The thermal time hypothesis: geometrical action of the modular group in 2D conformal field theory with boundary

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Outline:

1. Modular group as a flow of time

2. Double-cones in 2d boundary conformal field theory

3. Vacuum modular group for free Fermi fields

1. Time flow from the modular group

Modular group

"Von Neumann algebras naturally evolve with time" (*Connes*)

Let $\mathcal{A}$ be a von Neumann algebra equipped with a one parameter group of automorphism $\{\sigma_s, s \in \mathbb{R}\}$.
A weight (i.e. positive linear map) φ on $\mathcal{A}$ satisfies the *modular condition* iff

- $\varphi = \varphi \circ \sigma_s, \quad \forall s \in \mathbb{R}$,

- for every $a, b \in n_\varphi \cap n_\varphi^*$ ($n_\varphi = \{a \in \mathcal{A}, \varphi(a^* a) < +\infty\}$) there exists a bounded continuous function F_{ab}, analytic on the strip $0 \leq \text{Im } z < 1$ such that

$$F_{ab}(s) = \varphi(\sigma_s(a)b), \quad F_{ab}(s+i) = \varphi(b\sigma_s(a)).$$

- Each weight φ satisfies the modular condition with respect to at most one unique group of automorphism σ_s.

- a von Neumann algebra $\mathcal{A}$ acting on $\mathcal{H}$
- a vector Ω in $\mathcal{H}$ cyclic and separating

$\Rightarrow$ Tomita's operator: $S\, a\Omega \rightarrow a^*\Omega$

Polar decomposition: $\overline{S} = J\Delta^{\frac{1}{2}}$ where $\Delta = \Delta^* > 0$ and J is unitary, antilinear.

Tomita's Theorem: $\Delta^{it}\mathcal{A}\Delta^{-it} = \mathcal{A}$ hence

$$t \mapsto \sigma_s : a \mapsto \sigma_s(a) \doteq \Delta^{is} a \Delta^{-is}$$

is a 1 parameter group of automorphism. Moreover the state $\omega : a \mapsto \langle \Omega, a\Omega \rangle$ satisfies the modular condition with respect to σ_s.

- mathematical importance: $\Omega' \neq \Omega$ gives the same modular group, modulo inner automorphism. Classification of factors.
- physical importance: ω is KMS with respect to σ_s, with temperature -1,

$$\omega(\sigma_s(a)b) = \omega(b\sigma_{s-i}(a)).$$

Thermal-time hypothesis

Can σ_s be interpreted as a *real* physical time flow ?

$H = \ln \Delta$ yields $\sigma_s(a) = e^{iHs} a e^{-iHs}$

or

- $\mathcal{A}$ carries a representation of a symmetry group G of spacetime (e.g. Poincaré),

- σ_s is generated by elements of $\mathfrak{g} \Longrightarrow$ geometrical action of the modular group,

- the orbit of a point under this geometric action is timelike.

But the tangent vector ∂_s to these orbits must be normalised,

$$\partial_t \doteq \frac{\partial_s}{\beta} \text{ with } \beta \doteq \|\partial_s\| = \left\|\partial_t \frac{dt}{ds}\right\| = |\frac{dt}{ds}|.$$

Writing $\alpha_{-\beta s} \doteq \sigma_s$,

$$\omega((\alpha_{-\beta s} a) b) = \omega(b(\alpha_{-\beta s + i\beta} a)).$$

- ω is an equilibrium state at temperature β^{-1} with respect to the time evolution $t = -\beta s$.

Algebraic field theory

A net of algebras of local observables is a map

$$\mathcal{O} \in \mathcal{B}(\text{Minkovski}) \to \mathcal{A}(\mathcal{O})$$

where $\mathcal{A}(\mathcal{O})$'s are C^*-algebras fulfilling

- isotony: $\mathcal{O}_1 \subset \mathcal{O}_2 \Longrightarrow \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$,
- locality: $\mathcal{O}_1$ spacelike to $\mathcal{O}_2 \Longrightarrow [\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = 0$,

together with an irreducible representation π on an Hilbert space $\mathcal{H}$ such that

- Poincaré covariance: $U(\Lambda)\pi(\mathcal{A}(\mathcal{O}))U^*(\Lambda) = \pi(\mathcal{A}(\Lambda\mathcal{O}))$ for a unitary representation U of the Poincaré group G,
- vacuum: there exists a vector $\Omega \in \mathcal{H}$ such that $U(\Lambda)\Omega = U(\Lambda) \quad \forall \Lambda \in G$.

Ω defines the *vacuum state* $\omega : a \mapsto \langle \Omega, a\Omega \rangle$. In the associated GNS representation (*the vacuum representation*) one defines

$$\mathcal{M}(\mathcal{O}) = \pi(\mathcal{A}(\mathcal{O}))''$$

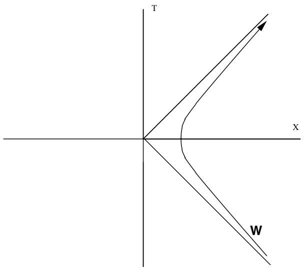
which is *the von Neumann algebra of local observables associated to* $\mathcal{O}$.

Wedge and Unruh temperature

Bisognano, Wichman, Sewell

$$W \longrightarrow \begin{cases} \text{algebra of observables } \mathcal{M}(W) \\ \text{vacuum modular group } \sigma_s^W \rightarrow \text{ boosts} \rightarrow \text{geometrical action} \end{cases}$$

uniformly accelerated observer's trajectory $\tau \in]-\infty, +\infty[$ = orbit of the modular group $s \in]-\infty, +\infty[$



$$\beta = |\frac{d\tau}{ds}| = |\frac{\tau}{s}| = \frac{2\pi}{a} = T_{\text{Unruh}}^{-1}.$$

- The temperature is constant along a given trajectory, and vanishes as $a \rightarrow 0$.

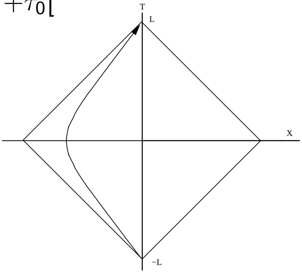
Double-cone in Minkowski space

Hislop, Longo; P.M., Rovelli

$$D \longrightarrow \begin{cases} \text{algebra of observables } \mathcal{M}(D) \\ \text{vacuum modular group } \sigma_s^D \end{cases}$$

$D = \varphi(W)$ for a some conformal map φ. For a Conformal Field Theory:

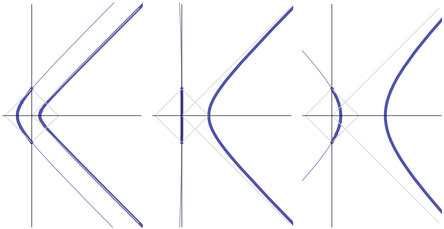
$$\begin{array}{c} \text{uniformly accelerated observer's trajectory} \\ \tau \in]-\tau_0, +\tau_0[\end{array} = \begin{array}{c} \text{orbit of the modular group} \\ s \in]-\infty, +\infty[\end{array}$$



$$\beta(\tau) = |\frac{d\tau}{ds}| = \frac{2\pi}{La^2}\left(\sqrt{1+a^2L^2} - \text{ch}\, a\tau\right).$$

- $T_D \doteq \frac{1}{\beta}$ is not constant along the orbit, and does not vanish for $a = 0$: $T_D(L)_{a=0} = \frac{\hbar}{\pi k_b L} \simeq \frac{10^{-11}}{L} K \rightarrow$ thermal effect for inertial observer.

Temperature, horizon, conformal factor



- Physical argument: for eternal observers, causal horizon $\Longleftrightarrow$ acceleration.
  For non-eternal observers, whatever a, there is a "life horizon"

$$D = \text{future(birth)} \bigcap \text{past(death)}.$$

- Mathematical argument: $\varphi : W \to D$ induces on W a metric $\tilde{g}$,

$$\tilde{g}(U, V) = g(\varphi_* U, \varphi_* V) = C^2 g(U, V).$$

  The double-cone temperature is proportional to the inverse of C,

$$\beta(x) = \frac{2\pi}{a'} C(\varphi^{-1}(x)).$$

  φ shrinks W to D, hence C cannot be infinite.

3. Double-cone in 2d boundary CFT

Longo, P. M., Rehren

Boundary CFT

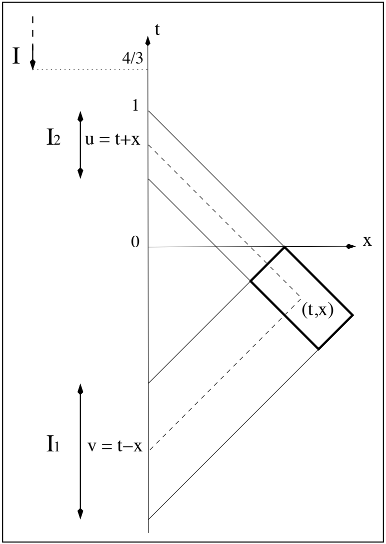
CFT on the half plane $(t, x > 0)$. Conservation of stress energy tensor T with zero-trace imply

$$\frac{1}{2}(T_{00} + T_{01}) = T_L(t + x),$$
$$\frac{1}{2}(T_{00} - T_{01}) = T_R(t - x).$$

Boundary condition (no energy flow across the boundary $x = 0$) implies

$$T_L = T_R = T.$$

T yields a chiral net of local v.Neumann algebras



$$\mathcal{I} = (A, B) \subset \mathbb{R} \mapsto \mathcal{A}(\mathcal{I}) := \{T(f), T(f)^* : \text{supp } f \subset \mathcal{I}\},$$

as well as a net of double-cone algebras

$$\mathcal{O} = I_1 \times I_2 \mapsto \mathcal{M}(\mathcal{O}) \doteq \mathcal{M}(I_1) \vee \mathcal{M}(I_2).$$

From the boundary to the circle

$\mathcal{A}$ extends to a chiral net over the intervals of the circle, via Cayley transform:

$$z = \frac{1+ix}{1-ix} \in S^1 \Longleftrightarrow x = \frac{(z-1)/i}{z+1} \in \mathbb{R} \cup \{\infty\}.$$

Square and square root:

$$z \mapsto z^2 \quad \Longleftrightarrow \quad x \mapsto \sigma(x) \doteq \frac{2x}{1-x^2},$$

$$z \mapsto \pm\sqrt{z} \quad \Longleftrightarrow \quad x \mapsto \rho_\pm(x) = \frac{\pm\sqrt{1+x^2}-1}{x}.$$

A pair of symmetric intervals:

$$I_1, I_2 \subset \mathbb{R} \text{ such that } \sigma(I_1) = \sigma(I_2) = I.$$

$$I_2 = (A, B) \Longrightarrow I_1 = (-\frac{1}{A}, -\frac{1}{B}).$$

Möbius covariance

In Minkowski space, the Poincaré group is both the covariance automorphism group and the group of invariance of the vacuum.

The net of algebra $\mathcal{A}(\mathcal{I})$ is covariant under an action of $\mathrm{Diff}(S^1)$. But the vacuum is only Möbius invariant where

$$\text{Möbius} = PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\left\{-1,1\right\}$$

acts on $\bar{\mathbb{R}}$ as

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \quad x \mapsto gx = \frac{ax+b}{cx+d}.$$

- Two equivalent points of view: S^1 or $\bar{\mathbb{R}}$; three important one-parameter subgroups of Möbius

$$R(\varphi) = \begin{pmatrix} \cos\frac{\varphi}{2} & \sin\frac{\varphi}{2} \\ -\sin\frac{\varphi}{2} & \cos\frac{\varphi}{2} \end{pmatrix},\ \delta(s) = \begin{pmatrix} e^{\frac{s}{2}} & 0 \\ 0 & e^{\frac{s}{2}} \end{pmatrix},\ \tau(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

acting as

$$R(\varphi)z = e^{i\varphi}z \text{ on } S^1, \quad \delta(s)x = e^s x \text{ on } \bar{\mathbb{R}}, \quad \tau(t)x = x+t \text{ on } \bar{\mathbb{R}}.$$

Modular group

Given a pair of symmetric intervals I_1, I_2 such that $I_1 \cap I_2 = \emptyset$. Consider the state

$$\varphi = (\varphi_1 \otimes \varphi_2) \circ \chi$$

where

$$\chi : \mathcal{A}(I_1) \vee \mathcal{A}(I_2) \to \mathcal{A}(I_1) \otimes \mathcal{A}(I_2) \text{ (split property)},$$
$$\varphi_k = \omega \circ \mathrm{Ad}\, U(\gamma_k) \text{ with } \omega \text{ the vacuum and } \gamma_k \text{ a diffeomorphism of } S^1$$
$$\text{such that } z \mapsto z^2 \text{ on } I_k.$$

The associated modular group has a geometrical action

$$(u, v) \in \mathcal{O} \;\mapsto\; (u_s, v_s) \in \mathcal{O} \qquad s \in \mathbb{R},$$

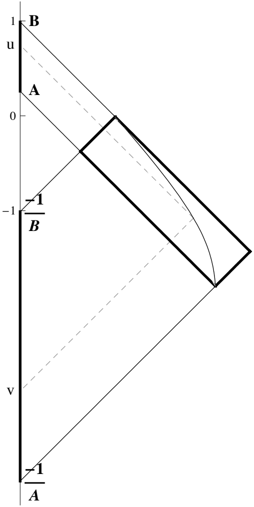
with orbits

$$\begin{aligned} u_s &= \rho_+ \circ m \circ \lambda_s \circ m^{-1} \circ \sigma(u) \in I_2, \\ v_s &= \rho_- \circ m \circ \lambda_s \circ m^{-1} \circ \sigma(v) \in I_1, \end{aligned}$$

where $\lambda_s(x) = e^s x$ is the dilation of $\mathbb{R}$, and m is a Möbius transformation which maps $\mathbb{R}_+$ to $I = \sigma(I_1) = \sigma(I_2)$.

Implicit equation of the orbits:

$$\frac{(u_s - A)(Au_s + 1)}{(u_s - B)(Bu_s + 1)} \cdot \frac{(v_s - B)(Bv_s + 1)}{(v_s - A)(Av_s + 1)} = \text{const},$$



- This equation only depends on the end points of $I_2 = (A, B),\ I_1 = (-\frac{1}{A}, -\frac{1}{B})$.
- All orbits are time-like, hence $\beta = |\frac{d\tau}{ds}|$ makes sense as a temperature.
- One and only one orbit is a boost (const $= 1$) and thus is the trajectory of a uniformly accelerated observer.

Explicit equation of the orbits:

$$I \in \mathbb{R}^+ \Longrightarrow I_2 = (A, B) \subset (0,1) \Longrightarrow A = \tanh \tfrac{\lambda_A}{2},\ B = \tanh \tfrac{\lambda_B}{2}.$$

$$u \in\ (A, B)\ = \tanh \frac{\lambda}{2} \quad \text{for } \lambda_A < \lambda < \lambda_B, \quad \sigma(u) = \sinh \lambda,$$

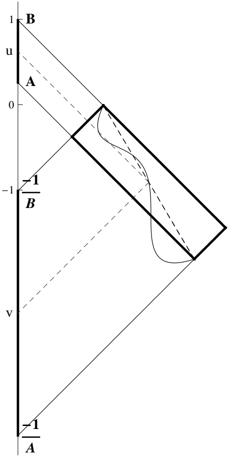
$$v \in\ (-\frac{1}{B}, -\frac{1}{A})\ = -\coth \frac{\lambda'}{2} \quad \text{for } \lambda_A < \lambda' < \lambda_B, \quad \sigma(v) = \sinh \lambda'.$$

$$u_s = \frac{\sqrt{(e^s k_a - k_b)^2 + (e^s k_{ab} - k_{ba})^2} - (e^s k_a - k_b)}{e^s k_{ab} - k_{ba}},$$

$$v_s = \frac{-\sqrt{(e^s k'_a - k'_b)^2 + (e^s k'_{ab} - k'_{ba})^2} - (e^s k'_a - k'_b)}{e^s k'_{ab} - k'_{ba}}$$

where $k_i \doteq \sinh \lambda - \sinh \lambda_i, k_{ij} \doteq k_i \sinh \lambda_j$.

- complicated dynamics (e. g. the sign of the acceleration may change).
- difficult to parametrize such a curve by its proper length τ, hence difficult to find the temperature $\frac{ds}{d\tau}$.



Temperature on the boost trajectory

Constant acceleration: $d\tau^2 = du\,dv$ hence

$$\beta = \frac{d\tau}{ds} = \sqrt{u'v'}$$

with $' = \frac{d}{ds}$. On the boost orbit, $v_s = -\frac{1}{u_s}$ hence

$$\beta = \frac{u'}{u} = \frac{d}{ds}\ln u_s \Longrightarrow \tau(s) = \ln u_s - \ln u_0 \Longrightarrow u_s = u_o e^{\tau(s)}.$$

Knowing

$$u_s' = f_{AB}(u_s) \doteq \frac{(u_s - A)(Au_s + 1)(B - u_s)(Bu_s + 1)}{(B - A)(1 + AB)\cdot(1 + u_s^2)}.$$

one finally gets

$$\beta(\tau) = \frac{f_{AB}(u_o e^{\tau})}{u_o e^{\tau}}.$$

Vacuum modular group for free Fermi fields

A pair of intervals $I_1 = (A_1, B_1)$, $I_2 = (A_2, B_2)$, with $x_1 = v \in I_1$, $x_2 = u \in I_2$. The action of the modular group σ_s of the vacuum, on monomials $\psi(x_i)$ is

$$\sqrt{\frac{dx_i}{d\zeta}}\,\sigma_s(\psi(x_i)) = \sum_{k=1,2} O_{ik}(s)\sqrt{\frac{dx_k}{d\zeta}}\,\psi(x_k(t)), \quad i = 1, 2,$$

where the geometrical action is

$$-\frac{x_i(\zeta) - A_1}{x_i(\zeta) - B_1} \cdot \frac{x_i(\zeta) - A_2}{x_i(\zeta) - B_2} = e^{\zeta}$$

with $\zeta(s) = \zeta_0 - 2\pi s$, and the "mixing" action is determined by the differential equation

$$\dot{O}(s) = K(s) O(s)$$

with

$$K_{ik}(s) = 2\pi \frac{\sqrt{\frac{dx_i}{d\zeta}}\sqrt{\frac{dx_k}{d\zeta}}}{x_i(s) - x_k(s)} \text{ for } i \neq k, \quad K_{ii}(s) = 0.$$

- The geometrical action is the same as the one in BCFT. The new feature is the mixing between the intervals.

Independant proof:

- because of the unicity of the KMS flow: enough to check that the vacuum is KMS with respect to σ_s.

- because the vacuum is quasi-free, enough to check on the 2-point functions, i.e. compute

$$\omega\left(\sigma_t(\psi(x_i))\sigma_s(\psi(y_j))\right)$$

using the propagator $\omega(\psi(x)\psi(y)) = \frac{-i}{x-y-i\epsilon}$.

One finds

$$\omega(\psi(x_i)\sigma_{-\frac{i}{2}}(\psi(y_j))) = \omega(\psi(y_j)\sigma_{-\frac{i}{2}}(\psi(x_i)))$$

Conclusion

BCFT: non-vacuum modular action on disjoint intervals is purely geometric,

free Fermi field: vacuum-modular action on disjoint intervals is a combination of the geometrical action of BCFT and some "mixing terms".

Connes cocycle between the vacuum and Longo's ad-hoc state is purely non-geometric.

One of the first examples in which there is an explicit control on the non-geometric part of the modular action.

Hint for modular action in double-cones for non-conformal theories (e.g. massive ones) ?