $\mathbf{k}_2\langle\langle X\rangle\rangle^{\mathrm{cont}} \equiv$ set of continuous power series over $\mathcal{C}l$.

   b) S is said to be *indiscernable* over $\mathcal{C}l$ iff, for any $\Phi \in \mathcal{C}l$, $\langle S \parallel \Phi\rangle = 0$.

2. Let χ_1 and χ_2 be real positive functions over X^*. Let $S \in \mathbf{k}_1\langle\langle X\rangle\rangle$.

   a) S satisfies the χ_1*−growth condition* of order g_1 if it satisfies
   $$\exists K \in \mathbb{R}_+, \exists n \in \mathbb{N}, \forall w \in X^{\geq n}, \quad \|\langle S|w\rangle\|_1 \leq K\chi_1(w)\, |\, w\, |!^{g_1}.$$
   We denote by $\mathbf{k}_1^{(\chi_1,g_1)}\langle\langle X\rangle\rangle$ the set of formal power series in $\mathbf{k}_1\langle\langle X\rangle\rangle$ satisfying the χ_1−growth condition of order g_1.

   b) If S is continuous over $\mathbf{k}_2^{(\chi_2,g_2)}\langle\langle X\rangle\rangle$ then it will be said to be (χ_2, g_2)-*continuous*. The set of formal power series which are (χ_2, g_2)-continuous is denoted by $\mathbf{k}_2^{(\chi_2,g_2)}\langle\langle X\rangle\rangle^{\mathrm{cont}}$.

Convergence condition

Proposition 1

Let χ_1 and χ_2 be real positive functions over X^*.
Let g_1 and $g_2 \in \mathbb{Z}$ such that $g_1 + g_2 \leq 0$.

1. *Let $\mathbf{k}_1^{(\chi_1,g_1)}\langle\langle X\rangle\rangle$ and let $P \in \mathbf{k}_1\langle X\rangle$.*
   The right residual of S by P belongs to $\mathbf{k}_1^{(\chi_1,g_1)}\langle\langle X\rangle\rangle$.
2. *Let $R \in \mathbf{k}_2^{(\chi_2,g_2)}\langle\langle X\rangle\rangle$ and let $Q \in \mathbf{k}_2\langle X\rangle$.*
   The concatenation QR belongs to $\mathbf{k}_2^{(\chi_2,g_2)}\langle\langle X\rangle\rangle$.
3. *χ_1, χ_2 are morphisms over X^* satisfying $\sum_{x\in X} \chi_1(x)\chi_2(x) < 1$.*
   If $F_1 \in \mathbf{k}_1^{(\chi_1,g_1)}\langle\langle X\rangle\rangle$ (resp. $F_2 \in \mathbf{k}_2^{(\chi_2,g_2)}\langle\langle X\rangle\rangle$) then F_1 (resp. F_2) is continuous over $\mathbf{k}_2^{(\chi_2,g_2)}\langle\langle X\rangle\rangle$ (resp. $\mathbf{k}_1^{(\chi_1,g_1)}\langle\langle X\rangle\rangle$).

Proposition 2

Let $\mathcal{C}l \subset \mathbf{k}_1\langle\langle X\rangle\rangle$ be a monoid containing $\{e^{tx}\}_{x\in X}^{t\in\mathbf{k}_1}$. Let $S \in \mathbf{k}_2\langle\langle X\rangle\rangle^{cont}$.

1. *If S is indiscernable over $\mathcal{C}l$ then for any $x \in X$, $x \triangleleft S$ and $S \triangleright x$ belong to $\mathbf{k}_2\langle\langle X\rangle\rangle^{cont}$ and they are indiscernable over $\mathcal{C}l$.*
2. *S is indiscernable over $\mathcal{C}l$ if and only if $S = 0$.*

Chen series of $\omega_0(z) = z^{-1}dz$ and $\omega_1(z) = (1-z)^{-1}dz$

Let $\gamma_0(\varepsilon)$ and $\gamma_1(\varepsilon)$ be the circular paths of radius ε encircling 0 and 1 clockwise, respectively. In particular, letting $\beta = \beta_1 - \beta_0$, one considers

$$\begin{array}{rcccl} \gamma_0(\varepsilon,\beta) & = & \varepsilon e^{\mathrm{i}\beta_0} \rightsquigarrow \varepsilon e^{\mathrm{i}\beta_1} & \subset & \gamma_0(\varepsilon), \\ \gamma_1(\varepsilon,\beta) & = & 1-\varepsilon e^{\mathrm{i}\beta_0} \rightsquigarrow 1-\varepsilon e^{\mathrm{i}\beta_1} & \subset & \gamma_1(\varepsilon). \end{array}$$

On the one hand, one has, for any $i = 0$ or 1 and $w \in X^+$,

$$|\langle C_{\gamma_i(\varepsilon,\beta)}|w\rangle| \le \varepsilon^{|w|_{x_i}} \beta^{|w|} |w|!^{-1}.$$

It follows then

$$C_{\gamma_i(\varepsilon,\beta)} = e^{\mathrm{i}\beta x_i} + o(\varepsilon) \quad \text{and} \quad C_{\gamma_i(\varepsilon)} = e^{2\mathrm{i}\pi x_i} + o(\varepsilon).$$

On the other hand, for $R \in \mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$ of minimal representation (λ,μ,η) of dimension n, one has, for any $w \in X^*$,

$$|\langle R|w\rangle| \le \|\lambda\|_\infty^{1,n} \|\mu(w)\|_\infty^{n,n} \|\eta\|_\infty^{n,1}.$$

Hence,

$$\alpha_{z_0}^z(R) := \langle R \parallel C_{z_0\rightsquigarrow z}\rangle = \lambda((\alpha_{z_0}^z \otimes \mu)\mathcal{D}_X)\eta = \lambda\Big(\prod_{l\in\mathcal{L}ynX}^{\searrow} e^{\alpha_{z_0}^z(S_l)\mu(P_l)}\Big)\eta.$$

Note that the map $\alpha_{z_0}^z : \mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle \to \mathcal{H}(\Omega)$ is not injective. For example,

$$\alpha_{z_0}^z(z_0 x_0^* + (1-z_0)(-x_1)^* - 1_{X^*}) = 0.$$

More about Chen series

Chen series $C_{z_0 \rightsquigarrow z}$ of $\{\omega_i\}_{i\geq 1}$ satisfies the following Freidrichs criterion

$$\forall u, v \in \mathcal{X}^*, \quad \langle C_{z_0 \rightsquigarrow z} | u ⧢ v\rangle = \langle C_{z_0 \rightsquigarrow z} | u\rangle \langle C_{z_0 \rightsquigarrow z} | v\rangle.$$

On the other hand, for any u and $v \in \mathcal{X}^*$,

$$\begin{aligned} \langle C_{z_0 \rightsquigarrow z} | u\rangle \langle C_{z_0 \rightsquigarrow z} | v\rangle &= \langle C_{z_0 \rightsquigarrow z} \otimes C_{z_0 \rightsquigarrow z} | u \otimes v\rangle, \\ \langle C_{z_0 \rightsquigarrow z} | u ⧢ v\rangle &= \langle \Delta_{⧢} C_{z_0 \rightsquigarrow z} | u \otimes v\rangle. \end{aligned}$$

Hence, $\Delta_{⧢} C_{z_0 \rightsquigarrow z} = C_{z_0 \rightsquigarrow z} \otimes C_{z_0 \rightsquigarrow z}$ and $\langle C_{z_0 \rightsquigarrow z} | 1_{\mathcal{X}^*}\rangle = 1$.

Note that $C_{z_0 \rightsquigarrow z}$ only depends on the homotopy class of $z_0 \rightsquigarrow z$ and the endpoints z_0, z. One has[14]

$$\begin{aligned} C_{z_0 \rightsquigarrow z} C_{z_1 \rightsquigarrow z_0} &= C_{z_1 \rightsquigarrow z}, \\ \iff \forall w \in \mathcal{X}^*, \quad \langle C_{z_1 \rightsquigarrow z} | w\rangle &= \sum_{u,v \in \mathcal{X}^*, uv = w} \langle C_{z_0 \rightsquigarrow z} | u\rangle \langle C_{z_1 \rightsquigarrow z_0} | v\rangle. \end{aligned}$$

Note also that, for $g \in \mathcal{H}(\Omega)$, one has $C_{g(z_0) \rightsquigarrow g(z)} = g_* C_{z_0 \rightsquigarrow z}$, *i.e.* the Chen series of $\{g^*\omega_i\}_{i\geq 1}$ along the path $g^*(z_0 \rightsquigarrow z)$.

Example 10 (with $\omega_0(z) = z^{-1}dz$ and $\omega_1(z) = (1-z)^{-1}dz$)

$g(z)$	z	z^{-1}	$(z-1)z^{-1}$	$z(z-1)^{-1}$	$(1-z)^{-1}$	$1-z$
$g^*\omega_0$	ω_0	$-\omega_0$	$-\omega_1 - \omega_0$	$\omega_1 + \omega_0$	ω_1	$-\omega_1$
$g^*\omega_1$	ω_1	$\omega_1 + \omega_0$	$-\omega_0$	$-\omega_1$	$-\omega_1 - \omega_0$	$-\omega_0$

[14] Although $\Delta_{\mathrm{conc}} w = \sum_{u,v \in \mathcal{X}^*, uv = w} u \otimes v$ but $\Delta_{\mathrm{conc}} C_{z_1 \rightsquigarrow z} \neq C_{z_0 \rightsquigarrow z} \otimes C_{z_1 \rightsquigarrow z_0}$.

NONCOMMUTATIVE PV THEORY AND INDEPENDENCE VIA WORDS

Noncommutative differential equations

Considering $A = (\mathcal{H}(\Omega), \partial)$ as the differential ring of holomorphic functions on Ω, equipped 1_Ω as the neutral element, the differential ring $(\mathcal{H}(\Omega)\langle\langle\mathcal{X}\rangle\rangle, \mathbf{d})$ is defined, for any $S \in \mathcal{H}(\Omega)\langle\langle\mathcal{X}\rangle\rangle$, by

$$\mathbf{d}S = \sum_{w\in\mathcal{X}^*} (\partial\langle S|w\rangle)w \in \mathcal{H}(\Omega)\langle\langle\mathcal{X}\rangle\rangle.$$

The Chen series $C_{z_0\rightsquigarrow z}$ satisfies the following differential equation

$$(NCDE) \quad \mathbf{d}S = MS, \quad \text{with} \quad M = \sum_{x\in\mathcal{X}} u_x x.$$

$$\Delta_{\sqcup\!\sqcup} M = \sum_{x\in\mathcal{X}} u_x(1_{\mathcal{X}^*} \otimes x + x \otimes 1_{\mathcal{X}^*}) = 1_{\mathcal{X}^*} \otimes M + M \otimes 1_{\mathcal{X}^*}.$$

More generally, for any $k \geq 1$, $C_{z_0\rightsquigarrow z}$ satisfies $\mathbf{d}^k S = Q_k S$ with $Q_k \in \mathbb{C}\{\{(u_x^{\pm1})_{x\in\mathcal{X}}\}\}\langle\mathcal{X}\rangle$ satisfying the recursion

$$Q_0 = 1 \text{ and } Q_k = Q_{k-1}M + \mathbf{d}Q_{k-1}.$$

Q_k can be computed explicitly by (summing over words $w = x_{i_1}\ldots x_{i_k}$ and derivation multiindices $\mathbf{r} = (r_1,\ldots,r_k)$ of degree $\deg \mathbf{r} = |w| = k$ and of weight $\mathtt{wgt}\ \mathbf{r} = k + r_1 + \ldots + r_k$)

$$Q_k = \sum_{\mathtt{wgt}\ \mathbf{r}=k, w\in\mathcal{X}^{\deg \mathbf{r}}} \prod_{j=1}^{\deg \mathbf{r}} \binom{\sum_{j=1}^{j} r_j + j - 1}{r_k} \tau_{\mathbf{r}}(w), \quad \text{where}$$

$$\tau_{\mathbf{r}}(w) = \tau_{r_1}(x_{i_1})\ldots\tau_{r_k}(x_{i_k}) = (\partial^{r_1}u_{x_{i_1}})x_{i_1}\ldots(\partial^{r_k}u_{x_{i_k}})x_{i_k} \in \mathbb{C}\{\{(u_x^{\pm1})_{x\in\mathcal{X}}\}\}\langle\mathcal{X}\rangle.$$

First step of noncommutative PV theory

1. The space of solutions of

$$(NCDE) \quad \mathbf{d}S = MS, \quad \text{with} \quad M = \sum_{x \in \mathcal{X}} u_x x.$$

is a right free $\mathbb{C}\langle\langle X\rangle\rangle$-module of rank 1.

2. By a theorem of Ree, $C_{z_0 \rightsquigarrow z}$ is a $\sqcup\!\sqcup$ −group-like solution of $(NCDE)$ and it can be obtained as the limit of a convergent Picard iteration, initialized at $\langle C_{z_0 \rightsquigarrow z} | 1_{\mathcal{X}^*}\rangle = 1_\Omega 1_{\mathcal{X}^*}$, for ultrametric distance.

3. If G and H are $\sqcup\!\sqcup$ −group-like solutions $(NCDE)$ there is a constant Lie series C such that $G = He^C$ (and conversely).

From this, it follows that

- the differential Galois group of $(NCDE)$ + $\sqcup\!\sqcup$ −group-like is the group[15] $\{e^C\}_{C \in \mathcal{L}ie_{\mathbb{C}.1_\Omega}\langle\langle \mathcal{X}\rangle\rangle}$.

Which leads us to the following definition

- the PV extension related to $(NCDE)$ is $\widehat{\mathcal{C}_0.\mathcal{X}}\{C_{z_0 \rightsquigarrow z}\}$.

It, of course, is such that $\mathrm{Const}(\mathcal{C}_0\langle\langle \mathcal{X}\rangle\rangle) = \ker \mathbf{d} = \mathbb{C}.1_\Omega\langle\langle \mathcal{X}\rangle\rangle$.

[15] In fact, the Hausdorff group (group of characters) of $\mathcal{H}_{\sqcup\!\sqcup}(\mathcal{X})$.

Basic triangular theorem over a differential ring

Suppose that the $\mathbb{C}$-commutative ring $\mathcal{A}$ is without zero divisors and equipped with a differential operator ∂ such that $\mathbb{C} = \ker \partial$.
Let $S \in \mathcal{A}\langle\langle \mathcal{X} \rangle\rangle$ be a group-like solution of $(NCDE)$ in the following form

$$S = \sum_{w \in \mathcal{X}^*} \langle S|w \rangle w = \sum_{w \in \mathcal{X}^*} \langle S|S_w \rangle P_w = \prod_{l \in \mathcal{L}yn\mathcal{X}}^{\searrow} e^{\langle S|S_l \rangle P_l}.$$

Then

1. If $H \in \mathcal{A}\langle\langle \mathcal{X} \rangle\rangle$ is another grouplike solution then there exists $C \in \mathcal{L}ie_{\mathcal{A}}\langle\langle \mathcal{X} \rangle\rangle$ such that $S = He^C$ (and conversely).
2. The following assertions are equivalent
   a) $\{\langle S|w \rangle\}_{w \in \mathcal{X}^*}$ is $\mathcal{C}_0$-linearly independent,
   b) $\{\langle S|l \rangle\}_{l \in \mathcal{L}yn\mathcal{X}}$ is $\mathcal{C}_0$-algebraically independent,
   c) $\{\langle S|x \rangle\}_{x \in \mathcal{X}}$ is $\mathcal{C}_0$-algebraically independent,
   d) $\{\langle S|x \rangle\}_{x \in \mathcal{X} \cup \{1_{\mathcal{X}^*}\}}$ is $\mathcal{C}_0$-linearly independent,
   e) $\{u_x\}_{x \in \mathcal{X}}$ is such that, for $f \in \mathrm{Frac}(\mathcal{C}_0)$ and $(c_x)_{x \in \mathcal{X}} \in \mathbb{C}^{(\mathcal{X})}$,
   $$\sum_{x \in \mathcal{X}} c_x u_x = \partial f \implies (\forall x \in \mathcal{X})(c_x = 0).$$
   f) $(u_x)_{x \in \mathcal{X}}$ is free over $\mathbb{C}$ and $\partial \mathrm{Frac}(\mathcal{C}_0) \cap \mathrm{span}_{\mathbb{C}}\{u_x\}_{x \in \mathcal{X}} = \{0\}$.

Examples of positive cases over $\mathcal{X} = \{x\}, \mathcal{A} = (\mathcal{H}(\Omega), \partial)$

1. $\Omega = \mathbb{C}, u_x(z) = 1_\Omega, \mathcal{C}_0 = \mathbb{C}\{\{u_x^{\pm 1}\}\} = \mathbb{C}.$
   $\alpha_0^z(x^n) = z^n/n!$, for $n \geq 1$. Thus, $\mathbf{d}S = xS$ and
   $$S = \sum_{n\geq 0} \alpha_0^z(x^n)x^n = \sum_{n\geq 0} \frac{z^n}{n!}x^n = e^{zx}.$$
   Moreover, $\alpha_0^z(x) = z$ which is transcendent over $\mathcal{C}_0$
   and the family $\{\alpha_0^z(x^n)\}_{n\geq 0}$ is $\mathcal{C}_0$-free. Let $f \in \mathcal{C}_0$ then $\partial f = 0$. Thus,
   if $\partial f = cu_x$ then $c = 0$.
2. $\Omega = \mathbb{C}\backslash]-\infty, 0], u_x(z) = z^{-1}, \mathcal{C}_0 = \mathbb{C}\{\{z^{\pm 1}\}\} = \mathbb{C}[z^{\pm 1}] \subset \mathbb{C}(z).$
   $\alpha_1^z(x^n) = \log^n(z)/n!$, for $n \geq 1$. Thus $\mathbf{d}S = z^{-1}xS$ and
   $$S = \sum_{n\geq 0} \alpha_1^z(x^n)x^n = \sum_{n\geq 0} \frac{\log^n(z)}{n!}x^n = z^x.$$
   Moreover, $\alpha_1^z(x) = \log(z)$ which is transcendent over $\mathbb{C}(z)$ then over $\mathbb{C}[z^{\pm 1}]$. The family the family $\{\alpha_1^z(x^n)\}_{n\geq 0}$ is $\mathbb{C}(z)$-free and then $\mathcal{C}_0$-free. Let $f \in \mathcal{C}_0$ then $\partial f \in \mathrm{span}_{\mathbb{C}}\{z^{\pm n}\}_{n\neq 1}$. Thus,
   if $\partial f = cu_x$ then $c = 0$.

Examples of negative cases over $\mathcal{X} = \{x\}, \mathcal{A} = (\mathcal{H}(\Omega), \partial)$

1. $\Omega = \mathbb{C}, u_x(z) = e^z, \mathcal{C}_0 = \mathbb{C}\{\{e^{\pm z}\}\} = \mathbb{C}[e^{\pm z}]$.
   $\alpha_0^z(x^n) = (e^z - 1)^n/n!$, for $n \geq 1$. Thus, $\mathbf{d}S = e^z x S$ and
   $$S = \sum_{n\geq 0} \alpha_0^z(x^n)x^n = \sum_{n\geq 0} \frac{(e^z-1)^n}{n!}x^n = e^{(e^z-1)x}.$$
   Moreover, $\alpha_0^z(x) = e^z - 1$ which is not transcendent over $\mathcal{C}_0$ and $\{\alpha_0^z(x^n)\}_{n\geq 0}$ is not $\mathcal{C}_0$-free. If $f(z) = ce^z \in \mathcal{C}_0$ $(c \neq 0)$ then $W(f, 1_\Omega) = \partial f(z) = ce^z = cu_x(z)$.
2. $\Omega = \mathbb{C}\backslash] - \infty, 0], u_x(z) = z^a (a \notin \mathbb{Q})$,
   $\mathcal{C}_0 = \mathbb{C}\{\{z, z^{\pm a}\}\} = \mathrm{span}_{\mathbb{C}}\{z^{ka+l}\}_{k,l\in\mathbb{Z}}$.
   $\alpha_0^z(x^n) = (a+1)^{-n}z^{n(a+1)}/n!$, for $n \geq 1$. Thus, $\mathbf{d}S = z^a x S$ and
   $$S = \sum_{n\geq 0} \alpha_0^z(x^n)x^n = \sum_{n\geq 0} \frac{z^{n(a+1)}}{(a+1)^n n!}x^n = e^{(a+1)^{-1}z^{(a+1)}x}.$$
   Moreover, $\alpha_0^z(x) = (a+1)^{-1}z^{a+1}$ which is not transcendent over $\mathcal{C}_0$ and $\{\alpha_0^z(x^n)\}_{n\geq 0}$ is not $\mathcal{C}_0$-free. If $f(z) = c(a+1)^{-1}z^{a+1} \in \mathcal{C}_0$ $(c \neq 0)$ then $W(f, 1_\Omega) = \partial f(z) = cz^a = cu_x(z)$.

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