THE BRICK POLYTOPE

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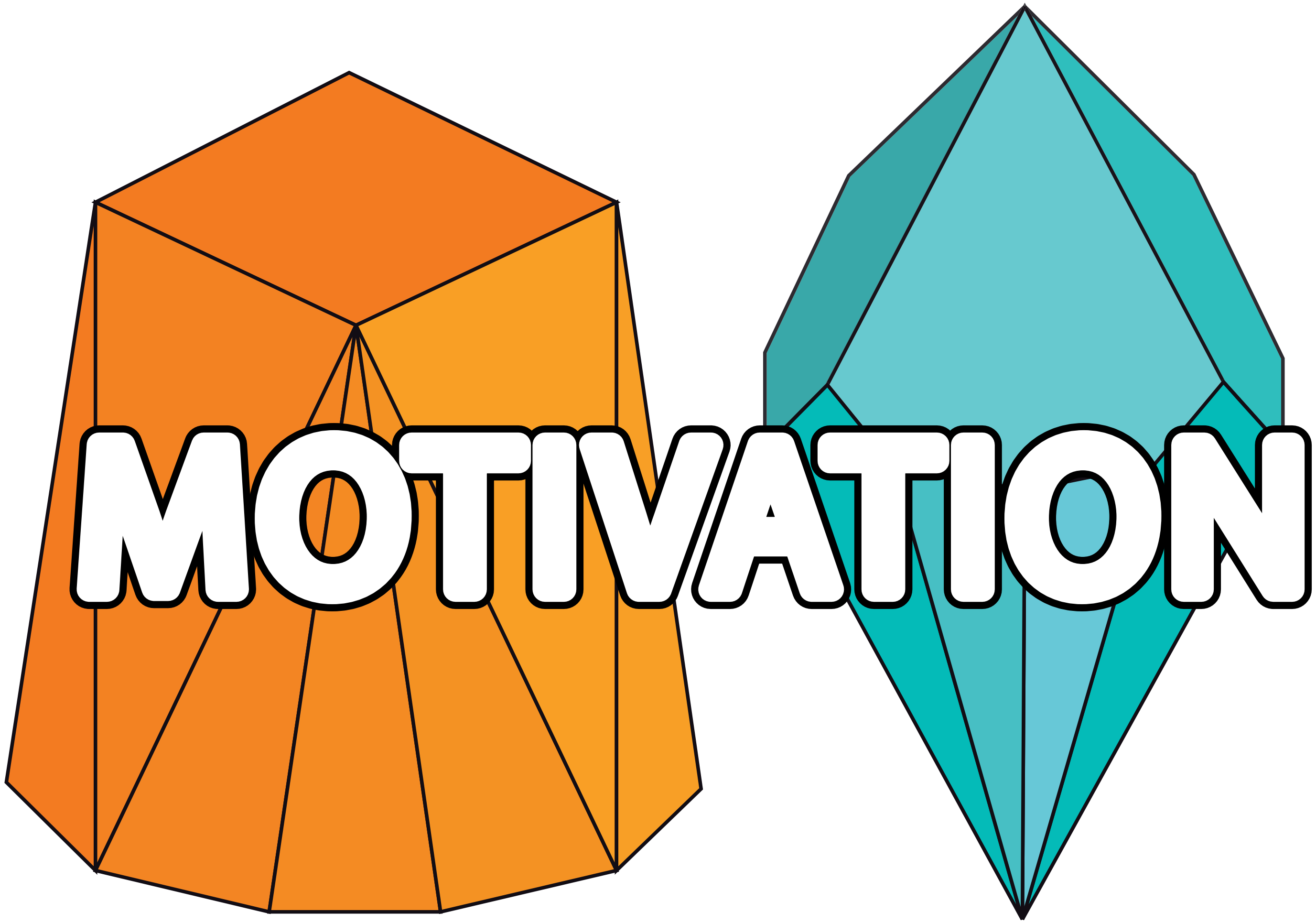

MOTIVATION

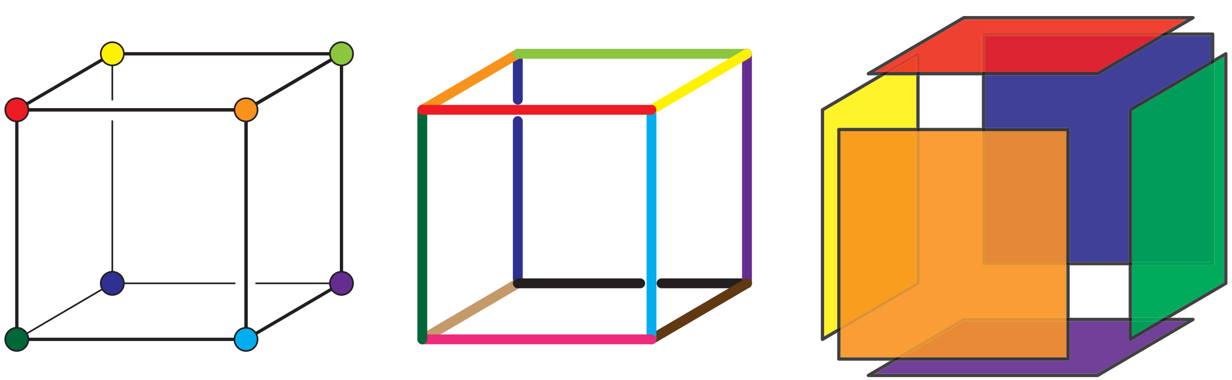
POLYTOPES WITH PRESCRIBED COMBINATORICS

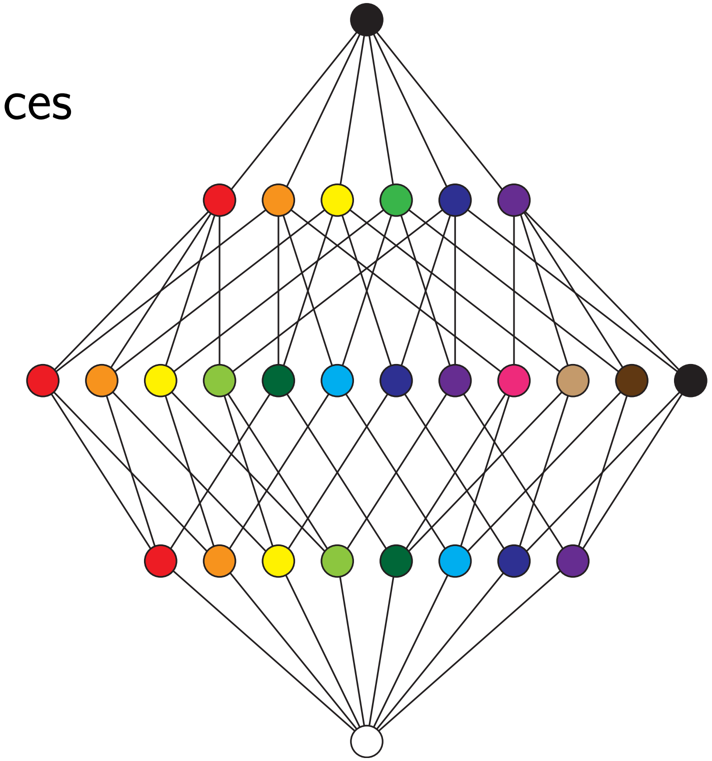
polytope = convex hull of a finite set of $\mathbb{R}^d$

= bounded intersection of finitely many half-spaces

face = intersection with a supporting hyperplane

face lattice = all the faces with their inclusion relations





Given a set of points, determine the face lattice of its convex hull.

Given a lattice, is there a polytope which realizes it?

POLYTOPES OF DIMENSION ≥ 4

Polytopes of dimension 3 $\longleftrightarrow$ planar 3-connected graphs

Various open conjectures in dimension 4:

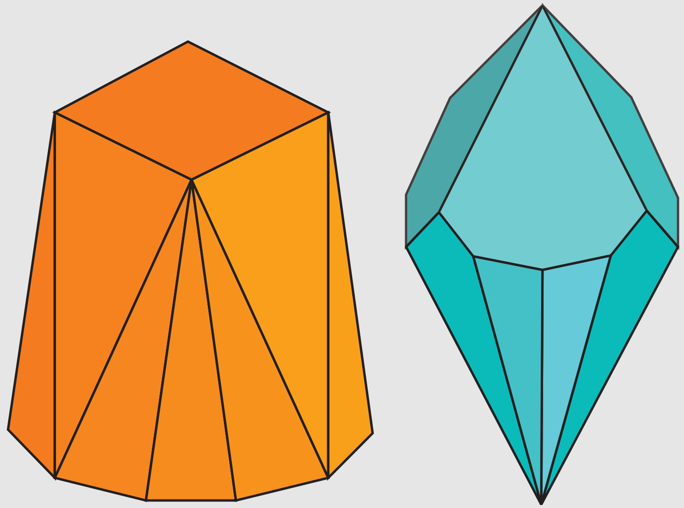
Hirsch conjecture

diameter $\leq$ #facets – dimension (Santos)

complexity of the simplex algorithm

3^d Conjecture (Kalai)

f-vecteur shape (Barany, Ziegler)



Prismatoïdes

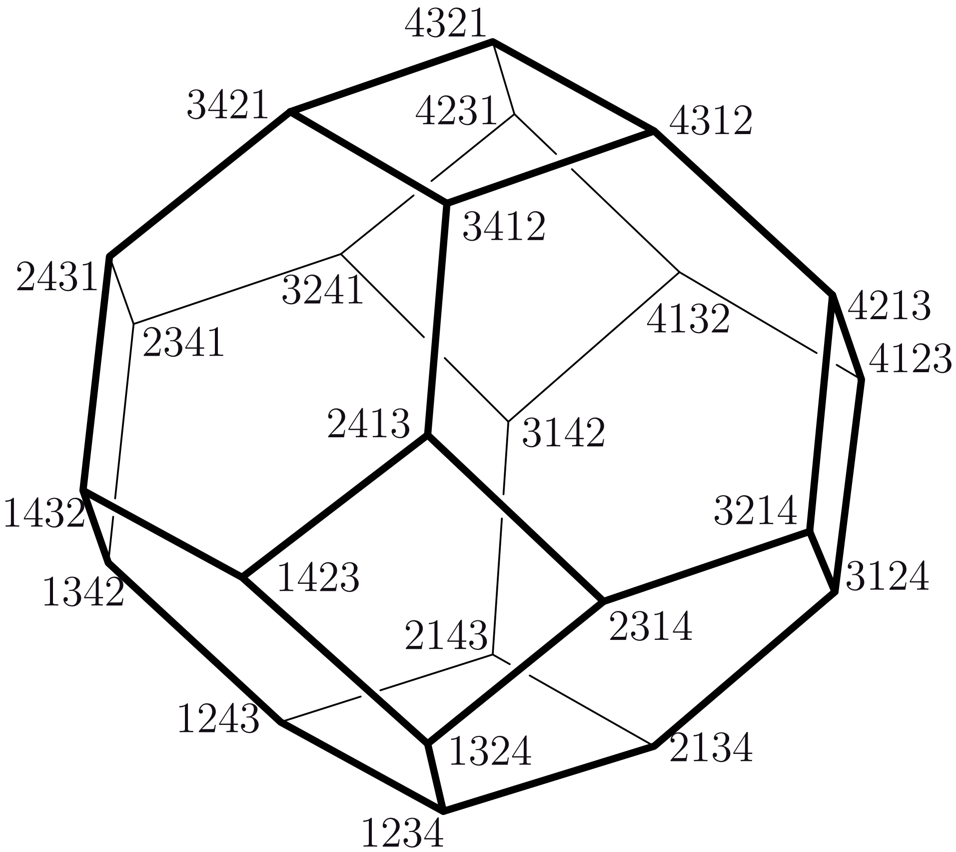
"Our main limits in understanding the combinatorial structure of polytopes still lie in our ability to raise the good questions and in the lack of examples, methods of constructing them, and means of classifying them."

Kalai. Handbook of Discrete and Computational Geometry (2004)

PERMUTAHEDRON

$\Pi_n = \text{conv}\left\{(\sigma(1), \ldots, \sigma(n))^T \;\middle|\; \sigma \in \mathfrak{S}_n\right\}$

$\partial\Pi_n =$ refinement poset on ordered partitions of $[n]$

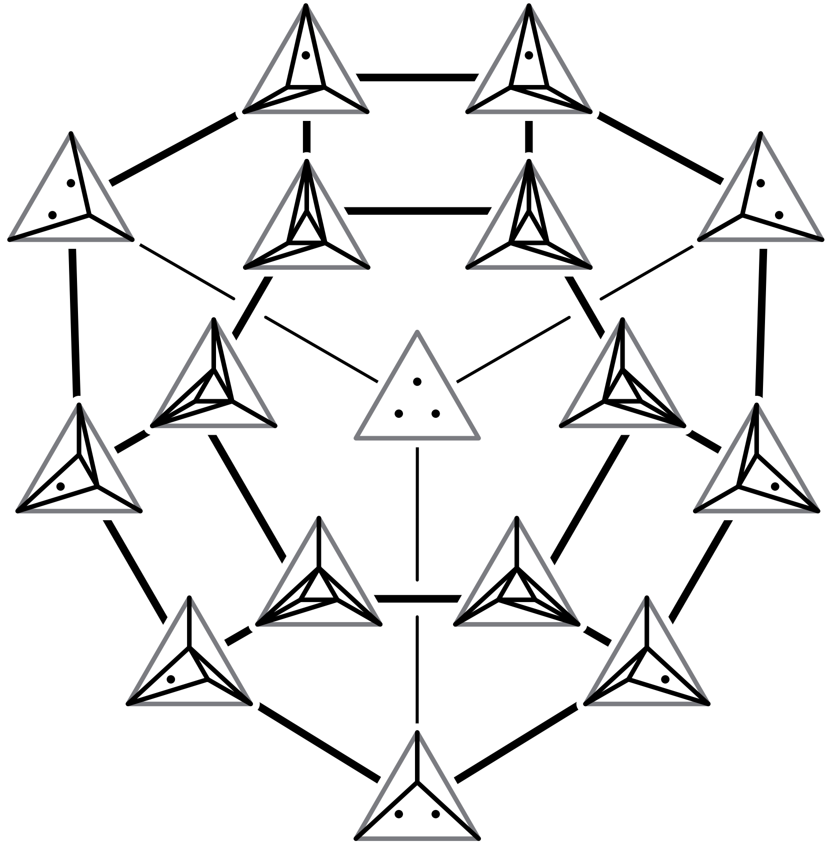


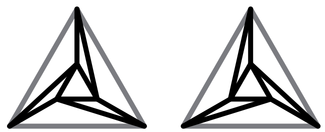
$\Pi_n = \sum_{i<j}[e_i, e_j]$ (Minkowski sum)

SECONDARY POLYTOPE

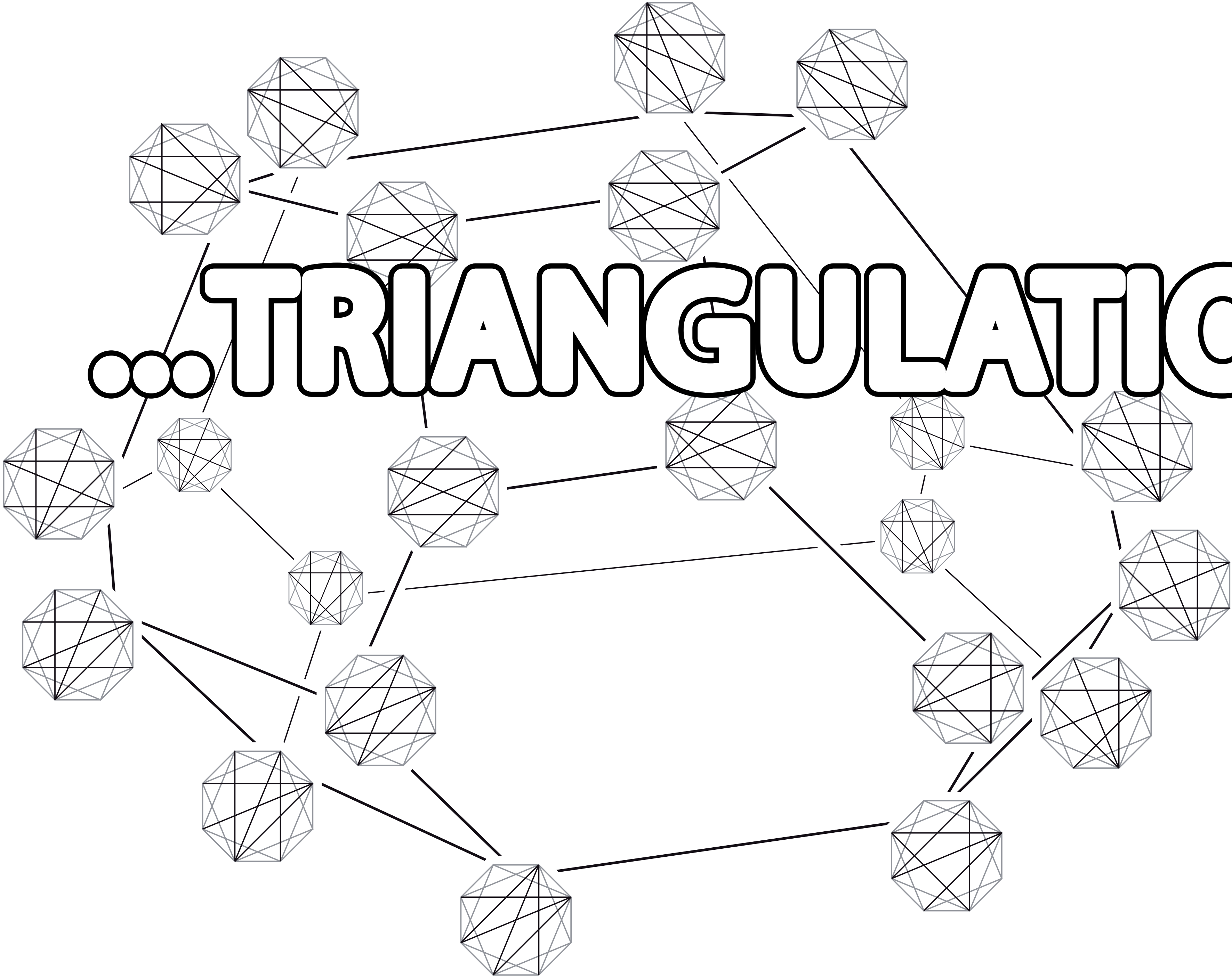
$\Sigma(P) = \text{conv}\left\{\sum_{p\in P} \text{vol}(T,p)e_p \;\middle|\; T \text{ triang. } P\right\}$

$\partial\Sigma(P) =$ refinement poset on regular polyhedral subdivisions of P

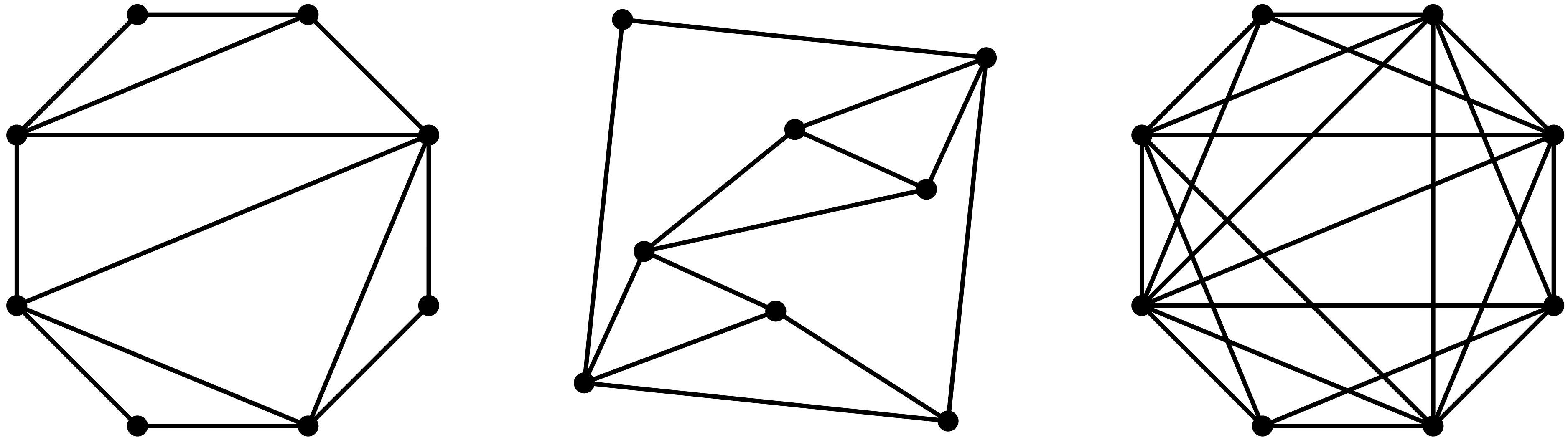


Triangulations  are non-regular

...TRIANGULATIONS



THREE GEOMETRIC STRUCTURES

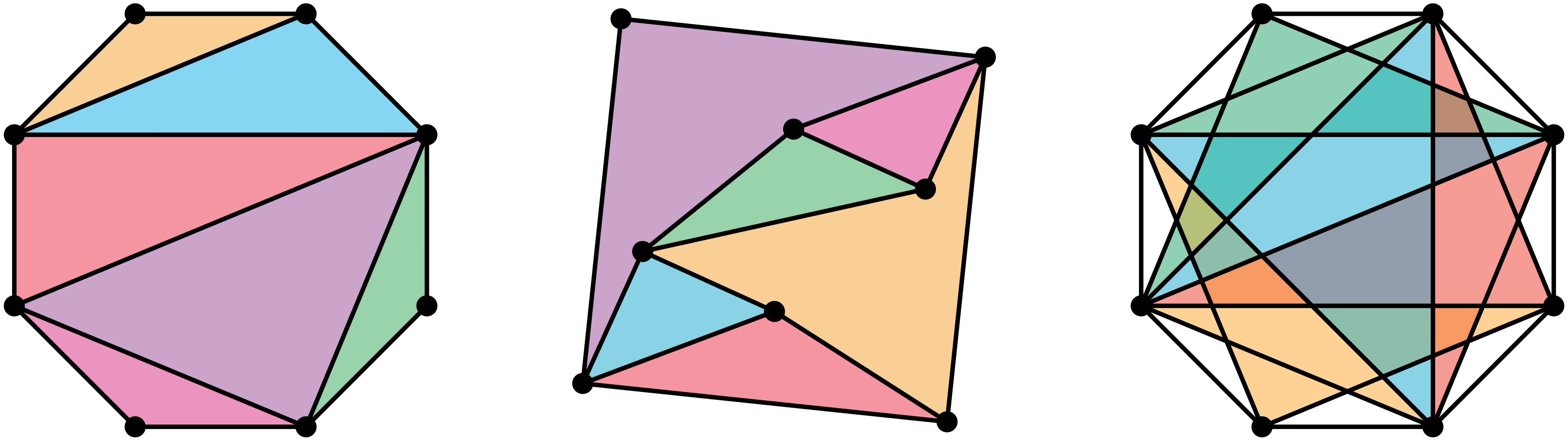


triangulation $=$ maximal crossing-free set of edges

pseudotriangulation $=$ maximal crossing-free pointed set of edges

k-triangulation $=$ maximal $(k+1)$-crossing-free set of edges

THREE GEOMETRIC STRUCTURES



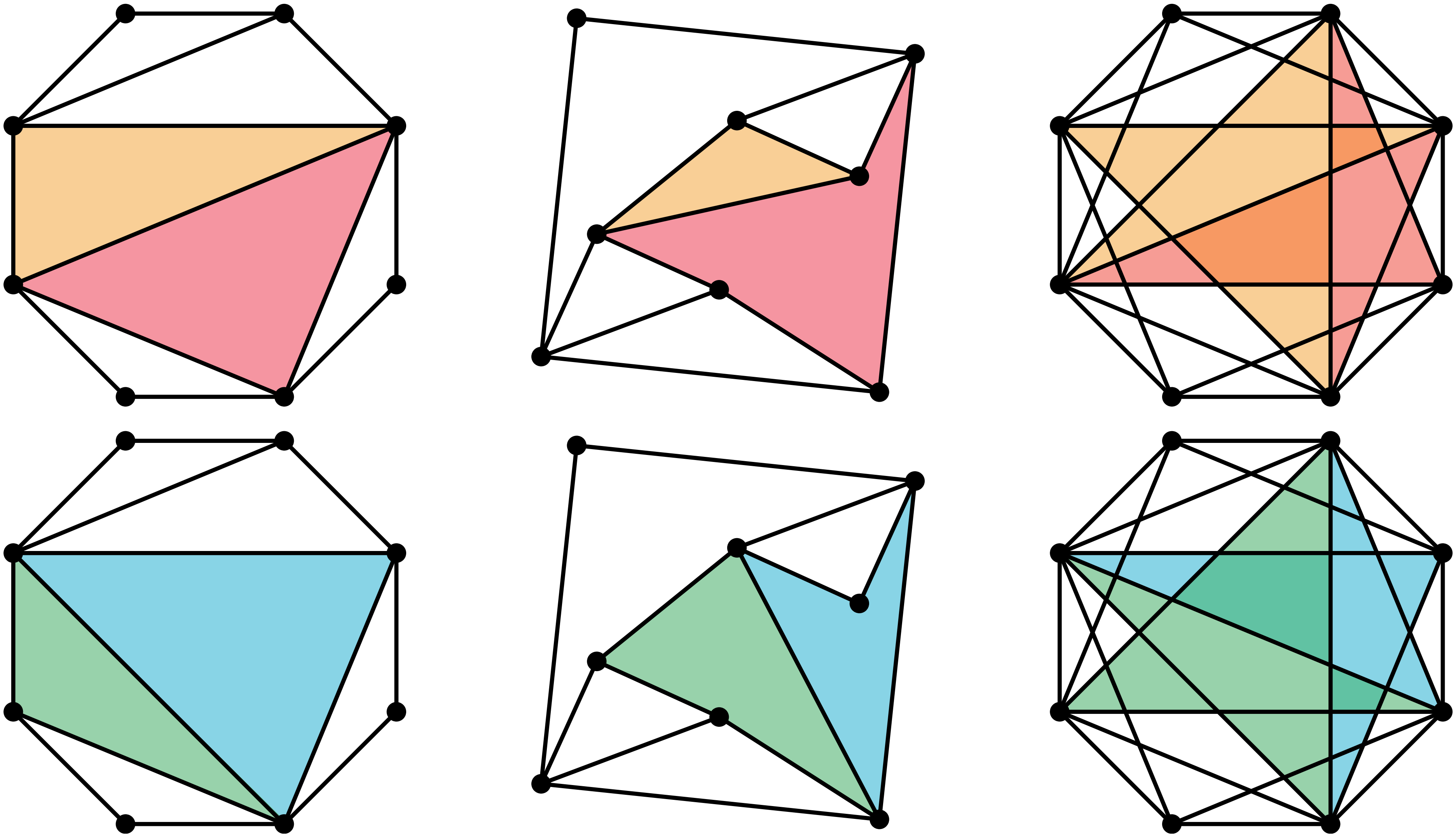
triangulation = maximal crossing-free set of edges
= decomposition into triangles

pseudotriangulation = maximal crossing-free pointed set of edges
= decomposition into pseudotriangles

k-triangulation = maximal $(k+1)$-crossing-free set of edges
= decomposition into k-stars

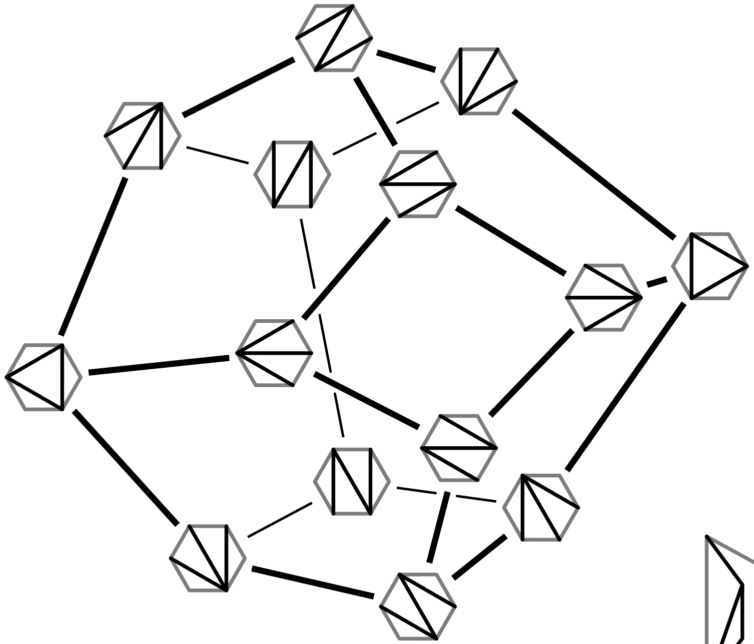
VP & F. Santos, Multitriangulations as complexes of star polygons, 2009.

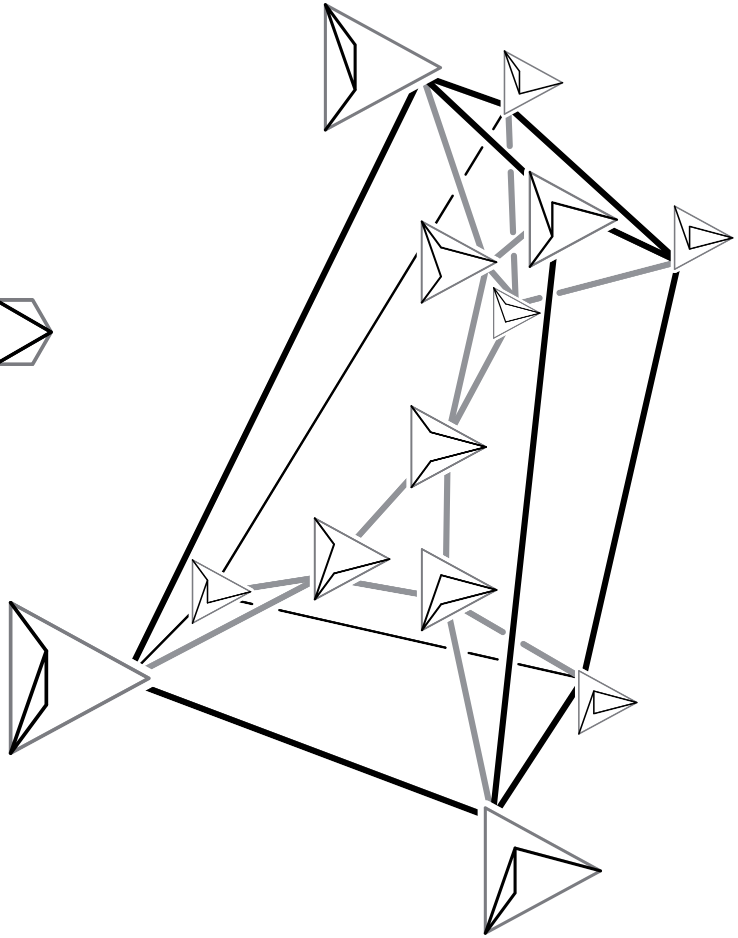
THREE GEOMETRIC STRUCTURES

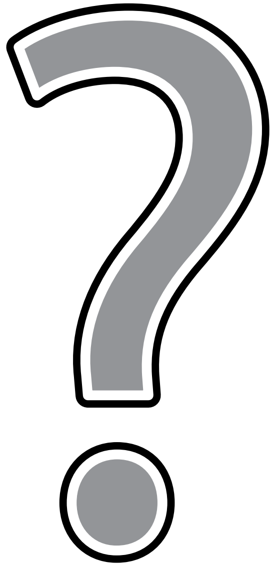


flip = exchange an internal edge with the common bisector of the two adjacent cells

THREE GEOMETRIC STRUCTURES



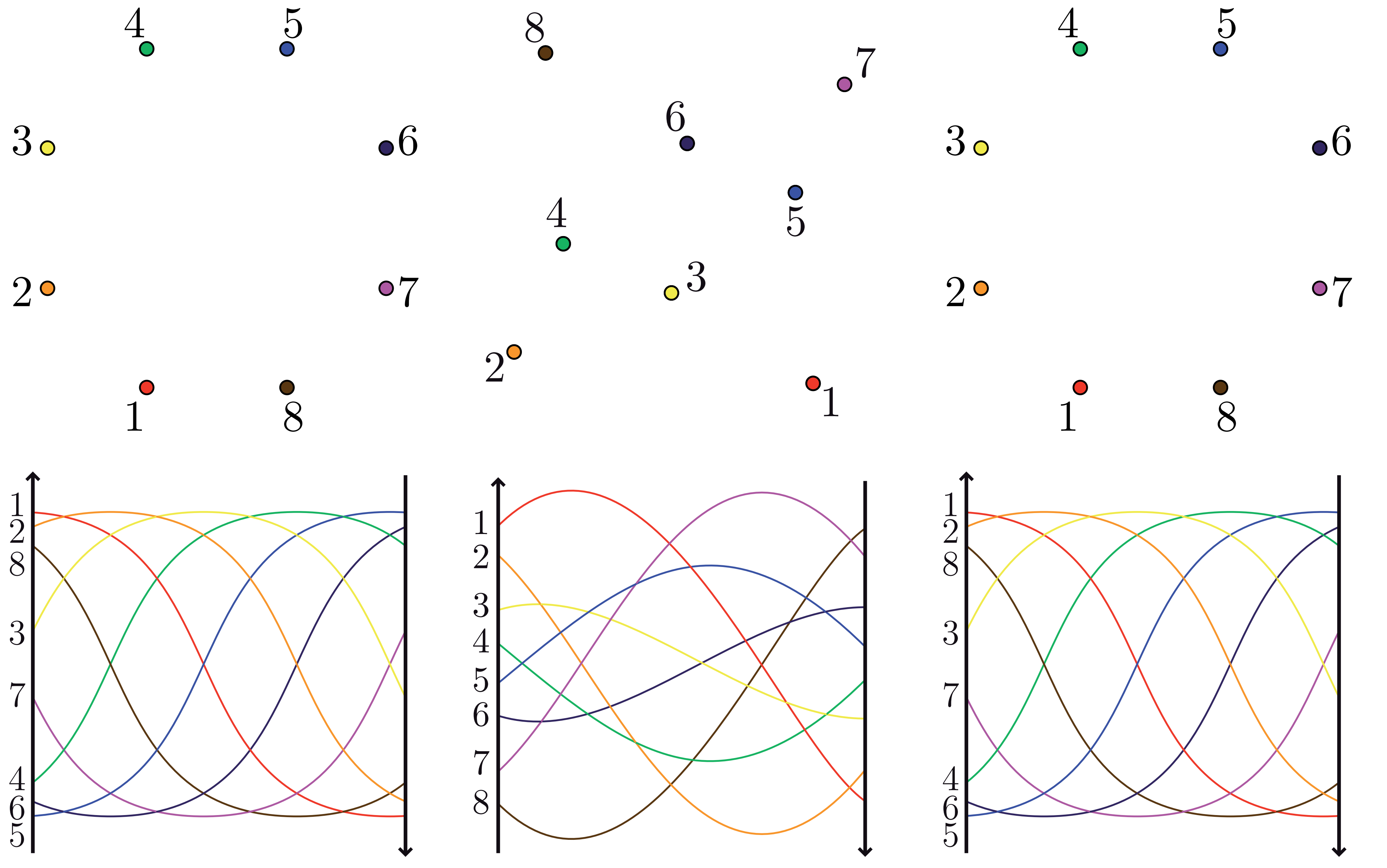




associahedron	$\longleftrightarrow$	crossing-free sets of internal edges
pseudotriangulations polytope	$\longleftrightarrow$	pointed crossing-free sets of internal edges
multiassociahedron	$\longleftrightarrow$	$(k+1)$-crossing-free sets of k-internal edges

DUALITY

DUALITY



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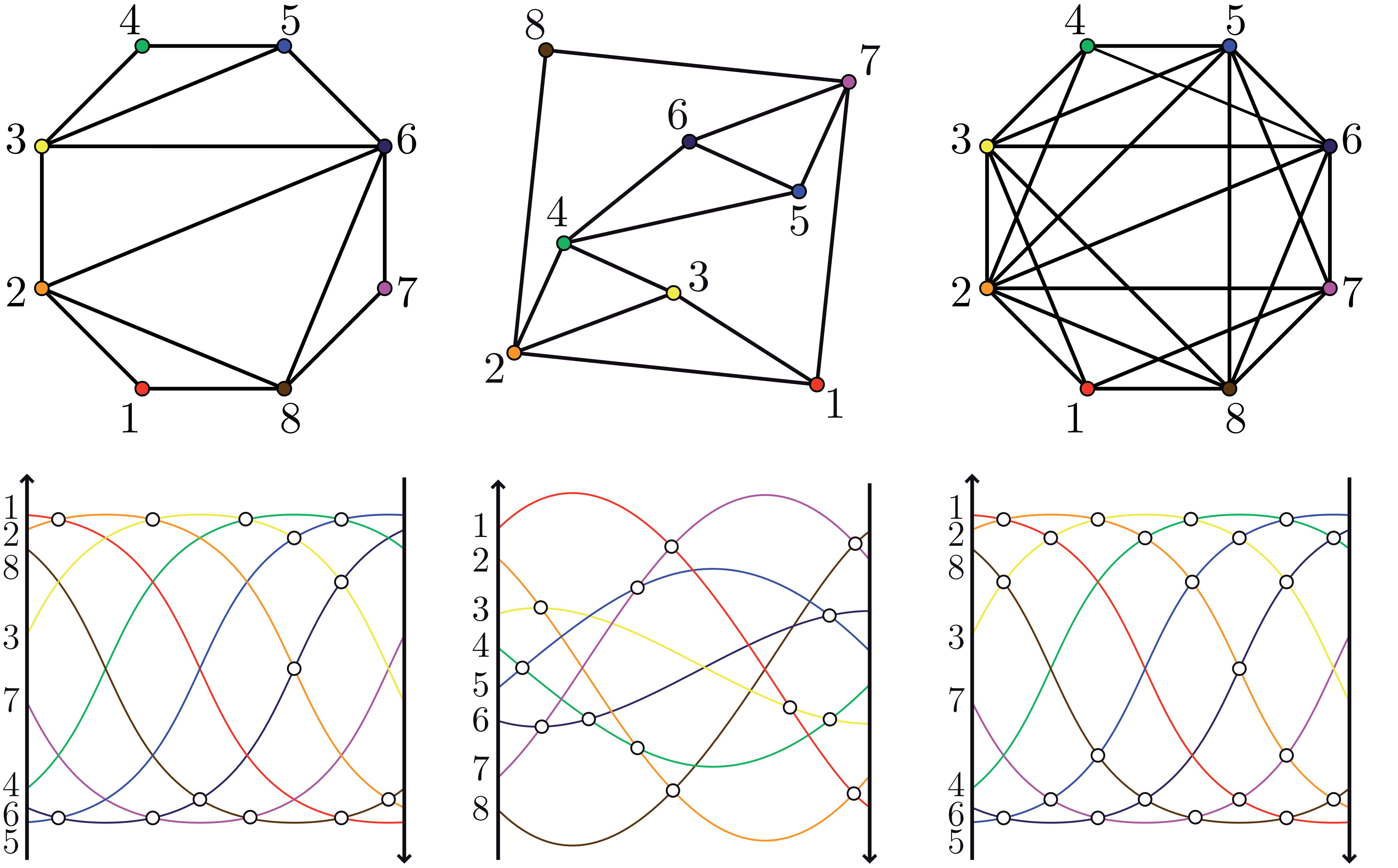

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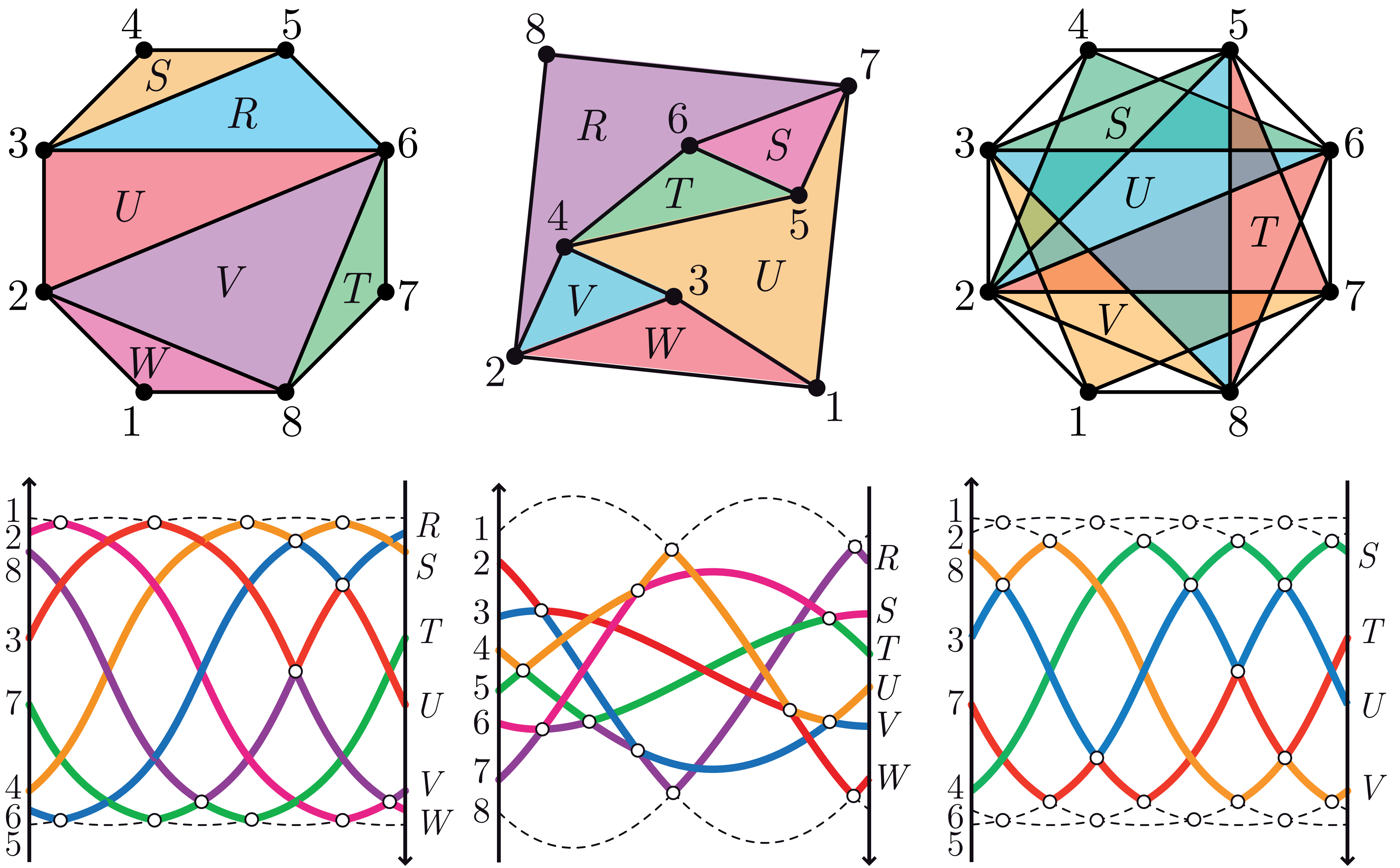
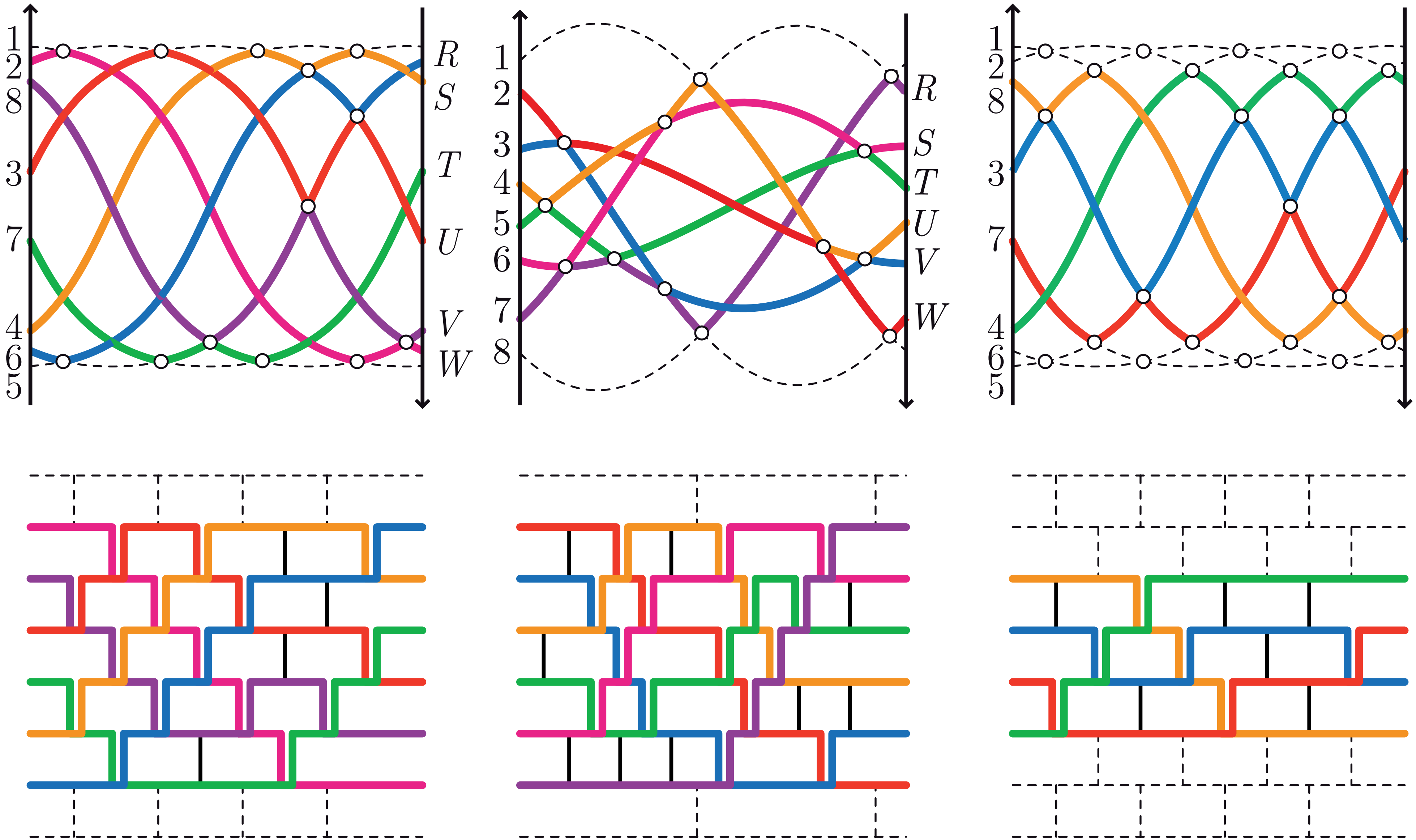
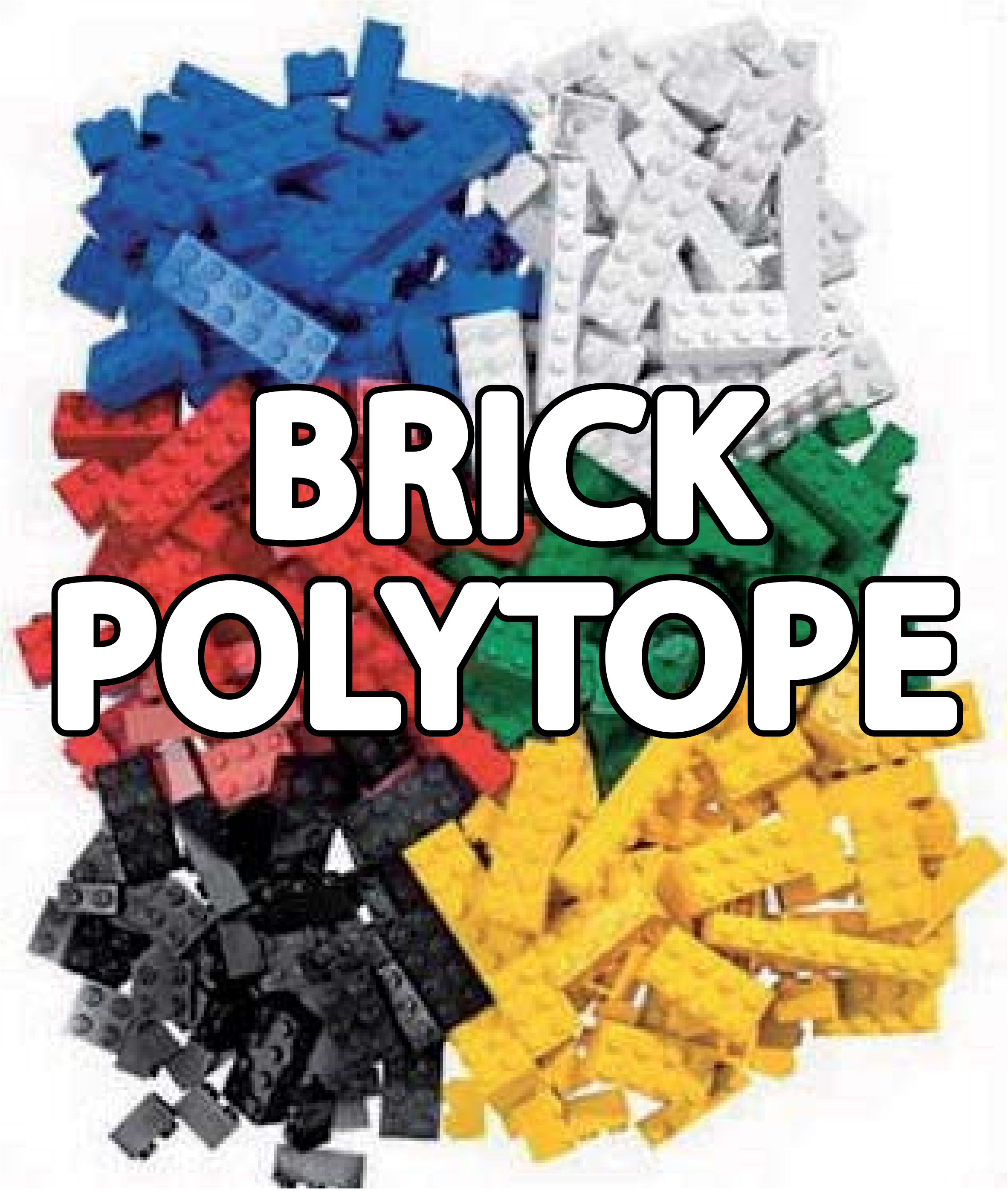


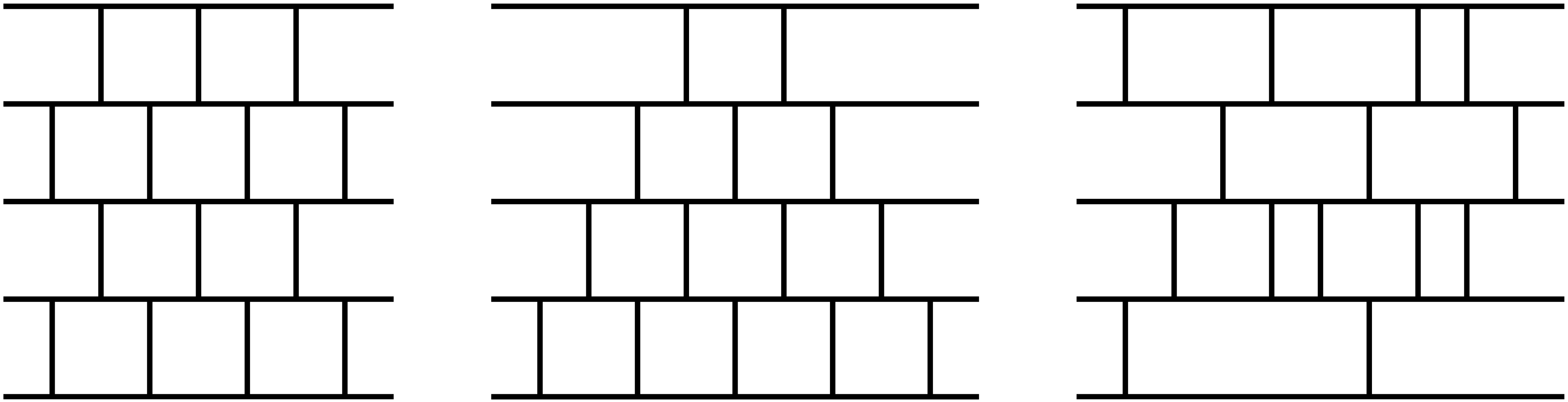
VP & M. Pocchiola, Multitriangulations, pseudotriangulations and primitive sorting networks, 2011.

DUALITY



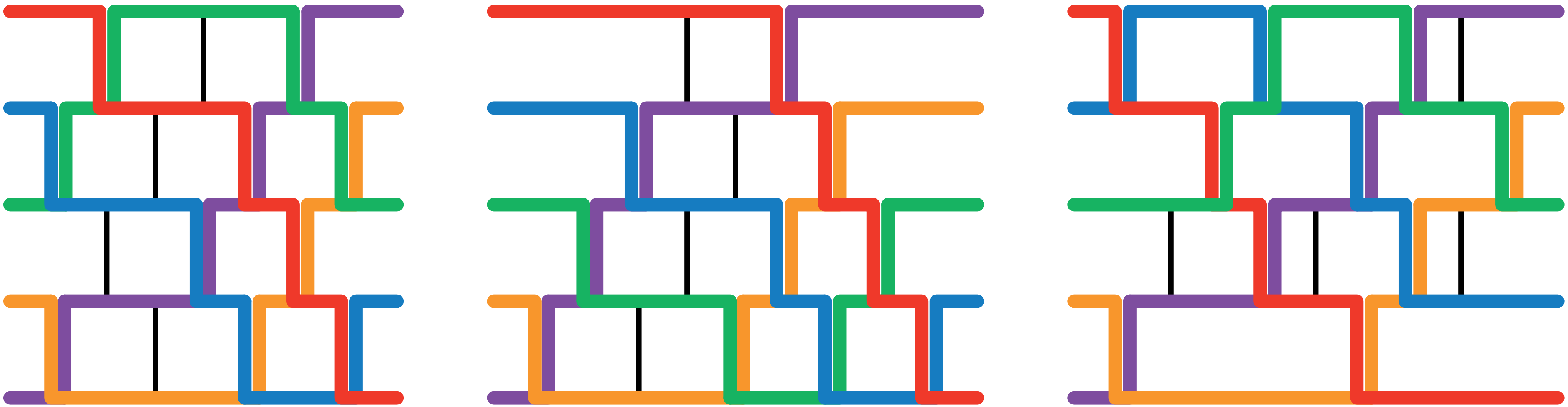

BRICK
POLYTOPE

NETWORKS & PSEUDOLINE ARRANGEMENTS



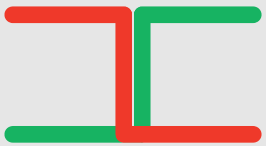
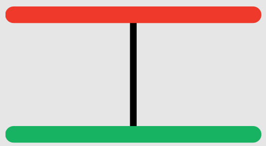
network $\mathcal{N}$ = n horizontal levels and m vertical commutators.
bricks of $\mathcal{N}$ = bounded cells.

NETWORKS & PSEUDOLINE ARRANGEMENTS



network $\mathcal{N}$ $=$ n horizontal levels and m vertical commutators.
bricks of $\mathcal{N}$ $=$ bounded cells.

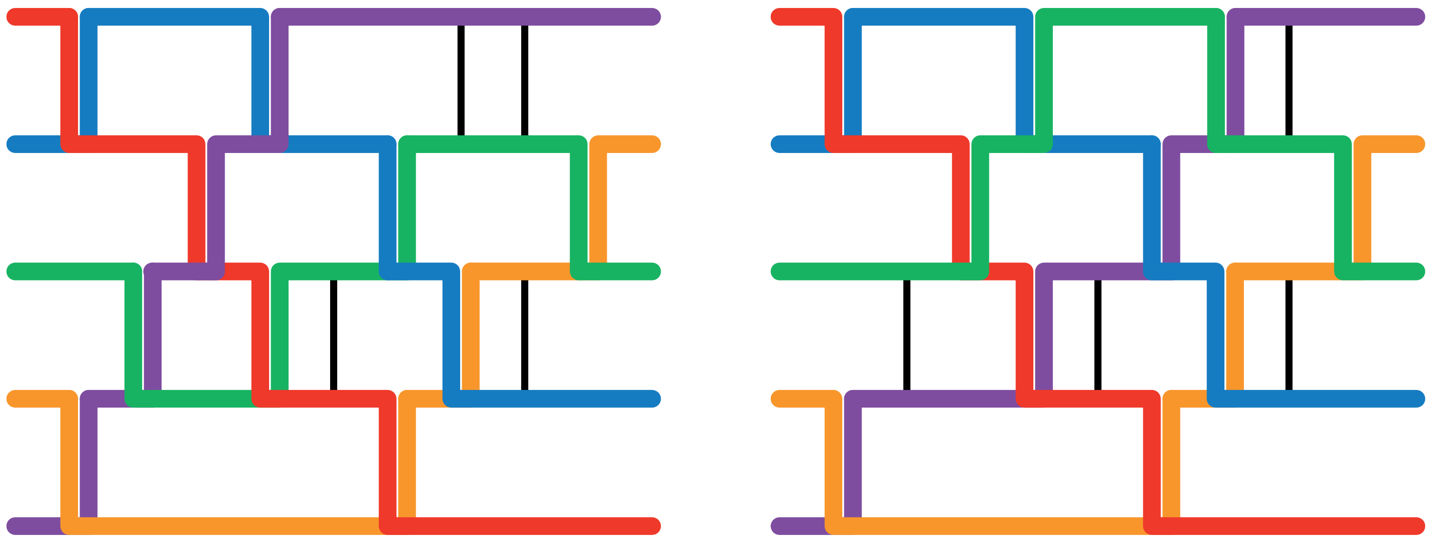
pseudoline $=$ x-monotone path which starts at a level l and ends at the level $n+1-l$.

crossing $=$  contact $=$ 

pseudoline arrangement (with contacts) $=$ n pseudolines supported by $\mathcal{N}$ which have pairwise exactly one crossing, eventually some contacts, and no other intersection.

FLIPS

flip $=$ exchange a contact with the corresponding crossing.



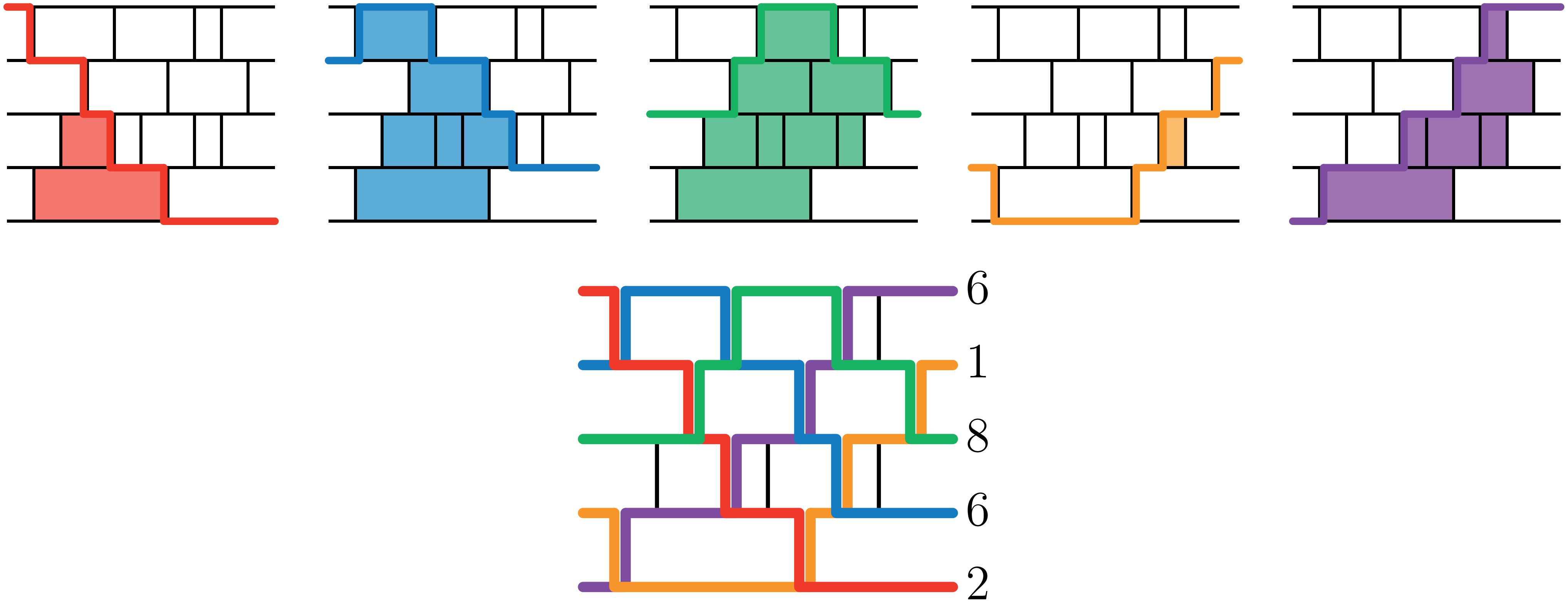
THEOREM. Let $\mathcal{N}$ be a sorting network with n levels and m commutators. The graph of flips $G(\mathcal{N})$ is $\left(m - \binom{n}{2}\right)$-regular and connected.

QUESTION. Is $G(\mathcal{N})$ the graph of a simple $\left(m - \binom{n}{2}\right)$-dimensional polytope?

A. Knutson & E. Miller, Subword complexes in Coxeter groups, 2004.

BRICK POLYTOPE

Λ pseudoline arrangement supported by $\mathcal{N}$ $\longmapsto$ brick vector $\omega(\Lambda) \in \mathbb{R}^n$.
$\omega(\Lambda)_j$ = number of bricks of $\mathcal{N}$ below the jth pseudoline of Λ.



Brick polytope $\Omega(\mathcal{N}) = \text{conv}\,\{\omega(\Lambda) \mid \Lambda$ pseudoline arrangement supported by $\mathcal{N}\}$.

BRICK POLYTOPE

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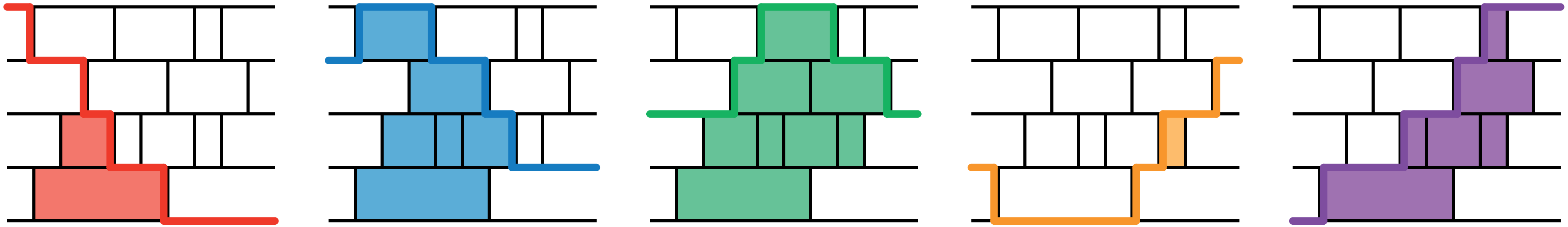


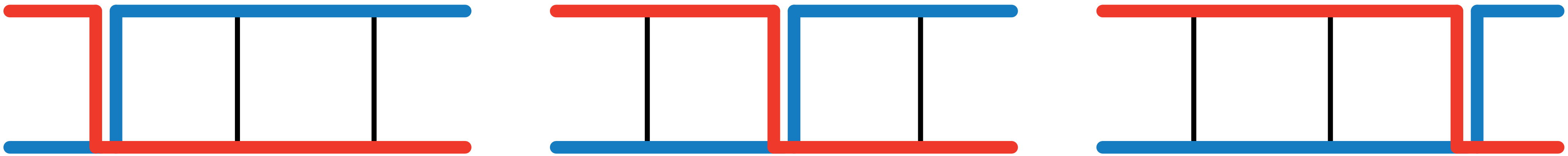
Brick polytope $\Omega(\mathcal{N})$ = conv $\{\omega(\Lambda) \mid \Lambda$ pseudoline arrangement supported by $\mathcal{N}\}$.

REMARK. The brick polytope is not full-dimensional:

$$\Omega(\mathcal{N}) \subset \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \,\middle|\, \sum_{i=1}^{n} x_i = \sum_{b \text{ brick of } \mathcal{N}} \text{depth}(b) \right\}.$$

EXAMPLE: 2-LEVELS NETWORKS

$\mathcal{X}_m$ = network with two levels and m commutators.

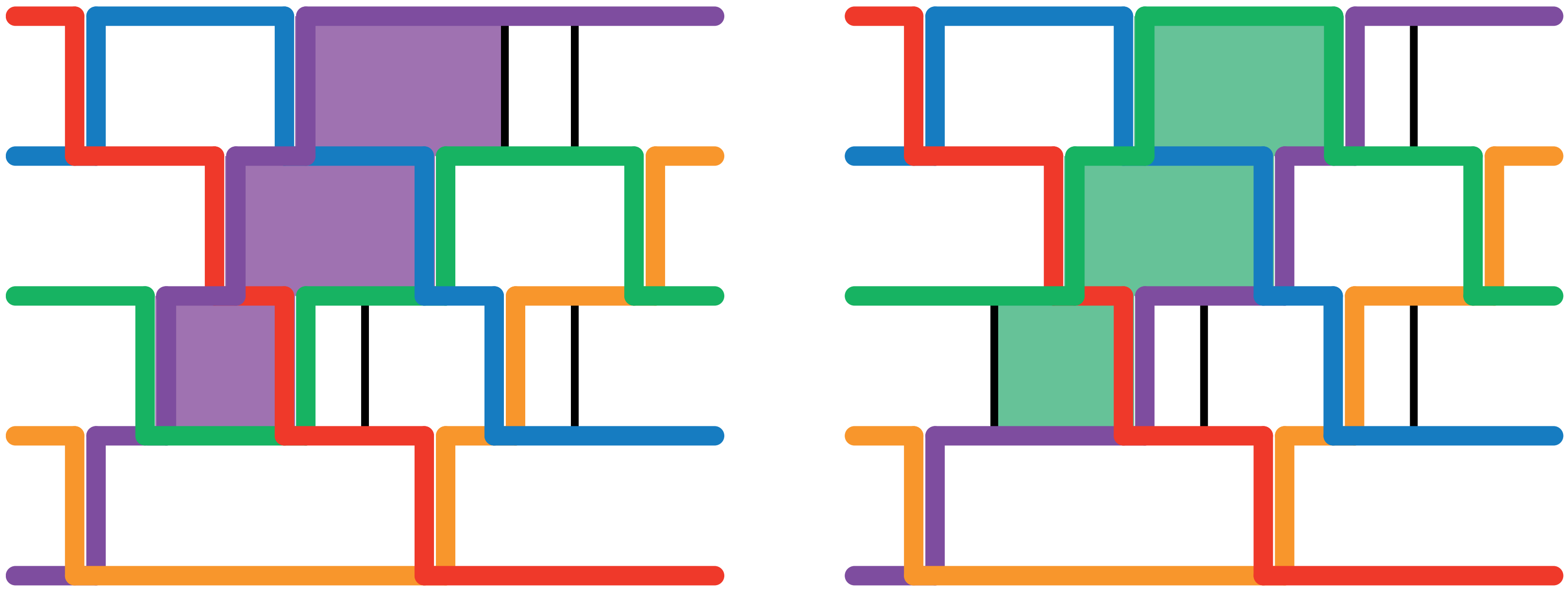


Graph of flips $G(\mathcal{X}_m)$ = complete graph K_m.

Brick polytope $\Omega(\mathcal{X}_m) = \text{conv}\left\{ \begin{pmatrix} m-i \\ i-1 \end{pmatrix} \,\middle|\, i \in [m] \right\} = \left[\begin{pmatrix} m-1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ m-1 \end{pmatrix} \right]$.

COMBINATORIAL DESCRIPTION

BRICK VECTORS AND FLIPS



REMARK. If Λ and Λ' are two pseudoline arrangements supported by $\mathcal{N}$ and related by a flip between their ith and jth pseudolines, then $\omega(\Lambda) - \omega(\Lambda') \in \mathbb{N}_{>0}\,(e_j - e_i)$.

COROLLARY. The cone generated by the vector configuration

$$\{e_j - e_i \mid \text{there is a contact between the } i\text{th and } j\text{th pseudolines of } \Lambda\}$$

is contained in the cone of the brick polytope $\Omega(S)$ at the brick vector $\omega(\Lambda)$.

INCIDENCE CONE OF A DIRECTED MULTIGRAPH

G directed (multi)graph $\longmapsto$ Incidence configuration $I(G) = \{e_j - e_i \mid (i,j) \in G\}$,
$\longmapsto$ Incidence cone $C(G)$ = cone generated by $I(G)$.

REMARK.

independant sets in $I(G)$	$\longleftrightarrow$	forests in G
spanning sets of $\langle\, \mathbb{1} \mid x \,\rangle = 0$	$\longleftrightarrow$	connected spanning subgraphs of G
basis of $\langle\, \mathbb{1} \mid x \,\rangle = 0$	$\longleftrightarrow$	spanning trees of G
circuits in $I(G)$	$\longleftrightarrow$	simple cycles in G
cocircuits in $I(G)$	$\longleftrightarrow$	minimal cuts in G

(and signs correspond to the orientations of the edges).

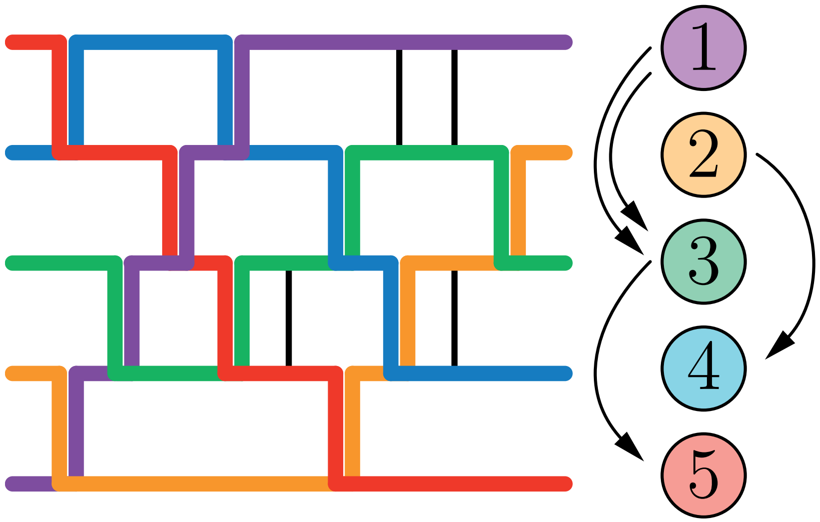
REMARK. H subgraph of G. Then $I(H)$ forms a k-face of $C(G)$ $\Longleftrightarrow$ H has $n-k$ connected components and G/H is acyclic. In particular:

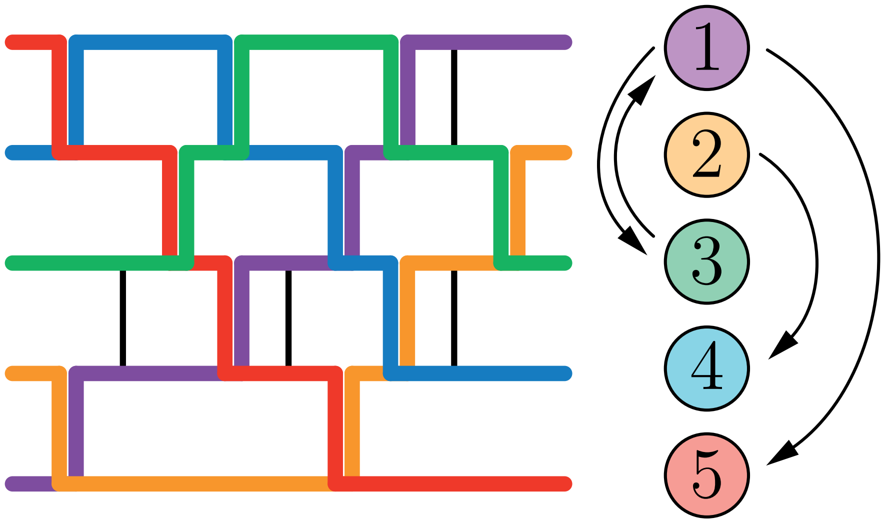
$C(G)$ is pointed	$\longleftrightarrow$	G is acyclic
facets of $C(G)$	$\longleftrightarrow$	complements of the minimal directed cuts of G

CONTACT GRAPH OF A PSEUDOLINE ARRANGEMENT

Contact graph $\Lambda^\#$ of a pseudoline arrangement $\Lambda =$

- a node for each pseudoline of Λ, and
- an arc for each contact point of Λ oriented from top to bottom.





THEOREM. The cone of the brick polytope $\Omega(S)$ at the brick vector $\omega(\Lambda)$ is the incidence cone $C(\Lambda^\#)$ of the contact graph of Λ.

COMBINATORIAL DESCRIPTION

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VERTICES OF $\Omega(\mathcal{N})$

The brick vector $\omega(\Lambda)$ is a vertex of $\Omega(\mathcal{N}) \iff$ the contact graph $\Lambda^{\#}$ is acyclic.

GRAPH OF $\Omega(\mathcal{N})$

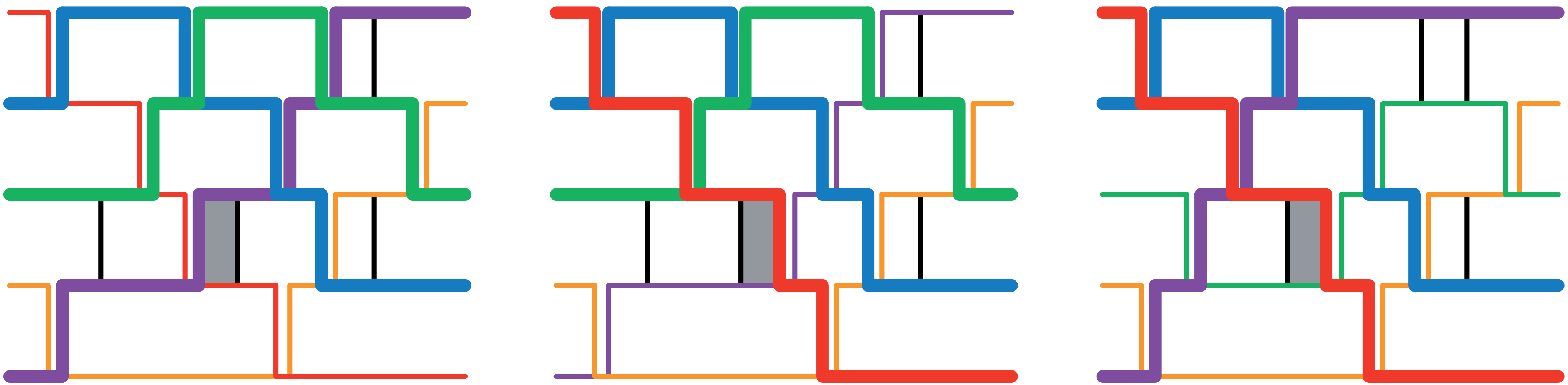
The graph of the brick polytope is a subgraph of $G(\mathcal{N})$ whose vertices are the pseudoline arrangements with acyclic contact graphs.

FACETS OF $\Omega(\mathcal{N})$

The facets of $\Omega(\mathcal{N})$ correspond to the minimal directed cuts of the contact graphs of the pseudoline arrangements supported by $\mathcal{N}$.

BRICK POLYTOPES AND MINKOWSKI SUMS

$\mathcal{N}$ network with n levels, b a brick of $\mathcal{N}$, Λ pseudoline arrangement supported by $\mathcal{N}$.
$\omega(\Lambda, b) \in \mathbb{R}^n$ characteristic vector of the pseudolines of Λ passing above b.
$\Omega(\mathcal{N}, b) = \text{conv}\,\{\omega(\Lambda, b) \mid \Lambda \text{ pseudoline arrangement supported by } \mathcal{N}\} \subset \mathbb{R}^n$.



THEOREM. The brick polytope $\Omega(\mathcal{N})$ is the Minkowski sum of the polytopes $\Omega(\mathcal{N}, b)$ associated to the bricks of $\mathcal{N}$:

$$\Omega(\mathcal{N}) = \sum_{b \text{ brick of } \mathcal{N}} \Omega(\mathcal{N}, b).$$

BRICK POLYTOPES AND GENERALIZED PERMUTOHEDRA

Generalized permutohedra = polytope whose inequality description is of the form

$$Z\left(\{z_I\}_{I\in[n]}\right) = \left\{ \begin{pmatrix} x_1 \\ \dots \\ x_n \end{pmatrix} \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n} x_i = z_{[n]} \text{ and } \sum_{i\in I} x_i \ge z_I \text{ for } I \subset [n] \right\}$$

for some family $\{z_I\}_{I\subset[n]} \in \mathbb{R}^{2^{[n]}}$.

A. Postnikov, Permutohedra, associahedra and beyond, 2009.

THEOREM. Any generalized permutahedron is a Minkowski sum of simplices:

$$Z\left(\{z_I\}_{I\in[n]}\right) = \sum_{I\subset[n]} y_I \Delta_I \quad \text{where} \quad y_I = \sum_{J\subset I} (-1)^{|I\smallsetminus J|} z_J \quad \left(\text{ie.} \quad z_I = \sum_{J\subset I} y_J\right).$$

F. Ardila, C. Benedetti & J. Doker, Matroid polytopes and their volumes, 2010.

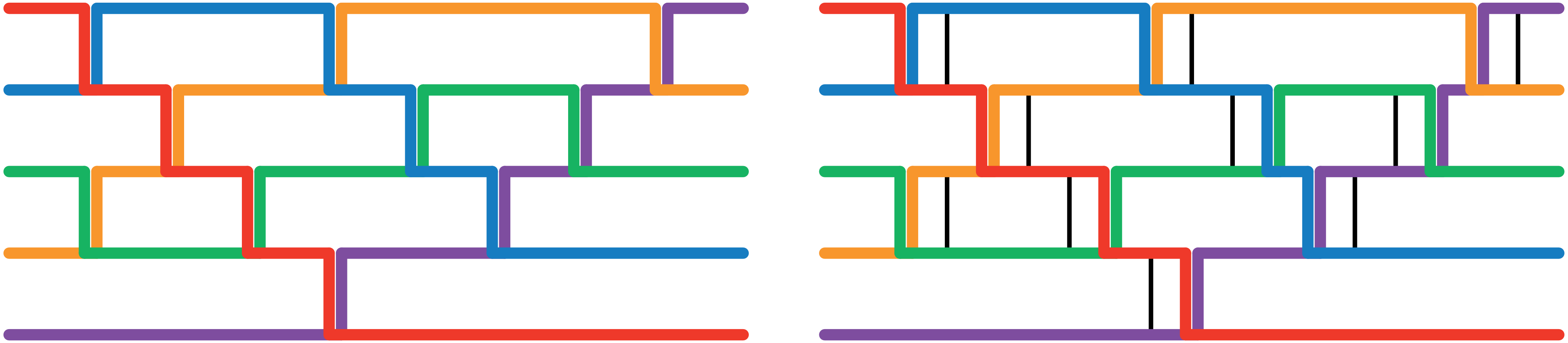
REMARK. All brick polytopes are generalized permutohedra. Compute $\{y_I\}_{I\subset[n]}$.
Which generalized permutohedra are brick polytopes?


ASSOCIAHEDRA
& PERMUTAHEDRA

DUPLICATED NETWORKS: PERMUTAHEDRA

Reduced network = network with n levels and $\binom{n}{2}$ commutators.
It supports only one pseudoline arrangement.

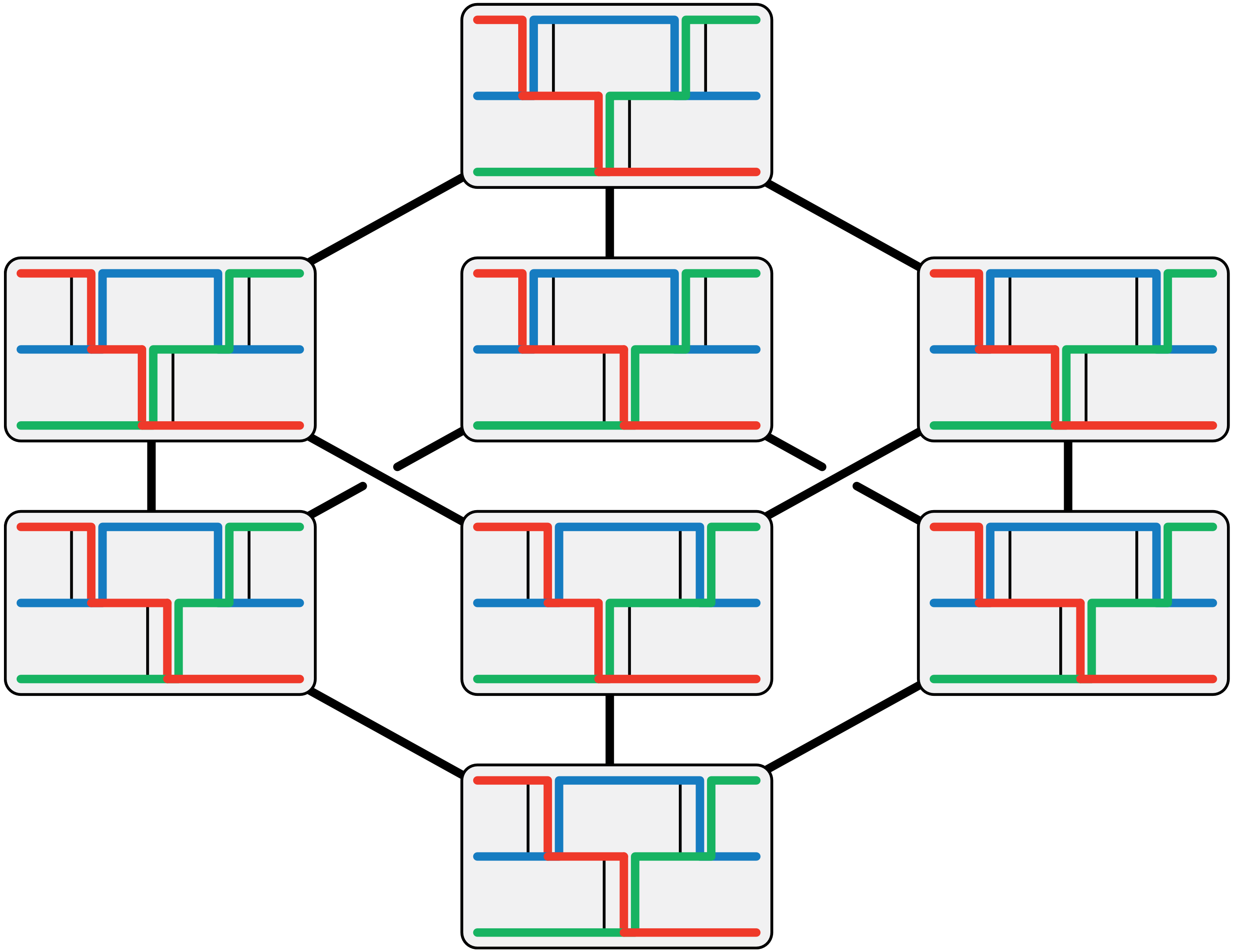
Duplicated network Π = network with n levels and $2\binom{n}{2}$ commutators obtained by duplicating each commutator of a reduced network.



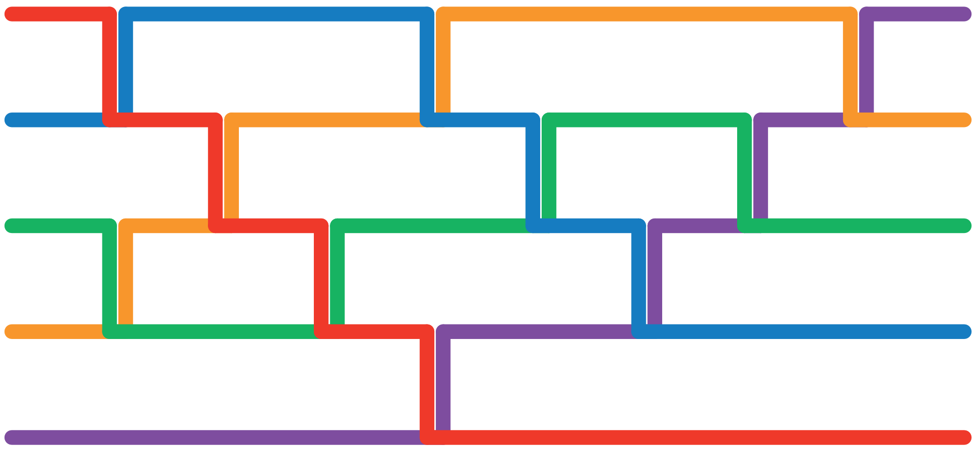
Any pseudoline arrangement supported by Π has one contact and one crossing among each pair of duplicated commutators.

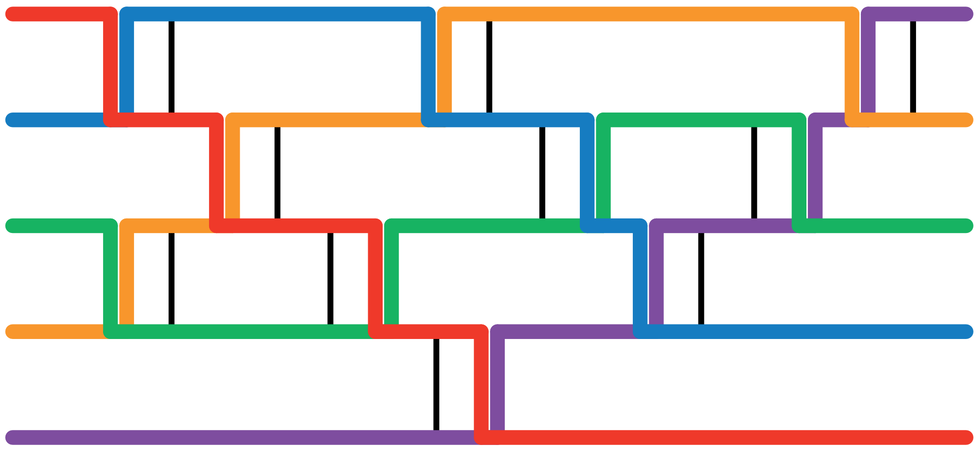
DUPLICATED NETWORKS: PERMUTAHEDRA

Graph of flips $G(\Pi) = \binom{n}{2}$-dimensional cube.



DUPLICATED NETWORKS: PERMUTAHEDRA





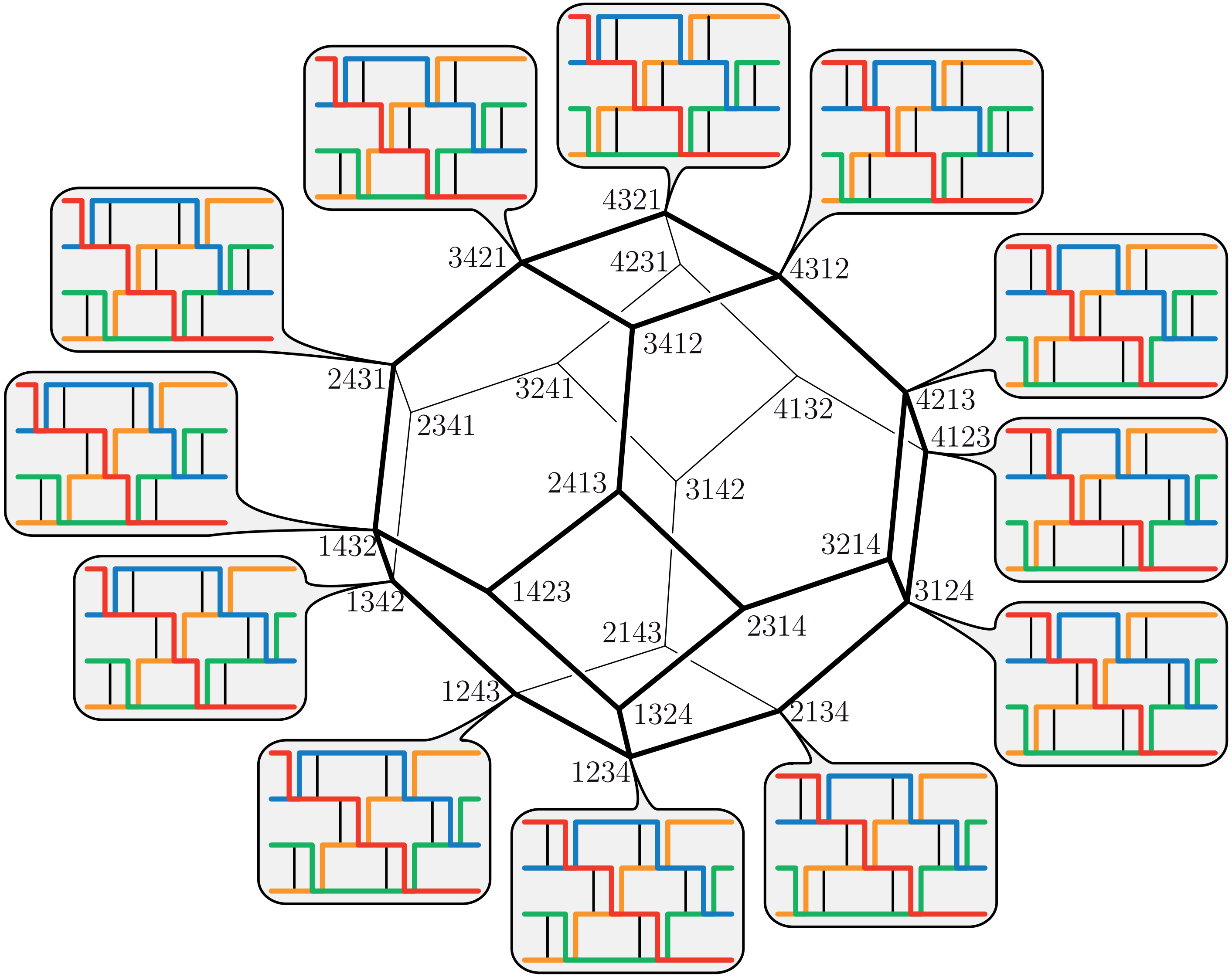
Any pseudoline arrangement supported by Π has one contact and one crossing among each pair of duplicated commutators. $\Longrightarrow$ The contact graph $\Lambda^{\#}$ is a tournament.

Vertices of $\Omega(\Pi)$	$\Longleftrightarrow$	acyclic tournaments	$\Longleftrightarrow$	permutations of $[n]$
Facets of $\Omega(\Pi)$	$\Longleftrightarrow$	cuts in a tournament	$\Longleftrightarrow$	ordered bipartitions of $[n]$

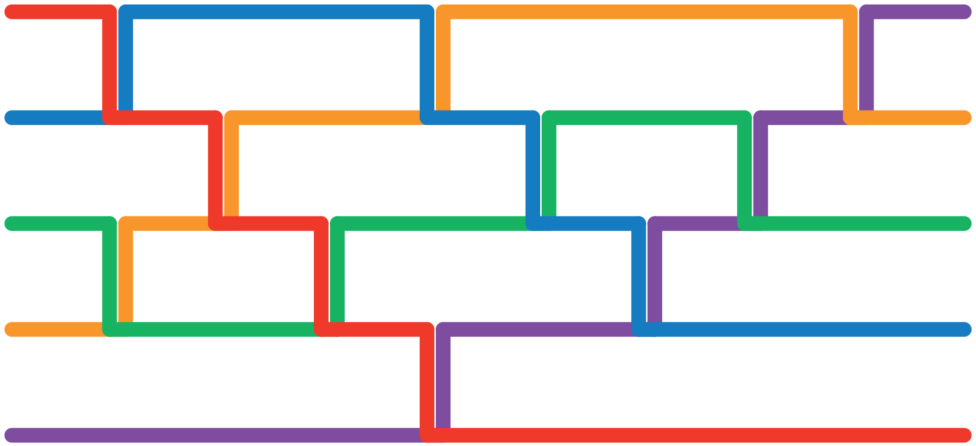
Brick polytope $\Omega(\Pi) =$ permutahedron

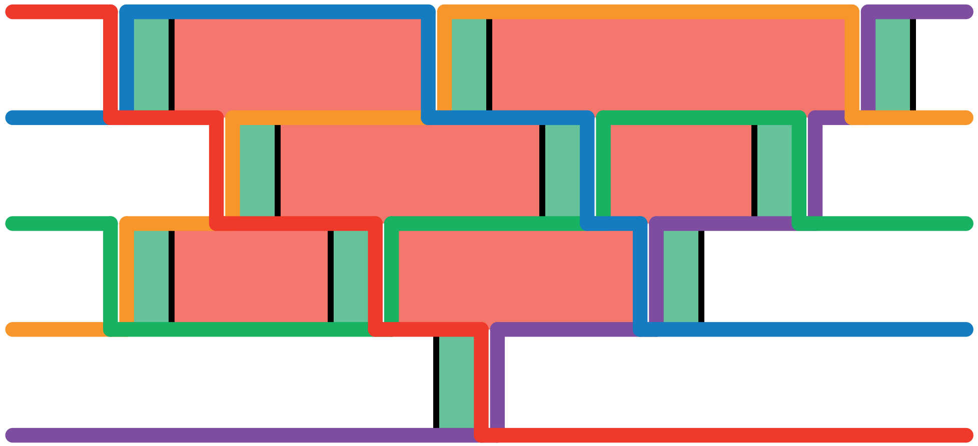
DUPLICATED NETWORKS: PERMUTAHEDRA

Brick polytope $\Omega(\Pi)$ = permutahedron



DUPLICATED NETWORKS: PERMUTAHEDRA





Minkowski sum decomposition

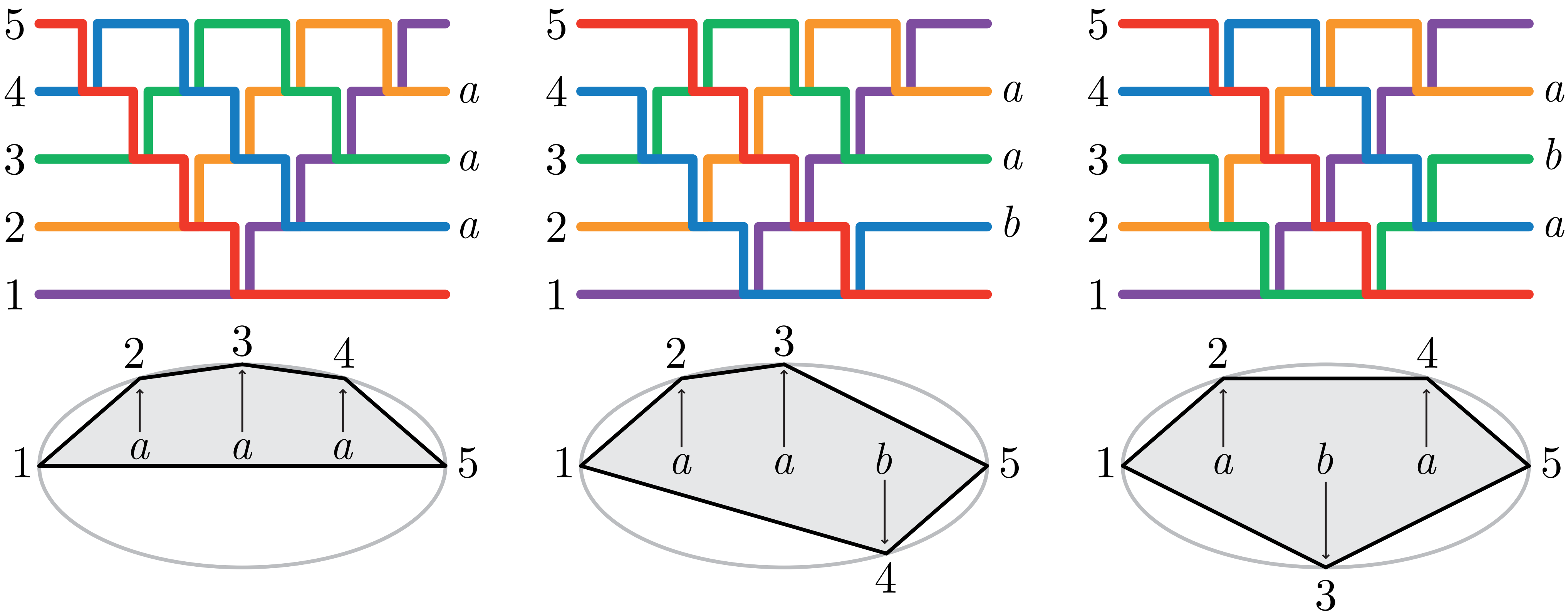
$$\Omega(\Pi) = \sum_{b \text{ brick of } \Pi} \Omega(\Pi, b) = \sum_{i<j} \text{segment } [e_i - e_j] + \sum \text{vertices} = \text{permutahedron}$$

$$P(0, 1, \dots, n-1) = \text{Newton}\left(\det \left[t_i^{j-1}\right]_{i,j \in [n]}\right) = \text{Newton}\left(\prod_{1 \le i < j \le n} (t_j - t_i)\right)$$

$$= \sum_{1 \le i < j \le n} \text{Newton}\,(t_j - t_i) = \sum_{1 \le i < j \le n} [e_j - e_i]$$

ALTERNATING NETWORK: ASSOCIAHEDRA

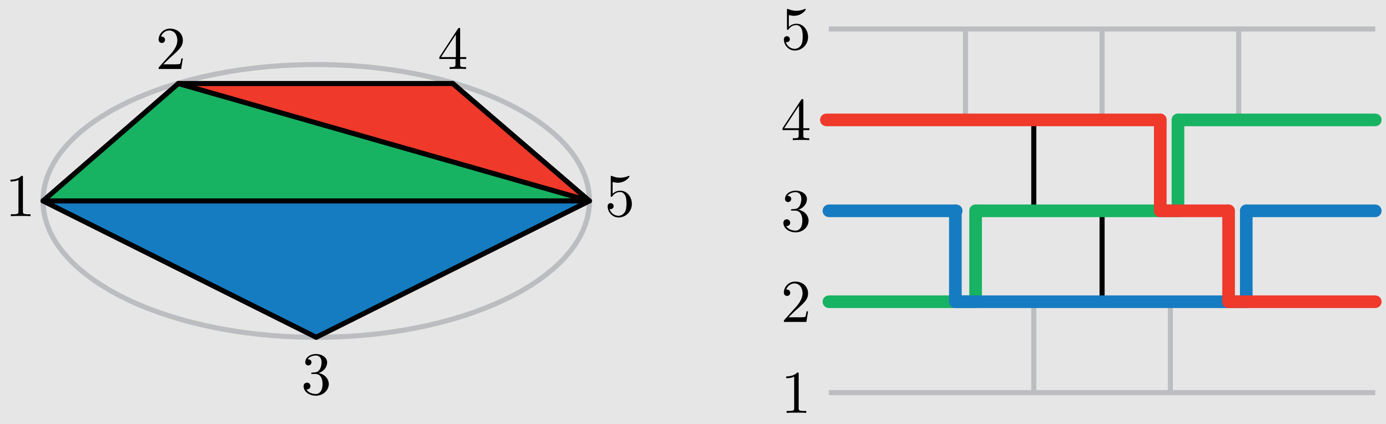
For $x \in \{a, b\}^{n-2}$, we define a reduced alternating network $\mathcal{N}_x$ and a polygon $\mathcal{P}_x$.



$\mathcal{N}_x$ is the dual pseudoline arrangement of the polygon $\mathcal{P}_x$.

ALTERNATING NETWORK: ASSOCIAHEDRA

THEOREM. There is a duality between the pseudoline arrangements supported by $\mathcal{N}_x^1$ and the triangulations of the polygon $\mathcal{P}_x$.



T triangulation of $\mathcal{P}_x$ $\longleftrightarrow$ T^* pseudoline arrangement supported by $\mathcal{N}_x^1$

Δ triangle of T $\longleftrightarrow$ Δ^* pseudoline of T^*

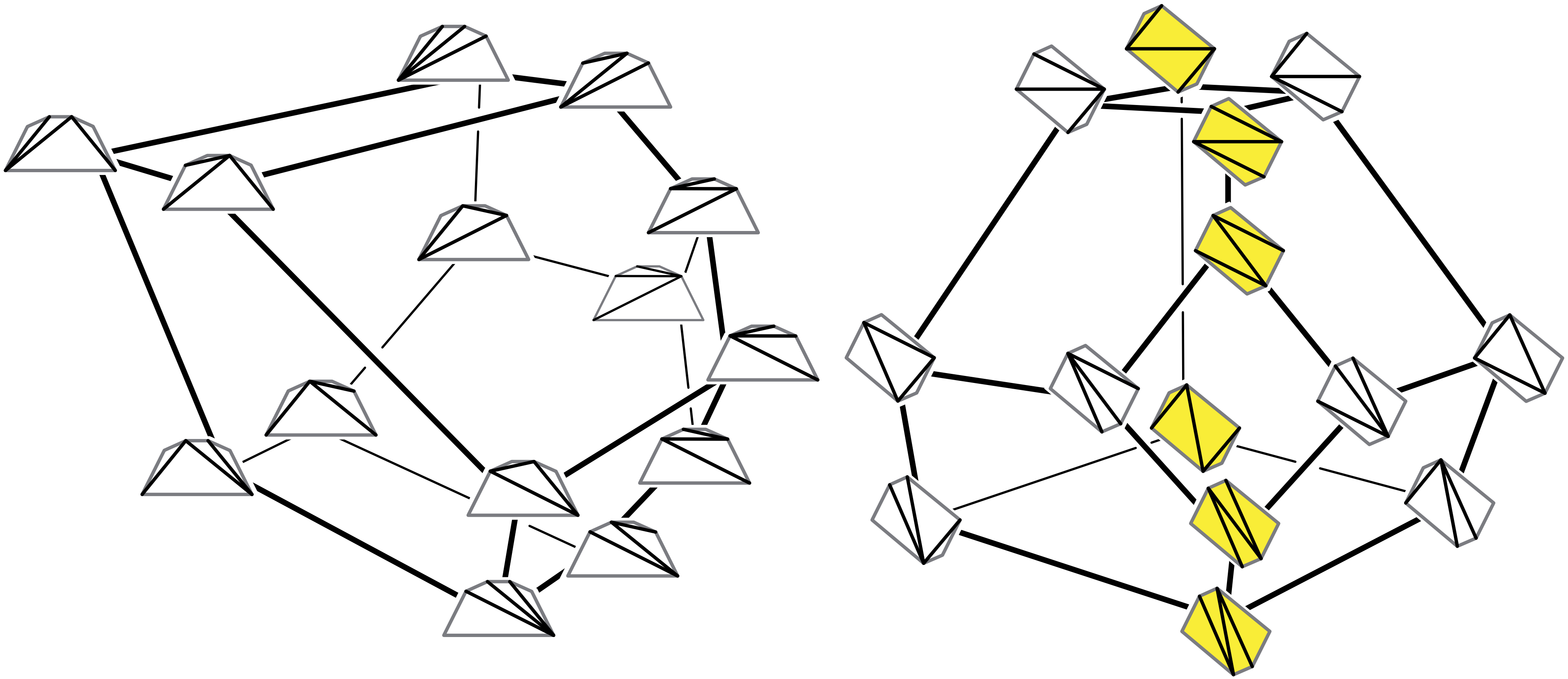
e common edge of Δ and Δ' $\longleftrightarrow$ e^* contact between Δ^* and Δ'^*

f common bissector of Δ and Δ' $\longleftrightarrow$ f^* crossing between Δ^* and Δ'^*

COROLLARY. (i) The graph of flips $G(\mathcal{N}_x^1)$ is (isomorphic to) the graph of flips $G(\mathcal{P}_x)$.
(ii) The contact graph $(T^*)^\#$ is (isomorphic to) the dual binary tree of T.

HOHLWEG & LANGE'S ASSOCIAHEDRA

THEOREM. For any word $x \in \{a, b\}^{n-2}$, the simplicial complex of crossing-free sets of internal diagonals of the convex n-gon $\mathcal{P}_x$ is (isomorphic to) the boundary complex of the polar of the brick polytope $\Omega(\mathcal{N}_x^1)$.



REMARK. Up to translation, we obtain Hohlweg & Lange's associahedra.

C. Hohlweg & C. Lange, Realizations of the associahedron and cyclohedron, 2007.

arXiv:1103.2731

THANK YOU