Algebraic equations for diagonals of bivariate rational functions

Alin Bostan[1] Louis Dumont[1] Bruno Salvy[2]

[1]INRIA, SpecFun

[2]INRIA, AriC

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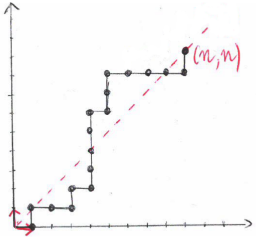
Introduction

Motivations

In combinatorics:
Lattice walks



$w_{n,m}$: number of walks that end at (n, m) $\longrightarrow$ generating power series: $W(x, y) = \sum_{n,m\geq 0} w_{n,m}x^n y^m$

$w_{n,n}$: number of walks that end on the diagonal at (n, n) $\longrightarrow$ diagonal series: $\mathrm{diag}\, W(x) = \sum_{n\geq 0} w_{n,n}x^n$

Diagonals also appear in statistical physics, number theory,...

Diagonals

$$F(x,y) = \frac{1}{1-x-y} = \sum_{n,m\geq 0} \binom{n+m}{n} x^n y^m$$

$$\mathrm{diag}(F) = \sum_{n\geq 0} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}}$$

1	1	1	1	1
1	2	3	4	5
1	3	6	10	15
1	4	10	20	35
1	5	15	35	70

$\mathrm{diag}(F)$ is algebraic: $(1-4x)\mathrm{diag}(F)^2 - 1 = 0$.

Theorem (Furstenberg, 1967)

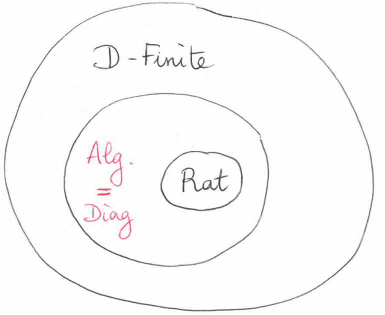
- *Diagonals of bivariate rational functions are algebraic*
- *Any algebraic univariate power series can be expressed as the diagonal of a bivariate rational function*

Hierarchy of univariate power series

Definition

A series $F(x) = \sum_{n \geq 0} u_n x^n$ is D-finite when

$$\sum_{i=0}^{r} p_i(x) \frac{d^i}{dx^i} F(x) = 0, \quad p_i \in \mathbb{Q}[x]$$



Abel ($\approx$ 1830): Alg $\subset$ D-Finite
BoChLeSaSc (2007): Alg $\subset$ D-Finite efficiently
Furstenberg (1967): Diag $=$ Alg

Aim of the talk

We have seen three data structures for a diagonal:

1. the rational function defining the diagonal;
2. a polynomial equation satisfied by the diagonal;
3. a differential equation satisfied by the diagonal.

We will:

- study the algorithmic change from the rational data structure to the algebraic one;
- show that the differential equation is a better data structure for the problem of expanding the diagonal series.

Today's guest star example

- $d \in \mathbb{N}$
- Step set $\mathfrak{S} = \{(1, i), 0 \leq i \leq d\} \cup \{(i, 1), 0 \leq i \leq d\}$
- $w_{n,m}$: number of walks with steps in $\mathfrak{S}$ that end at (n, m)

Proposition

$$W(x, y) = \sum_{n,m \geq 0} w_{n,m} x^n y^m = \frac{1}{1 - \sum_{(i,j) \in \mathfrak{S}} x^i y^j}$$

Problem

Let N be a non-negative integer. Compute the expansion of $\operatorname{diag} W(x)$ at order N:

$$\operatorname{diag} W(x) = \sum_{n=0}^{N} w_{n,n} x^n + O(x^{N+1})$$

Two classical methods

Common strategy:

1. Compute the power series expansion of $W(x, y)$
2. Keep the diagonal, throw away the rest

Two classical methods

- Linear recurrence with constant coefficients:

$$w_{n,m} = \sum_{(i,j)\in\mathfrak{S}} w_{n-i,m-j}$$

Resulting complexity: $\mathcal{O}(dN^2)$ arithmetical operations

- Newton iteration

Resulting complexity: $\tilde{\mathcal{O}}(N^2)$ arith. ops.

A linear complexity algorithm

Strategy:

1. Compute a polynomial that cancels the diagonal
2. Deduce a differential equation that cancels the diagonal
3. Deduce a linear recurrence with polynomial coefficients for $w_{n,n}$
4. Compute enough initial conditions using one of the elementary methods.
5. Compute the desired amount of terms using the recurrence

Resulting complexity: $\mathcal{O}(N)$
Complexity of the pre-processing: ???

Algebraic equations for diagonals

Theorem (Polya (1921), Furstenberg (1967))

The diagonal of a bivariate rational function is algebraic.

- $G(x,y) = \frac{1}{y}F(x/y,y) \to \mathrm{diag}(F) = [y^{-1}]G(x,y)$
- $y_1(x),\ldots,y_r(x)$: distinct poles of $G(x,y) \in \mathbb{Q}(x)(y)$
- $\alpha_1(x),\ldots,\alpha_r(x)$: residues of G at the $y_i's$

$$\mathrm{diag}F(x) = \sum_{\lim_{x\to 0} y_i(x)=0} \alpha_i(x)$$

The y_i's whose limit is 0 at 0 are called the *small branches* of G.

Example

$d = 0$, steps (1,0) and (0,1).

$$F(x,y) = \frac{1}{1-x-y} \longrightarrow G(x,y) = \frac{1}{y-x-y^2}$$

roots of the denominator of G:

$$x_1 = \frac{1-\sqrt{1-4x}}{2}, \quad x_2 = \frac{1+\sqrt{1-4x}}{2}$$

residue at x_1:

$$\mathrm{diag}(F) = \frac{1}{1-2x_1} = \frac{1}{\sqrt{1-4x}}$$

Algebraic equations for diagonals

reminder : the α_i's are the residues of $G(x, y) = \frac{1}{y} F(x/y, y)$

We divide the problem of finding an algebraic equation for the diagonal into three subproblems:

1. compute the polynomial $R = \prod_{i=1}^{r} (y - \alpha_i(x)) \in \mathbb{Q}(x)[y]$
2. compute the number c of small branches of G
3. R being known, compute the polynomial $\Sigma_c R$ defined by

$$\Sigma_c R = \prod_{i_1 < \ldots < i_c} (y - (\alpha_{i_1} + \ldots + \alpha_{i_c})) \in \mathbb{Q}(x)[y]$$

First step: polynomial cancelling the residues of G

Write $G(x, y) = P(x, y)/Q(x, y)$.

1. If y_i is a **simple** pole, then $\alpha_i = \frac{P(x,y_i)}{Q_y(x,y_i)}$. α_i is cancelled by the Rothstein-Trager resultant:

$$\mathrm{Res}_z(Q_y(x, y)z - P(x, y), Q(x, y))$$

2. if y_i is a **multiple** pole: Bronstein resultants

Third step: Polynomial cancelling the sums of residues

Main tool: Newton sums

Definition

Let $R = a\prod_{i=1}^{r}(x-\alpha_i) \in \mathbb{Q}[x]$ be a polynomial. On définit la série génératrice des sommes de Newton de R par:

$$\mathcal{N}(R) = \sum_{n\geq 0}(\alpha_1^n + \ldots + \alpha_r^n)x^n$$

si $R = a_0 + a_1x + \ldots + a_rx^r$, on note $\mathrm{rec}(R) = a_0x^r + a_1x^{r-1} + \ldots + a_r$.

Proposition

Let $R \in \mathbb{Q}[x]$ be a polynomial of degree r. Then

- $\mathcal{N}(R) = \mathrm{rec}(R')/\mathrm{rec}(R)$
- $\mathrm{rec}(R) = \exp\left(\int \frac{r-\mathcal{N}(R)}{x}\right)$

Size of the polynomial, complexity of the pre-processing

Theorem (Bostan, D., Salvy (2014))

Let $A/B \in \mathbb{Q}(x,y)$ be a rational function that is not singular at 0, and such that B has bidegree (d_x, d_y). There exists $a(x) \in \mathbb{Q}(x)$ such that, with the same notations as above and $\Phi = a\Sigma_c R$,

- $\Phi \in \mathbb{Q}[x,y]$
- $\Phi(x, \mathrm{diag} F(x)) = 0$
- *Φ has degree at most $\binom{d_x+d_y}{d_x}$ in y*
- *"generically", Φ is irreducible over $\mathbb{Q}(x)$*
- *"generically", Φ is computed in quasi-optimal time*

In particular, the pre-processing of the algebraic equation for the diagonal has a complexity that is exponential in the size of the input rational function.

A linear complexity algorithm
with polynomial time pre-processing

Strategy:

1. Directly compute the minimal differential equation for $\operatorname{diag} W$
2. Deduce a linear recurrence with polynomial coefficients for $w_{n,n}$
3. Compute enough initial conditions using one of the elementary methods.
4. Compute the desired amount of terms using the recurrence

Resulting complexity: $\mathcal{O}(N)$
Pre-processing of polynomial cost in d.

Diagonals satisfy small differential equations

Theorem (Bostan, Chen, Chyzak, Li (2010))

Let $A/B \in \mathbb{Q}(x, y)$ be a rational function such that B has bidegree (d_x, d_y) and

- *the degrees in x and y of A are less than those of B*
- *A is prime to B*
- *B is primitive with respect to y*

Then there exists a differential operator $L(x, \partial_x)$ such that

- $L(x, \partial_x) \cdot \mathrm{diag}(A/B) = 0$
- *L has order at most d_y and degree $\mathcal{O}(d_x d_y^2)$*