Cells in the box and a hyperplane

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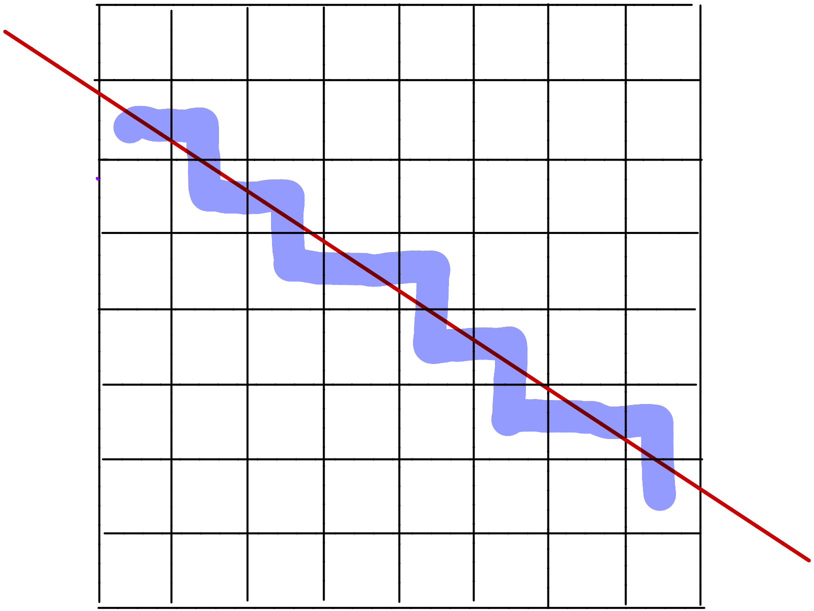
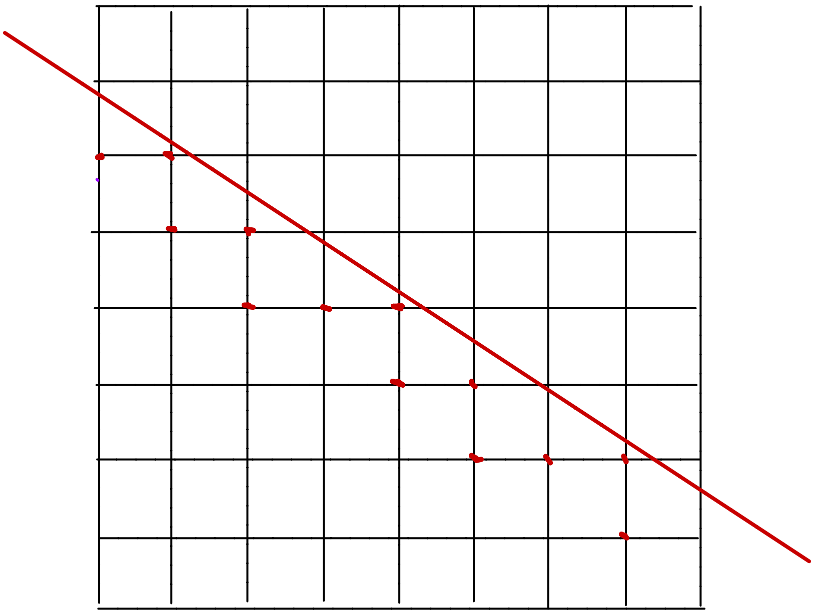
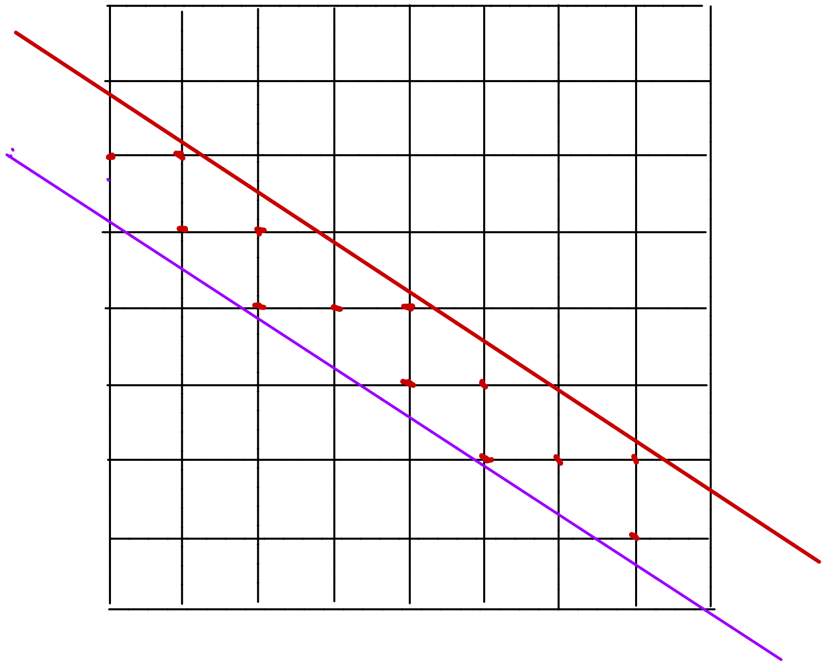
Fact: a line intersects at most $2n-1$ cells (squares) of the $n \times n$ chessboard. (gen pb)







Question: How many cells of the $\overbrace{n \times \dots \times n}^{d \text{ times}}$ chessboard can a hyperplane intersect?

$d=3$ $\qquad M_n$ = max number of cells in $\mathbb{R}^3$

Thm 1. $M_n = \frac{9}{4} n^2 + O(n)$

(higher dim later)

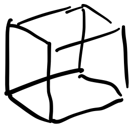
More precisely

$$M_n \le \frac{9}{4} n^2 + 2n + 1$$

$$M_n \ge \frac{9}{4} n^2 + n - \begin{cases} 5 & n \text{ páros} \\ \frac{17}{4} & n \text{ ptlan} \end{cases}$$

$$M_2 = 7, \quad M_3 = 19, \quad M_4 = 35$$

$$230 \le M_{10} \le 246$$

$K_n = [0,n]^3$, $C(z) =$  $z \in \mathbb{Z}^3$ unit cube (cell)

z

P a plane with equation $ax + by + z = d$

$0 < a < b < 1$

Lower bound $m = \frac{3n}{2}$ (n even) $= \frac{3n-1}{2}$ (n odd)

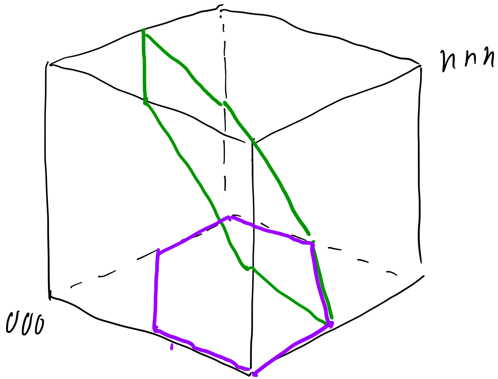
$P = \{ x + y + z = m + \varepsilon \}$ ($\varepsilon > 0$ small)

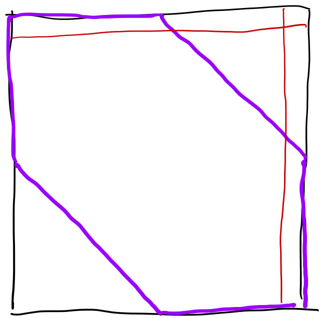
intersects $\frac{9n^2-8}{4}$ (n even) $\frac{9n^2-5}{4}$ (odd) cells

Proof. $\# (x,y,z) \in \mathbb{Z}^3, \quad 0 \le x,y,z \le n-1$

then $x+y+z = m, m-1, m-2$

because $x+y+z < m+\varepsilon$ and $x+1+y+1+z+1 > m+\varepsilon$





$\square$

improve

Upper bound $ax + by + cz = d$ the maximizer $\underbrace{\text{plane}}_{P}$

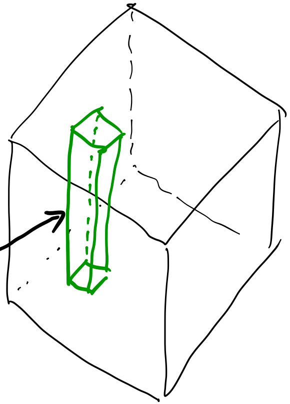
wlog $0 < a < b < c = 1$

Claim 1. $a + b > 1$.

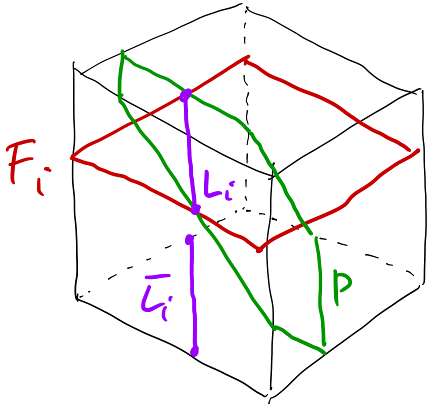
Otherwise $a + b \le 1$ and

P intersects at most

2 cells in a stack



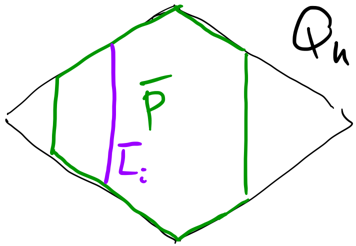
$F_i = \{(x,y,i) \in K_n,\ i \in \mathbb{Z}\}$ "floor"



$L_i = P \cap F_i$

$\bar{L}_i$ its projection to $F_0 = Q_n$

$L_i = \emptyset$ possible but



Claim 1 implies that either

$L_0 \neq \emptyset$ or $L_p \neq \emptyset$ or both.

Assume $L_n \neq \phi$. Then

$$L_p, \dots, L_n \neq \phi \quad \text{and} \quad L_0 = \dots = L_{p-1} = \phi$$

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l_i is the length of (L_i and of $\overline{L_i}$)

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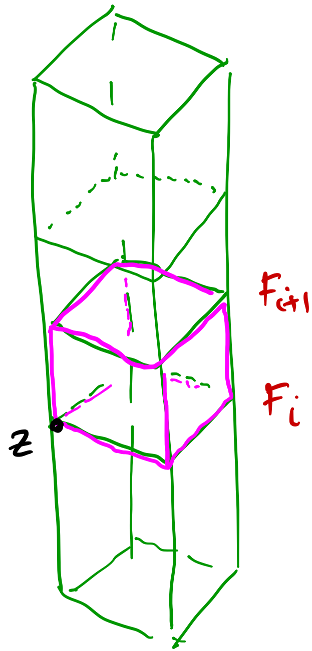
m_i = # cells of Q_n hit by $\overline{L_i}$

m = # cells of Q_n intersect $\overline{P}$

Lemma # cells hit by P = $m + m_{p+1} + \dots + m_{n-1}$

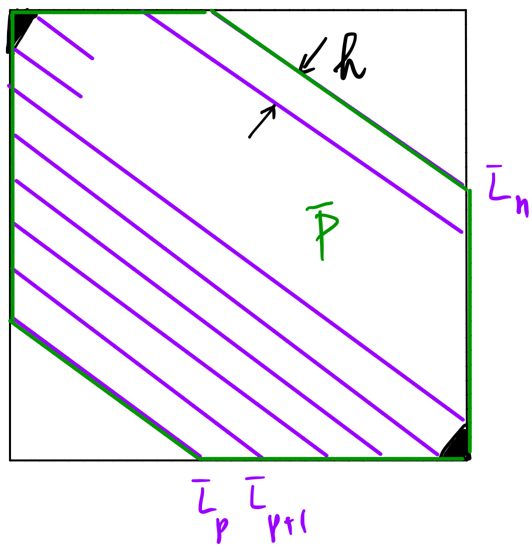
Proof: count $P \cap C(z)$ on the bottom face of $C(z)$ if P hits the bottom face

If it does not, then $P \cap C(z)$ is counted in m $\square$



Upper bound on M_n

$$h = \frac{1}{\sqrt{a^2+b^2}}$$



$$m \leq \text{Area}\,\bar{P} + m_p + m_n$$

$$\text{Area}\,\bar{P} = \sum_{i=p}^{n-1} h\,\frac{l_i + l_{i+1}}{2} + \left(\blacktriangleleft + \blacktriangleleft\right)$$

$$m_i \leq \frac{a+b}{\sqrt{a^2+l^2}}\, l_i + 1 = (a+b)\,h\,l_i + 1$$

. . . . . leads to . . .

Lemma. If $0 \le a \le b \le 1$ then

$$(a+b+1)\left(1 - \frac{(a+b-1)^2}{4ab}\right) \le \frac{9}{4},$$

equality iff $a = b = 1$. $\square$

stability

$d \geqq 3$

$M_n^d =$ max # of cells in $K_n = [0, n]^d$
that a (gen. pos.) hyperplane intersects

$M_n^2 = 2n - 1$

$M_n^3 = \frac{9}{4} n^2 + O(n)$

$d \geqq 3$

$v \in \mathbb{R}^d$ unit vector $\|v\|_2 = 1$. P_v is the hyperplane orthogonal to v and containing the centre of $[0,1]^d$

<u>Define</u> $$V_d = \max_{v\ldots} \|v\|_1 \, \mathrm{vol}_{d-1}\left([0,1]^d \cap P_v\right)$$

L_1 norm

central section

$$1 \leq \mathrm{vol}_{d-1}\left([0,1]^d \cap P_v\right) \leq \sqrt{2} \quad (\text{K. Ball})$$

$\Rightarrow \quad \sqrt{d} \leq V_d \leq \sqrt{2d}$ but

Thm (I. Aliev, 2020) The maximum is attained on $v = \frac{1}{\sqrt{d}}(1, \dots, 1)$.

$V_2 = 2$, $V_3 = \frac{9}{4}$, $V_4 = \frac{8}{3}$, ... increasing

$V_d \longrightarrow \sqrt{\frac{6d}{\pi}}$

Thm 2. $\boxed{M_n^d = V_d\, n^{d-1}(1+o(1))}$

$M_n^d(v)$ = max # of lattice points in K_n between two hyperplanes orthogonal to v and at distance $\|v\|$,

$S(v)$ is the part of K_n between these hyperplanes alternative definition:

$$\boxed{M_n^d = \max\left\{ M_n^d(v) \;:\; \|v\|_2 = 1 \right\}}$$

Here

should be

$$M_n^d(v) = \underbrace{\|v\|_1 \operatorname{vol}_{d-1}\left([0,1]^d \cap P_v\right)}_{\approx \operatorname{vol} S(v)} n^{d-1}\left(1+o(1)\right)$$

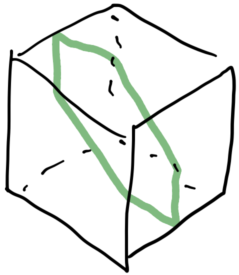
because of a metatheorem:

$$\boxed{\text{for convex } K \quad \# \, \mathbb{Z}^d \cap K \approx \operatorname{vol} K}$$

valid when K is well positioned, that is, when $\operatorname{vol}_d K$ is large and $\operatorname{vol}_{d-1} \operatorname{bd} K$ is small

BUT: this is not the case

$S(v)$ is a very thin slice



$K \subset \mathbb{R}^d$ convex body, $C(z)$ $(z \in \mathbb{Z}^d)$ cell is

inside if $C(z) \subset K$

outside if $C(z) \cap K = \emptyset$

bdry otherwise

$$\boxed{\bigcup_{\text{inside}} C(z) \subset K \subset \bigcup_{\substack{\text{inside,}\\ \text{bdry}}} C(z)} \implies$$

$$\#\underset{\text{inside}}{C(z)} \leq \operatorname{vol} K \leq \#\underset{\text{inside or bdry}}{C(z)}$$

and

$$\#\underset{\text{inside}}{C(z)} \leq \# K \cap \mathbb{Z}^d \leq \#\underset{\text{inside or bdry}}{C(z)}$$

Thm A $\quad \left| \operatorname{vol} K - |K \cap \mathbb{Z}^d| \right| \leq \#$ bdry cells

Given a basis $F = \{f_1, \dots, f_d\}$ of $\mathbb{Z}^d$, an F-cell is a basic parallelotope in basis F

Thm A* $\quad \left| \operatorname{vol} K - |K \cap \mathbb{Z}^d| \right| \leq \#$ bdry F-cells surprise:

Thm B K, L convex bodies in $\mathbb{R}^d$, $K \subset L$

$\Rightarrow$ # bdry cells of K $\leq$ # bdry cells of L

a lattice analogue of

$$\mathrm{vol}_{d-1}\, \partial K \leq \mathrm{vol}_{d-1}\, \partial L$$

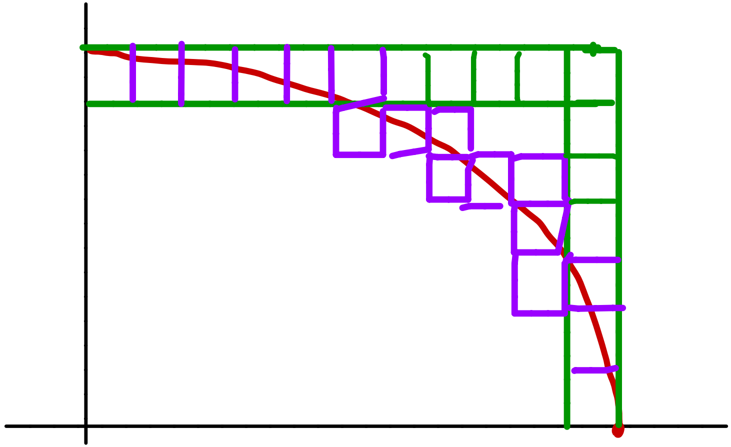
Proof is easy in

2-dim

in $\mathbb{R}^d$ a

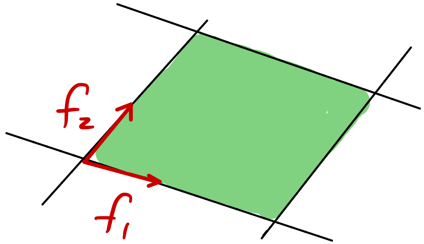
homotopy argument

works $\square$



Next ingredient: Given a basis $F = \{f_1, \dots, f_d\}$ of $\mathbb{Z}^d$

an F-box is



$$B(\underline{\alpha}, \underline{\beta}, F) = \left\{ x = \sum_{i=1}^{d} x_i f_i : \alpha_i \leq x_i \leq \beta_i \ \forall i \right\}$$

$B(K, F)$ = min F-box containing K

Thm C (B. Vershik '92) for every convex body $K \subset \mathbb{R}^d$

$\exists$ basis F of $\mathbb{Z}^d$ such that

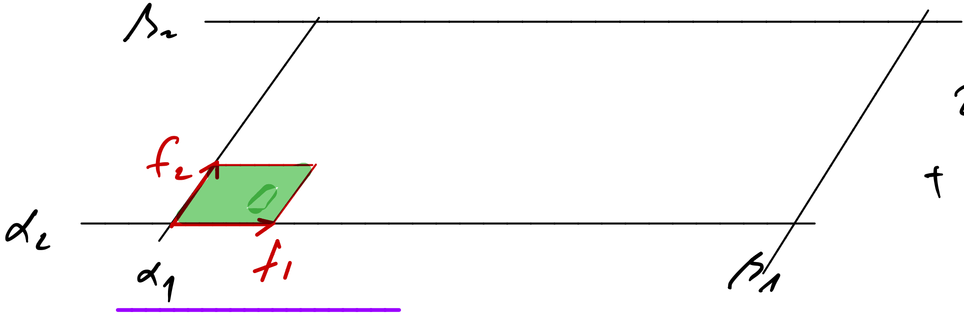
$$\mathrm{vol}\, B(K,F) \ll_d \mathrm{vol}\, K$$

Corollary $K \subset \mathbb{R}^d$ convex, F a basis. Then

$$\#\text{ bdry } F\text{-cells of } K \leq \#\text{ bdry } F\text{-cells of } B(K,F)$$

Advantage: determining the

bdry F-cells of $B(K,F)$ is easy:



$2(\beta_1-\alpha_1)$
$+2(\beta_2-\alpha_2)$

in $\mathbb{R}^3$

$$2(\beta_1-\alpha_1)(\beta_2-\alpha_2)+2(\beta_2-\alpha_2)(\beta_3-\alpha_3)+2(\beta_3-\alpha_3)(\beta_1-\alpha_1)$$

(*) $K \cap \mathbb{Z}^d$ contains $d+1$ affinely independent points

$B(K,F) = B(\alpha, \beta, F)$ $\quad \alpha_i < \beta_i$ $\quad \gamma_i = \beta_i - \alpha_i \geq 1$

integer (assume)

$$\# \text{ bdry } F\text{-cells of } B(K,F) \approx \prod_{i=1}^{d} \gamma_i \left(\frac{1}{\gamma_1} + \dots + \frac{1}{\gamma_d} \right)$$

This is not $B(K,F)$

Thm 3. $K \subset \mathbb{R}^d$ convex, $\exists$ a basis F s.t.

$$\left| \operatorname{vol} K - |K \cap \mathbb{Z}^d| \right| \ll_d \operatorname{vol} K \left(\frac{1}{\gamma_1} + \dots + \frac{1}{\gamma_d} \right)$$

where $\gamma_1, \dots, \gamma_d$ come from the minimal box $B(K, F)$.

K convex in $\mathbb{R}^d$, Λ a lattice in $\mathbb{R}^d$,

K satisfies (*) with $d+1$ points in Λ $\Longrightarrow$

Thm 4. $\exists$ basis F of Λ such that

$$\left| \frac{1}{\det \Lambda} \operatorname{vol} K - |K \cap \Lambda| \right| \ll_d \frac{1}{\det \Lambda} \operatorname{vol} K \left(\frac{1}{\gamma_1} + \dots + \frac{1}{\gamma_d} \right)$$

where $\gamma_1, \dots, \gamma_d$ come from the minimal box $B(K, F)$.

Lower bound on M_d^n via $M_d^n(z)$ with $z \in \mathbb{Z}^d$ fixed. $M_d^n(e_1) = n^{d-1} \Rightarrow$ $M_d^n \geq n^{d-1}$.

$M_d^n(z)$ is reached on $\|z\|_1$ concentric lattice hyperplanes $\perp$ to z, in the lattice $L \subset \mathbb{Z}^d$ with $\det L = \|z\|_2$. Then 4 applies (in $\mathbb{R}^{d-1}$) with $C = [0,n]^d \cap P(z,t)$ $\left(P(z,t) = \{x \in \mathbb{R}^d : z.x = t\} \right)$

$$\left| \frac{1}{\|z\|_2} \mathrm{vol}\, K - |C \cap \mathbb{Z}^d| \right| \ll \frac{1}{\|z\|_2} \mathrm{vol}_{d-1} C \left(\frac{1}{\gamma_1} + \dots + \frac{1}{\gamma_{d-1}} \right)$$

$$= O(n^{d-2})$$

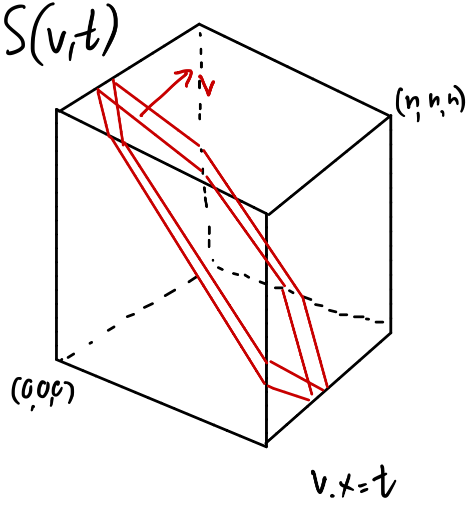
and $\mathrm{vol}_{d-1} C = n^{d-1} \mathrm{vol}_{d-1} P(z)$

$$\Rightarrow \quad M_d^n(t) \geq \underbrace{\frac{\|z\|_1}{\|z\|_2} \mathrm{vol}_{d-1} P(z)}_{V_d(z) \longrightarrow V_d} \, n^{d-1} \left(1 + O\left(\frac{1}{n}\right) \right) \qquad \square$$

$z = (1, 1, \dots, 1)$ ...

Upper bound

thin slice



target

$$\boxed{\left|S(v,t)\cap \mathbb{Z}^d\right| \le (V_d+\varepsilon)\, n^{d-1}}$$

maximiser

$(\varepsilon > 0 \text{ fixed})$

$$S(v,t) = S(v_n,t_n) = S_n$$

consider the basis $F = \{f_1, \dots, f_d\}$ from Thm B

and the minimal box $B(S_n, F)$ $(F = F_n)$.

By Thm 4

$$\boxed{\left| \operatorname{vol} S_n - |S_n \cap \mathbb{Z}^d| \right| \ll \operatorname{vol} S_n \left(\frac{1}{\gamma_1} + \dots + \frac{1}{\gamma_d} \right)}$$

where $\gamma_1, \dots, \gamma_d \geqq 1$ are integers, $\gamma_i = \gamma_i(n)$

Simple case $\quad \gamma_i(n) \to \infty \quad \forall i \in [d]$

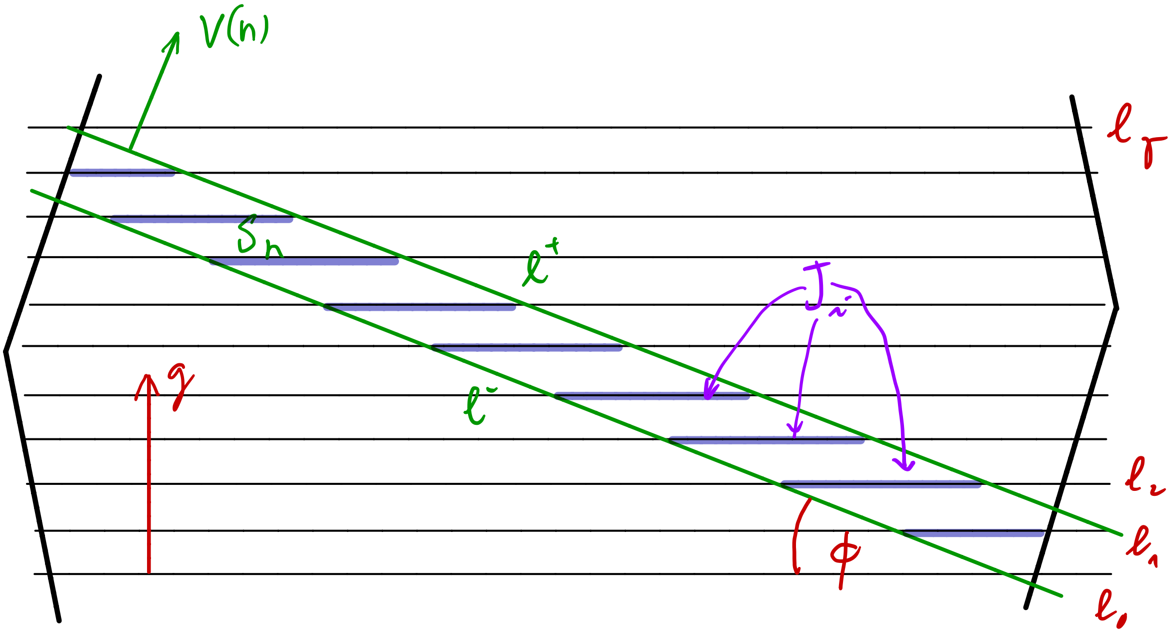
If not, some $\gamma_i(n)$ is bounded along a subsequence, then the corresponding dual basis vector $g_i(n)$ is fixed along another subsequence. $i = 1$ say.

So $\quad \gamma = \gamma_1(n) = \text{const} \quad$ and $\quad g = g_1(n) = \text{const}.$

$$P_n = P := \operatorname{span}\{v(n), g\} \quad - \text{2-dim plane}$$

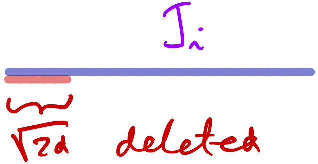
Project K_n and S_n to P

$$\Pi_n = \Pi : \mathbb{R}^d \to P$$



$V(n)$
S_n
ℓ^+
ℓ^-
J_i
g
ϕ
ℓ_2
ℓ_1
ℓ_0

$\phi = \phi_n$ tends to zero

$J_i^* =$ 

Lemma A vertical line intersects at most $|g|_1 + 1$ segments J_i, and at most $|g|_1$ segments J_i^*.

# of lattice points in S_n =

$$= \sum_{i=1}^{r} \# \text{ lattice points in } \underbrace{\pi^{-1}(J_i) \cap K_n}_{\text{a polytope !!}}$$

$$\approx \sum_{i=1}^{r} \mathrm{vol}_{d-1}\left(\pi^{-1}(J_i) \cap K_n\right) \quad \cdots\cdots$$

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a technical proof. $\square$

Question:

How many lines are needed to hit all the cells of an $n \times n$ chessboard?

n always suffice

Thanks!