Hungarian matching algorithm, tropical determinants and Jacobi's bound for differential systems

François Ollivier (CNRS), LIX, École polytechnique

Carl Gustav Jacob (Jacques Simon) Jacobi (1804-1851)

1812-1822 Edict of emancipation
Professor at Königsberg 1827-1843
1841 Bankrupt
Travel in Italy 1843-1844
1844- Berlin, Preußischen Akademie der Wissenschaften
1848 Revolution

Differential equations
Algebraic equations
Number theory
Tropical determinant, Graph theory and shortest paths (1836-1845)

From Königsberg to Berlin via Roma

Moritz Hermann (Moses) Jacobi (1801-1874) Борис Семёнович Якоби

1835 Professor of architecture, Dorpat (Tartu) university

1837- Saint Petersburg Russian Academy of Sciences

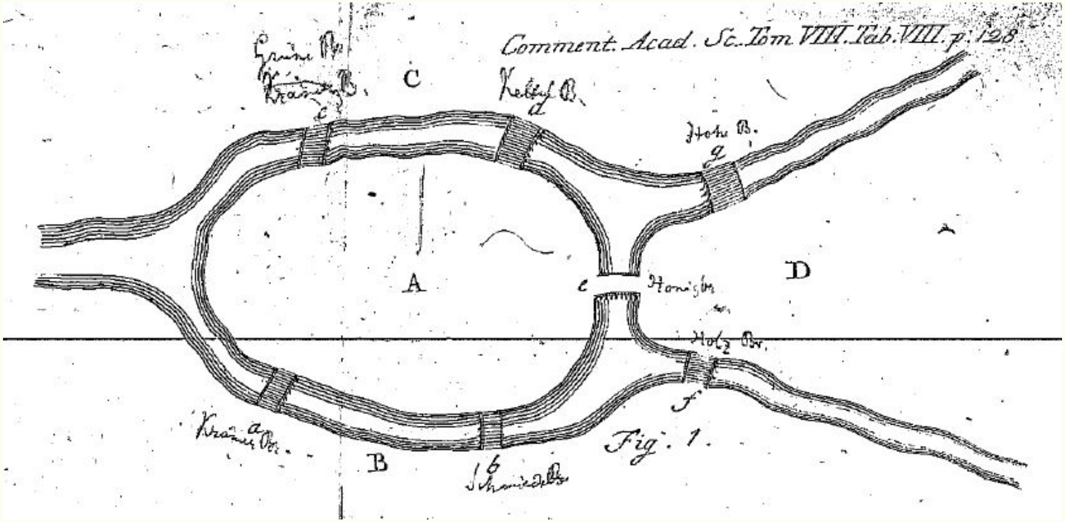
Electrotyping (galvanoplasty)

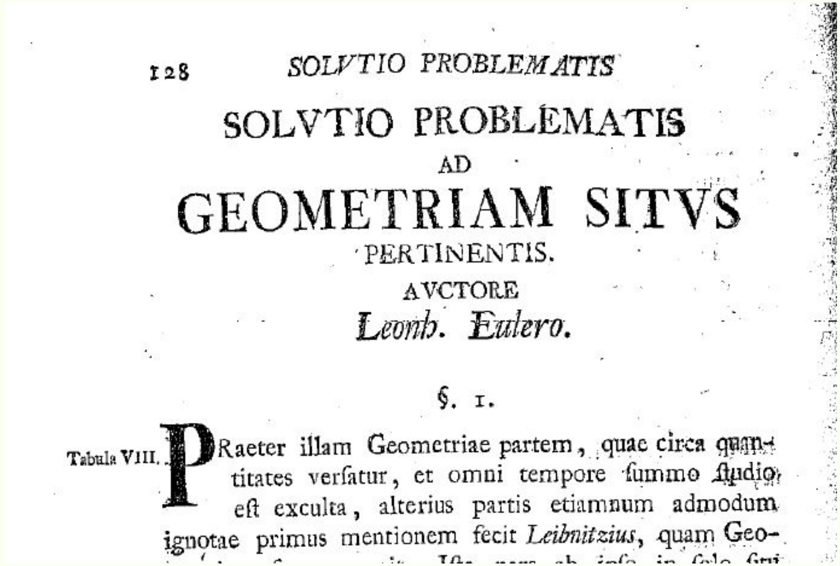
Telegraphy (1842-1845 Tsarskoe Selo)

Electric engine. 1839 Electroboat on the Neva

Maximum power theorem

The bridges of Königsberg





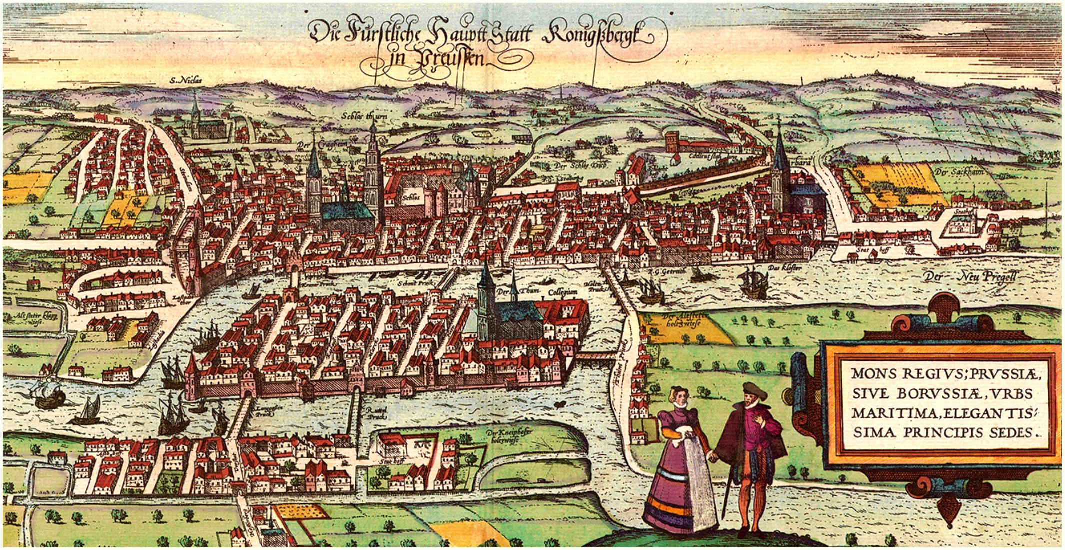
128 *SOLVTIO PROBLEMATIS*

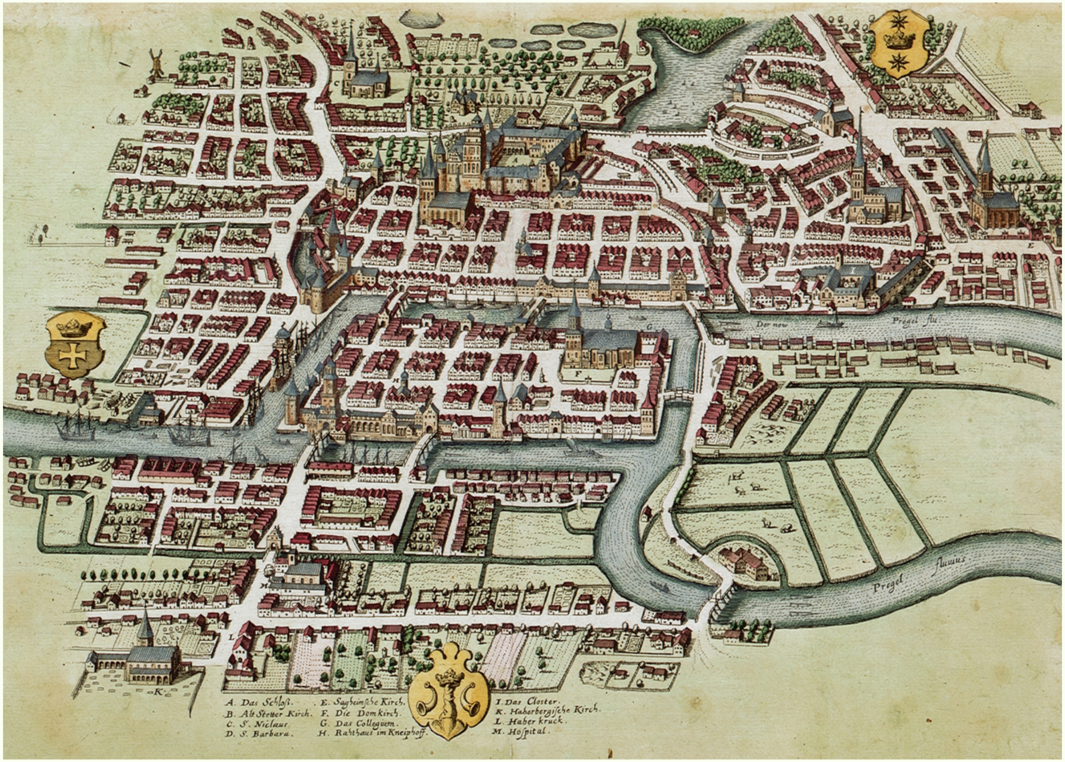
SOLVTIO PROBLEMATIS
AD
GEOMETRIAM SITVS
PERTINENTIS.

AVCTORE
Leonh. Eulero.

§. 1.

Tabula VIII. Praeter illam Geometriae partem, quae circa quantitates verſatur, et omni tempore ſummo ſtudio eſt exculta, alterius partis etiamnum admodum ignotae primus mentionem fecit *Leibnitzius*, quam Geo-


Die Fürstliche Hauptt Statt Konigßbergk in Preussen
MONS REGIVS; PRVSSIÆ, SIVE BORVSSIÆ, VRBS MARITIMA, ELEGANTISSIMA PRINCIPIS SEDES.
Der Neu Pregell


A. Das Schloß.
B. Alt Stetter Kirch.
C. S. Niclaus.
D. S. Barbara.
E. Sagheimsche Kirch.
F. Die Domkirch.
G. Das Collegium.
H. Rahthaus im Kneiphoff.
I. Das Closter.
K. Haberbergische Kirch.
L. Haber kruck.
M. Hospital.
Pregel fluvius

Königsberg (Regiomons) (1255-1945) Albertina University (1544)

Kaliningrad (Kalininopolis)

Differential equations. 1836-1845

Last multiplier. Analog of Euler's multiplier.
Isoperimeric equations: $\int U(x(t))\mathrm{d}t$ must be minimal or maximal; $\mathrm{ord}_{x_j} P_i = e_i + e_j$, where $e_i = \mathrm{ord}_{x_i} U$.
The order of the system is $2\sum_{i=1}^n e_i$
if $|\partial P_i/\partial x_j^{(e_i+e_j)}| = |\partial^2 U/\partial x_i \partial x_j| \neq 0$.

General case: $a_{i,j} := \mathrm{ord}_{x_j} P_i \leq \mu_i + \nu_j$.
The order is at most $\max_{\sigma\in S_n} \sum_{i=1}^n a_{i,\sigma(i)} \left(= \sum_{i=1}^n (\mu_i + \nu_j)\right)$,
with equality if $\nabla := |\partial P_i/\partial x_j^{(\mu_i+\nu_j)}| \neq 0$.
In the generic case, one needs to differentiate P_i at least $\ell_i = (\max_{k=1}^n \mu_k) - \mu_i$ to compute a normal form.
At most, if $\nabla \neq 0$.

Posterity

Nanson 1876 and Jordan 1883. Heuristic approach for computing a resolvent under foggy genericity hypotheses.

Chrystal 1895. Linear case with constant coefficient.

Ritt 1935, 1950 General linear case and $n \leq 2$.

Volevich 1960. Linear case (differential operators).

Kondratieva, Mikhalev and Pankratiev 1982. (Johnson's regularity hypothesis)

F.O. Sadik 2006. Diffieties. ∇.

Shaleninov 1990, Pryce 2001 Shortest reduction.

Tropical geometry

Max/min algebras. Replace $\times$ by $+$ and $+/-$ by max or min.

Named in honor of the Brasilian (Hungarian-born) mathematician Imre Simon (1943–2009).

Jacobi's bound $\mathcal{O} := \max_{\sigma \in S_n} \sum_{i=1}^{n} a_{i,\sigma(i)}$ is a tropical determinant.

Some history

Frobenius 1912. Marriage problem.

Denes Kőnig; Jenő Egerváry 1931. Weighted marriage problem.

Assignment problem, *ab* 1944.

Von Neumann claims that a polynomial algorithm exists.

Kuhn 1955. Hungarian method.

Cohn 1983. Rediscovers Jacobi's contribution to the assignment problem. Remains unnoticed.

F.O. Translation of Jacobi's papers *ab* january 2003.

Contacts with Kuhn at the end of 2005.

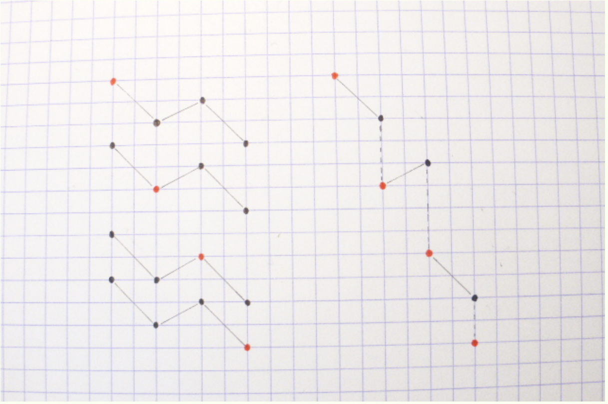
Assignment problem

Assuming that the productivity of worker i at task j is $a_{i,j}$, find an assignment σ of the n workers to the n tasks so that the total productivity $\mathcal{O} := \max_{\sigma \in S_n} \sum_{i=1}^{n} a_{i,\sigma(i)}$ is maximal.

First mentionned for optimizing the reassignment of RAF soldiers during WWII.

"[he] said that from the point of view of a mathematician there was no problem. Since the number of permutations was finite, one had only to try them all and chose the best. [...] This is really cold comfort for the psychologist, however, when one considers that only ten men and ten jobs mean over three and a half million of permutations."

Computation of the tropical determinant



Key of Jacobi's algorithm: computation of a canon. Replace $a_{i,j}$ by $a_{i,j}+\ell_i$, so that $a_{i,\sigma(i)}+\ell_i$ is maximal in its column. Easy if $a_{i,j} = \mu_i + \nu_j$: $\ell_i = (\max_{k=1}^{n} \mu_k) - \mu_i$.

We assume dense representation, with n rows and n columns.

Paths

Preparation process: increase each row so that it contains a maximum (in its column). Then, there are at least 2 transversal maxima (all maxima in the union of 1 line and 1 column). At each step of the algorithm, assume that we have r transversal maxima (with $*$), that we may assume to be $a_{i,i}$, $1 \leq i \leq r$, up to a permutation. (Starred maxima).

$$
\begin{array}{rc}
 & \begin{array}{cc} \overbrace{\hphantom{* \quad \ddots \quad *}}^{\textit{left}} & \overbrace{\hphantom{* \quad \ddots \quad *}}^{\textit{right}} \end{array} \\
\begin{array}{r} \textit{upper}\left\{\vphantom{\begin{array}{c} * \\ \ddots \\ * \end{array}}\right. \\ \textit{lower}\left\{\vphantom{\begin{array}{c} \\ \\ \\ \end{array}}\right. \end{array} &
\left(\begin{array}{ccc|ccc}
* & & & & & \\
 & \ddots & & & & \\
 & & * & & & \\
\hline
 & & & & & \\
 & & & & & \\
 & & & & &
\end{array}\right)
\end{array}
$$

We divide the rows in upper and lower rows, the columns in left and right.

We then define the path relation. There is a path from row $1 \leq i \leq r$ to row $i^{'}$ if $a_{i',i} = a_{i,i}$ (row $i^{'}$ contains a maximum in the same column as the starred maximum of row i).

First class rows are rows that contain a right maximum, or to which there is a path from a first class row.

Third class rows are lower rows or rows from which there is a path to a third class row.

Second class rows are the remaining rows.

Increasing path. Kőnig step

If there is a lower first class row, then the set of transversal maxima can be increased. Add to the list of starred maxima the corresponding lower left maximum. In the chain from an upper row with a right maximum to a lower row, replace the starred elements by the elements in red.

$$
\left(\begin{array}{ccccc|c}
a^*_{i_0,i_0} & \cdots & & & & \mathbf{a}_{\mathbf{i_0},\boldsymbol{\beta}} \\
\shortparallel & & & & & \\
\mathbf{a}_{\mathbf{i_1},\mathbf{i_0}} & \cdots & a^*_{i_1,i_1} & & & \\
 & & \shortparallel & & & \\
 & & \mathbf{a}_{\mathbf{i_2},\mathbf{i_1}} & \ddots & & \\
 & & & \vdots & & \\
 & & & \ddots & a^*_{i_{p-1},i_{p-1}} & \\
\hline
 & & & & \shortparallel & \\
 & & & & \mathbf{a}_{\mathbf{i_p},\mathbf{i_{p-1}}} & \\
\end{array}\right)
$$

Creating increasing paths. Egerváry step

If there is no lower first class row, increase the third class row by the minimal quantity so that (at least) one of them become a first class or a second class row become a third class row.

Some lower third class row must become a first class row at some point. Then, there is a increasing path.

Jacobi *vs* Hungarians

Egerváry. If μ_i, ν_j is not a minimal cover, then the set of $a_{i,j}$ such that $a_{i,j} = \mu_i + \nu_j$ belong to p rows (say rows $1, \dots, p$) and $m - p$ columns, with $m < n$ (say columns $1, \dots, m - p$).

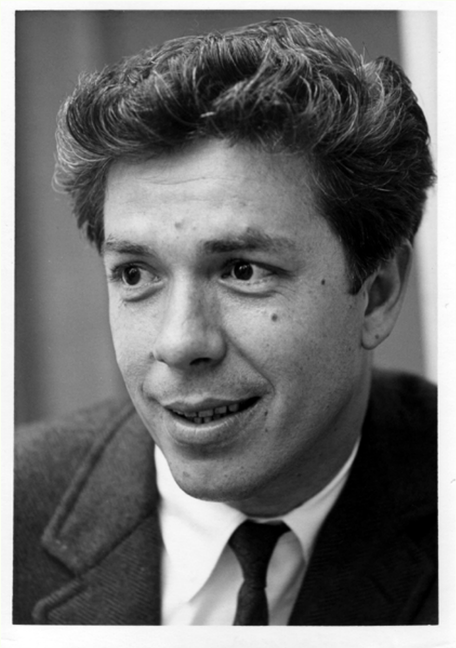
Let $e := \min_{i=p+1}^{n} \min_{j=m-p+1}^{n} \mu_i + \nu_j - a_{i,j}$; $e > 0$.

Let $\mu_i^{'} = \mu_i$ if $1 \leq i \leq p$, $\mu_i^{'} = \mu_i - e$ if $p < i \leq n$, $\nu_j^{'} = \nu_j + e$ if $1 \leq j \leq m - p$, $\nu_j^{'} = \nu_j$ if $m - p < j \leq n$, then $\sum_{i=1}^{n}(\mu_i^{'} + \nu_i^{'}) = \sum_{i=1}^{n}(\mu_i + \nu_i) - (n - m)e$.

Exponential complexity with integer coefficients, can never stops with coefficients in $\mathbf{Q}(\sqrt{5})$ (Jüttner 2004).

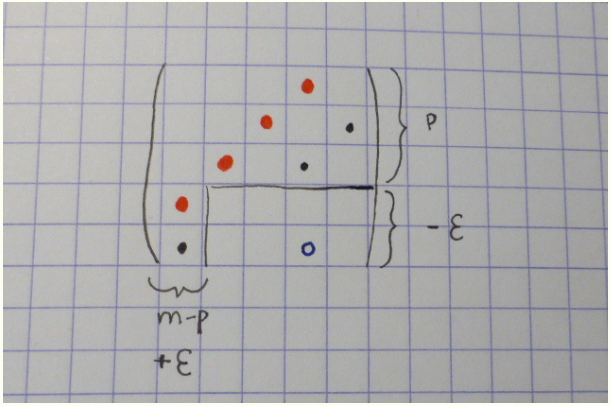
Kuhn. Basically equivalent to Jacobi's algorithm when the set of $n - p$ rows for which μ_i is decreased is chosen to be minimal. They turn to be third class rows.

Kuhn's "Hungarian Method" 1955.



Harold W. Kuhn (1925-2014)

Builds special types of minimal covers, so that the set of rows with decreased μ_i is minimal. Allows transpositions.



Jenő Egerváry's "covers"

Jenő Egerváry (1891-1958)

Cover: μ, ν such that $a_{i,j} \leq \mu_i + \nu_j$.

Theorem. — *The cover μ, ν is minimal ($\sum_{i=1}^{n} \mu_i + \nu_i$ minimal) iff there exists a permutation σ_0 such that $a_{i,\sigma_0(i)} = \mu_i + \nu_{\sigma_0(i)}$.*
Then, $\max_{\sigma \in S_n} \sum_{i=1}^{n} a_{i,\sigma(i)} = \sum_{i=1}^{n} a_{i,\sigma_0(i)} = \sum_{i=1}^{n} (\mu_i + \nu_i)$.

Kőnig's theorem

Dénes Kőnig (1884-1944)

Analysis situs, Graph Theory.

Kőnig theorem. — *If for any permutation σ there are at most $m < n$ indices i such that $a_{i,\sigma(i)} = \mu_i + \nu_{\sigma(i)}$, then the (i, j) with $a_{i,j} = \mu_i + \nu_j$ belong to p rows and $m - p$ columns.*

Some warnings about asymptotic complexity

Depends on the computational model and the size of data.

Asymptotic complexity $\neq$ practical complexity.

During the XIX$^{\text{th}}$ century, computational tools are pen and paper.

So, all considerations on the complexity of Jacobi's algorithms are fully anachronic.

Complexity

Munkres $O(n^4)$ 1957.

Tomizawa 71, Edmonds and Karp 1972 $O(n^3)$.

For the marriage problem (a matrix of 0 and 1), one may use many independent increasing paths without recomputing the path relation: this gives a $O(n^{5/2})$ algorithm. Hopcroft and Karp 1973.

For integer matrices, one may use Hopcroft and Karp to recursiveley produce an approximated result. This gives a $O(\ln(Cn)n^{5/2})$ algorithm, where $C := \max a_{i,j}$. Gabow and Tarjan 1989.

One may obtain the exponent ω of matrix multiplication with probabilistic algorithms (Sankowski 2009) and near-linear complexity with $(1-\epsilon)$-approximate algorithms (Duan Petie 2010).

Complexity analysis

Building path relations and classes: $O(n^2)$

Repeated $O(n)$ times to create an augmenting path, due to possible second class rows

$O(n^4)$ Munkres.

When a second class row goes to third class, rebuilding path relations and classes is faster *i.e.* $O(n)$, leading to a $O(n^3)$ total complexity. (Equivalent to Tomizawa, Edmonds and Karp)

Hopcroft and Karp's algorithm

The main idea is to build a maximal set (in the sense that it is not strictly included in another such set) of disjoint paths of minimal length going to a lower first class row, before building a new path relation.

The main step of the algorithm is not to produce a single augmenting path, but, at each stage k, a maximal set of disjoint paths of the same minimal length γ_k.

Theorem. — *The sequence γ_k is strictly increasing.*

Elementary path relations at stage $k-1$ are indicated by $\vdots$ and at stage k by $|$; after rearrangement by $[$. The starred ones of stage $k-1$ by $\mathit{1}$ and those of stage k by $\mathbf{1}$.

$$
\begin{array}{c}
\left(\begin{array}{cccccc}
\mathit{1} & \cdots & & \cdots & & \cdots \\
\vdots & & & & & \\
\vdots & & & & \mathbf{1} & \cdots \\
\vdots & & & & | & \\
\mathbf{1} & \cdots & \mathit{1} & \cdots & 1 & \cdots \\
| & & \vdots & & & \\
| & & 1 & \cdots & & \cdots \\
| & & & & & \\
1 & \cdots & & \cdots & & \cdots
\end{array}\right)
\end{array}
\begin{array}{l}
\mathit{row}\ \phi^{-1}(i_{h_k}) \\
\\
\mathit{row}\ i_{h_k-1} \\
\\
\mathit{row}\ i_{h_k} \\
\\
\mathit{row}\ \phi(i_{h_k}) \\
\\
\mathit{row}\ i_{h_k+1}
\end{array}
$$

$$
\begin{pmatrix}
0 & \mathit{1} & \mathbf{1} & 0 \\
\mathit{1} & \mathbf{1} & 0 & 0 \\
\mathbf{1} & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{pmatrix}
$$

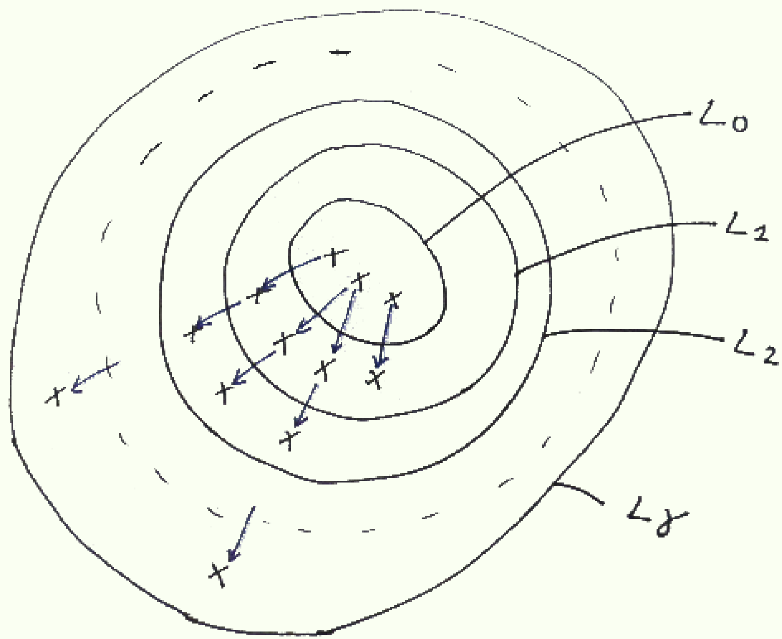
$(1_{1,3} - 1_{1,2} + 1_{2,2} - 1_{2,1} + 1_{3,1}) + (1_{3,4} - 1_{3,1} + 1_{4,1}) = (1_{3,4}) + (1_{1,3} - 1_{1,2} + 1_{2,2} - 1_{2,1} + 1_{4,1})$

Hopcroft–Karp. — **Input data:** a matrix A of zeros and ones.
Output: the elements of a maximal transversal sum of A and a minimal cover.

Step 1) As stated above, we repeatedly look for lower right ones, producing first at least $\lceil \mathcal{O}/2 \rceil$ transversal ones T.

Step 2) Let $(\gamma, L) := \mathit{Length}(A, T)$. If $\gamma =$ "*failed*", then return T.
If not, let $F := [1, s] \setminus L_0$ and **for** $i_0 \in L_0$ **do**: If $J := \mathit{Path}(\gamma, [i_0]) \neq$ "*failed*" then $T := \mathit{Increase}(T, J)$.
Repeat *Step 2)*.

Lengths and paths



Complexity: $O(s^2)$.

Jacobi and shortest paths

Jacobi gave two extra algoritms in order to compute the minimal canon in the case when:

A canon is known, but it is not minimal. Let $A + \ell$ be a canon for A; assume that the $a_{i,i}$ form a maximal transversal sum. Then, we define a weighted directed graph G on the set $\{0, 1, \dots, n\}$, by associating the weight $w_{i,j} := a_{i,i} + \ell_i - a_{j,i} - \ell_j \geq 0$ to the ordered pair (i, j), and $w_{i,0} := \ell_i$ to the ordered pair $(i, 0)$.
Reciprocally, we may associate to any such directed graph with positive weight a square matrix A and a canon $A + \ell$, defined by $a_{i,i} = C$, for $C \geq 2 \max_{(i,j)\in[0,n]^2} w_{i,j}$, $\ell_i = w_{i,0}$ and $a_{j,i} := C - w_{i,j} + \ell_i - \ell_j$.

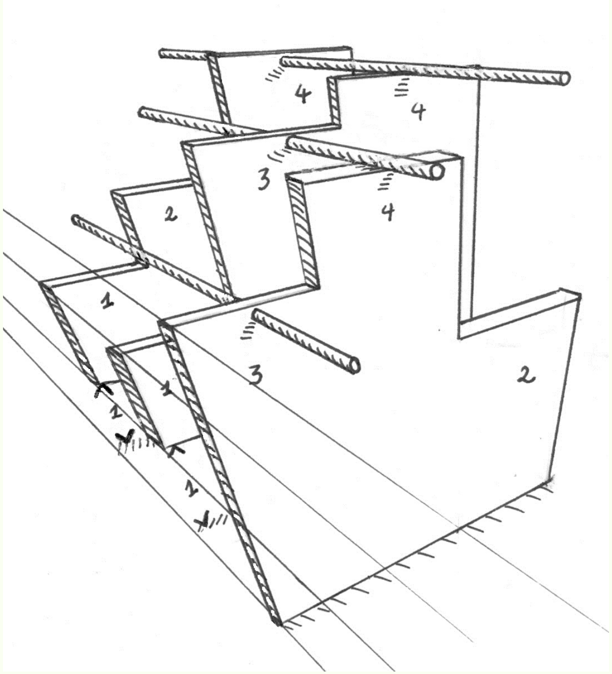
Equivalent to Dijkstra's algorithm.

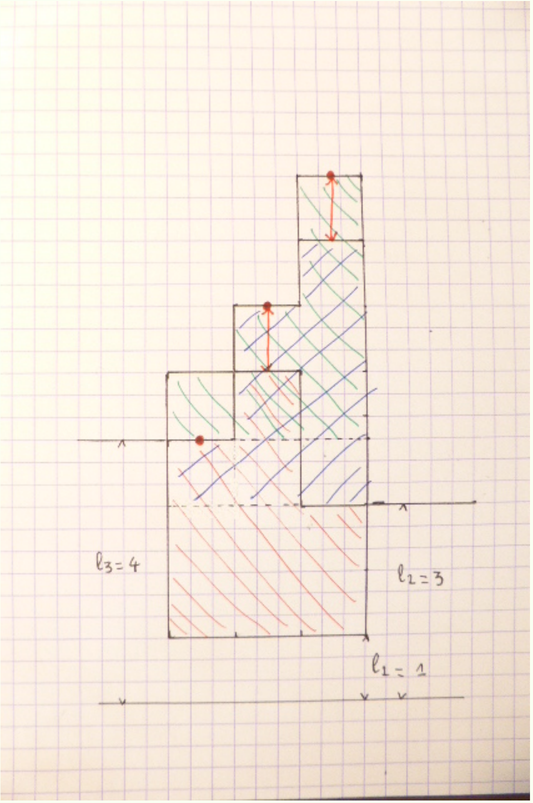
Dijkstra (1959) algorithm

Edsger Wybe Dijkstra (1930-202)

Repeatedly decrease all rows from which there is not a path to a row with $\ell_i = 0$. Equivalent to computing shortest paths when all paths are non-negative. $O(n^3)$, can be turned to $O(\ln(n)n^2)$ (D. Johnson 1977) or even $O(n^2)$ (Fredman and Tarjan 1987).

Mechanical solution



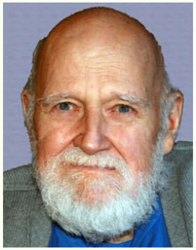


The elements of a maximal sum $\sum_{i=1}^{n} a_{i,\sigma(i)}$ are known. This is equivalent to a shortest path problem. With $\sigma = \mathrm{Id}$ Define an oriented weighted graph on the set of vertices $\{0, 1, \dots, n\}$ by setting $w_{i,0} := 0$ on edge $(i, 0)$ and $w_{i,j} := a_{i,i} - a_{j,i}$ on edge (i, j). Reciprocally, define for any such weighted graph a matrix A with $a_{i,i} := C := \max(0, \max_{i,j} w_{i,j})$ and $a_{j,i} := C - w_{i,j}$.

Equivalent to Bellman – Ford algorithm.

Bellman (1956) – Ford (1958) algorithm

Richard Bellman (1920-1984)



Lester Randolph Ford junior (1927-2017)

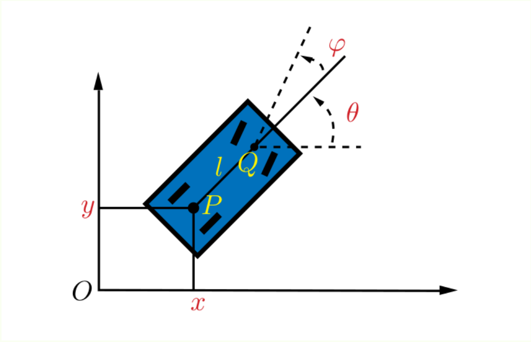
Until a canon i reached, repeat:
For $i = 1$ to $i = n$, increase row i of $\max_{j=1}^{n} a_{j,i} - a_{i,i}$.

Equivalent to computing shortest paths when some paths may be negative. The sum is maximal iff there is no negative cycle. Then, the main loop is iterated at most $n - 1$ times. $O(n^3)$.

Application to flat systems

Flat systems are differential systems such that their general solution can be parametrized by the arbitrary choice of m functions, the *flat* outputs, where m is the number of controls (differential dimension).

The car example.



$\theta = \arctan \frac{y'}{x'}.$

Block triangular systems

Kaminski and F.O. 2022 Generalization of Gstöttner *et al.* 2022.

One may test in polynomial time if a system is block triangular.

There exists a subset of variables with Jacobi's bound 0. If the system determinant ∇ does not vanish, the system is flat.

A simplified aircraft model enters in this category.



A question

Let $s < n$. Is there a polynomial time algorithm to compute

$$\min_{R\subset[1,n]\ \sharp R=s}\ \max_{\sigma\in \mathrm{Bij}([1,s],R)} \sum_{i=1}^{s} a_{i,\sigma(i)}?$$

To test if it is 0?

THANKS!