Combinatorial functional equations and Jacobi theta functions

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Strategy: Recursive method

Aim: Determine the generating function $F(t)$ for some class of combinatorial objects, e.g. lattice walks, planar maps, trees, permutations etc.

- **Step 1:** Find a recursive decomposition of each object in your class
- **Step 2:** Write functional equations which characterise the generating function $F(t)$
- **Step 3:** Solve the functional equations

STRATEGY: THETA FUNCTION METHOD

Aim: Determine the generating function $F(t)$ for some class of combinatorial objects, e.g. lattice walks, planar maps, trees, permutations etc.

- **Step 1:** Find a recursive decomposition of each object in your class
- **Step 2:** Write functional equations which characterise the generating function $F(t)$
- **Step 3:** Solve the functional equations using theta functions!

Problems solved with this method (so far)

- Quadrant walks [Kurkova, Raschel, 2012] + [Bernardi, Bousquet-Mélou, Raschel, 2017]
- Walks avoiding a quadrant [Raschel, Trotignon, 2019]
- Walks by winding number [E.P., 2020+] (generalising results of [Budd, 2020])
- Six vertex model on 4-valent maps [Kostov, 2000], [E.P., Zinn-Justin, 2020+], [Bousquet-Mélou, E.P., 2020+]
- Properly coloured triangulations [E.P., 2020+], Previously shown to be D-algebraic by Tutte.

JACOBI THETA FUNCTION

All results are in terms of the series:

$$T_k(u,q) = \sum_{n=0}^{\infty}(-1)^n(2n+1)^k q^{n(n+1)/2}(u^{n+1} - (-1)^k u^{-n})$$
$$= (u \pm 1) - 3^k q(u^2 \pm u^{-1}) + 5^k q^3(u^3 \pm u^{-2}) + O(q^6).$$

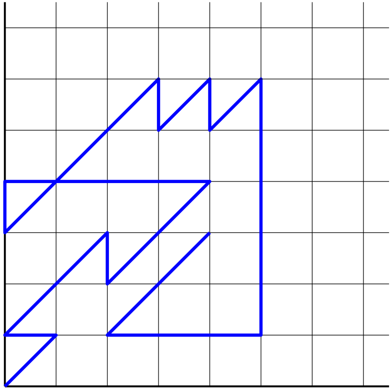
Related to Jacobi Theta function $\vartheta(z,\tau) \equiv \vartheta_{11}(z,\tau)$ by

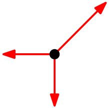
$$\vartheta^{(k)}(z,\tau) \equiv \left(\frac{\partial}{\partial z}\right)^k \vartheta(z,\tau) = e^{\frac{(\pi\tau-2z)i}{2}} i^k T_k(e^{2iz}, e^{2i\pi\tau}).$$

Nice properties of $\vartheta(z) \equiv \vartheta(z,\tau)$:

- $\vartheta(z) = -\vartheta(z+\pi) = -\vartheta(-z) = -e^{(2z+\pi\tau)i}\vartheta(z+\pi\tau).$
- holomorphic, zeros only when $z \in \pi\mathbb{Z} + \pi\tau\mathbb{Z}$.
- differentially algebraic.

PREVIEW: KREWERAS EXCURSIONS





$$\mathsf{Q}(t) := \sum_{\substack{\text{paths from}\\ (0,0) \text{ to } (0,0)}} t^{\#\text{steps}}.$$

PREVIEW: KREWERAS EXCURSIONS IN QUADRANT

Define $T_k(u,q) = \sum_{n=0}^{\infty}(-1)^n(2n+1)^k q^{n(n+1)/2}(u^{n+1}-(-1)^k u^{-n})$

$$= (u \pm 1) - 3^k q(u^2 \pm u^{-1}) + 5^k q^3(u^3 \pm u^{-2}) + O(q^6).$$

Let $q(t) \equiv q = t^3 + 15t^6 + 279t^9 + \cdots$ satisfy

$$t = q^{1/3}\frac{T_1(1,q^3)}{4T_0(q,q^3)+6T_1(q,q^3)}.$$

The gf for Kreweras excursions (in the quadrant) is:

$$\mathsf{Q}(t) = -\frac{q^{-\frac{2}{3}}}{t}\frac{T_0(q,q^3)^2}{T_1(1,q^3)^2}\left(\frac{T_1(q,q^3)^2}{T_0(q,q^3)^2} - \frac{T_2(q,q^3)}{T_0(q,q^3)} - \frac{T_2(-1,q)}{2T_0(-1,q)} + \frac{T_3(1,q)}{6T_1(1,q)} + \frac{T_3(1,q^3)}{3T_1(1,q^3)}\right).$$

Complexity heirarchy

For a series (or a function) $F(t)$, the following properties satisfy

$$\text{Rational} \Rightarrow \text{Algebraic} \Rightarrow \text{D-finite} \Rightarrow \text{D-Algebraic}:$$

- **Rational:** $F(t) = \frac{P(t)}{Q(t)}$ for polynomials $P(t)$ and $Q(t)$.
- **Algebraic:** $P(F(t)) = 0$ for some non-zero polynomial $P(x)$.
- **D-finite:** $F(t)$ satisfies some non-trivial linear differential equation. E.g.

$$t^3F''(t) + t^2F'(t) + (t+1)F(t) - 1 = 0$$

- **D-algebraic:** $F(t)$ satisfies some non-trivial algebraic differential equation. E.g.

$$t^2F'(t) + F'(t)F(t) + tF(t) = 0$$

The theta function $\vartheta(z, \tau)$ is D-algebraic.

Outline of talk

Part 1: Quadrant walks (with small steps)

- problem and functional equations
- background
- solution to functional equations ([Kurkova, Raschel, 2012] and [Bernardi, Bousquet-Mélou, Raschel, 2017], slighly rephrased)

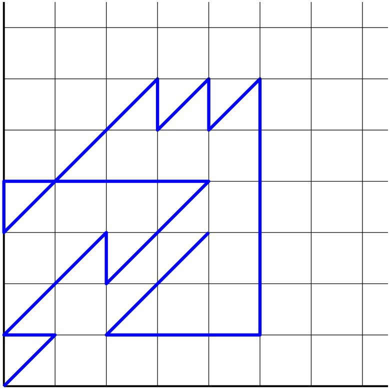
Part 2: Walks by winding angle ([E.P., 2020+], generalising [Budd, 2020])

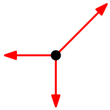
- problem and functional equations
- background
- solution to functional equations

Part 3: Walks in other two-dimensional cones

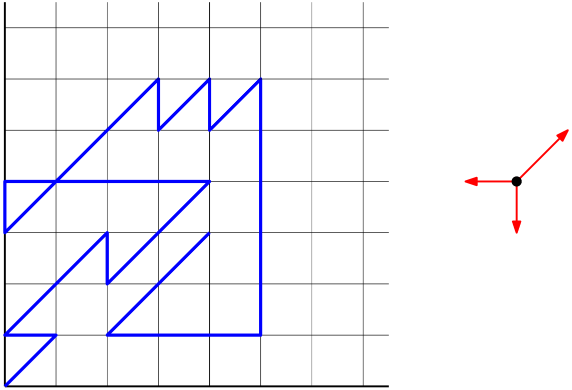
- Certain walks avoiding a quadrant ([Raschel, Trotignon, 2019])
- Corollaries of winding angle results on infinitely many cones.

Part 1: Quadrant walks



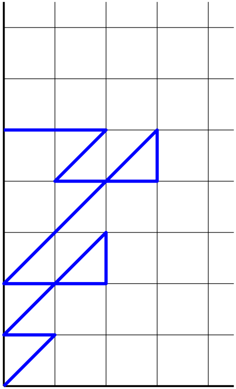


EXAMPLE: KREWERAS PATHS



$$\mathsf{Q}(x,y) \equiv \mathsf{Q}(t,x,y) := \sum_{a,b=0}^{\infty} \sum_{\substack{\text{paths from} \\ (0,0) \text{ to } (a,b)}} t^{\#\text{steps}} x^a y^b.$$

KREWERAS PATHS



$$Q(x,y) = 1 + xytQ(x,y) + \frac{t}{x}(Q(x,y) - Q(0,y)) + \frac{t}{y}(Q(x,y) - Q(x,0))$$

FUNCTIONAL EQUATIONS FOR QUADRANT WALKS

Kreweras paths: The generating function $\mathsf{Q}(t,x,y) \equiv \mathsf{Q}(x,y)$ is characterised by

$$\mathsf{Q}(x,y) = 1 + xyt\mathsf{Q}(x,y) + \frac{t}{x}\left(\mathsf{Q}(x,y) - \mathsf{Q}(0,y)\right) + \frac{t}{y}\left(\mathsf{Q}(x,y) - \mathsf{Q}(x,0)\right).$$

Aim: Solve this equation

More generally: Take a step set $S \subset \{-1,0,1\}^2$ and write

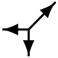
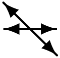
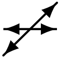
$$P_S(x,y) = \sum_{(i,j)\in S} x^{i+1}y^{j+1}.$$

The generating function $\mathsf{Q}_S(t,x,y) \equiv \mathsf{Q}(x,y)$ is characterised by

$$\begin{aligned} xy\mathsf{Q}(x,y) = xy + tP_S(x,y)\mathsf{Q}(x,y) - tP_S(0,y)\mathsf{Q}(0,y) \\ - tP_S(x,0)\mathsf{Q}(x,0) + tP_S(0,0)\mathsf{Q}(0,0). \end{aligned}$$

Quadrant walks background

Ad-hoc methods for specific cases, e.g.:

-  [Kreweras, 1965], [Gessel, 1986]
-  [Gouyou-Beauchamps, 1986]
-  [Kauers, Koutschan and Zeilberger, 2009], [Bostan, Kauers, 2010]

Systematic methods:

- Algebraic using group of the walk: 22 D-finite cases solved [Bousquet-Mélou, Mishna, 2010]
- Computer algebra: All 23 D-finite cases solved explicitly [Bostan, Chyzak, Van Hoeij, Kauers, Pech, 2017]
- Complex analysis: Solutions as integral expressions in all cases [Kurkova, Raschel, 2012], 9 D-algebraic (non D-finite) cases solved [Bernardi, Bousquet-Mélou, Raschel, 2017]

Quadrant walks background

In total: 79 different non-trivial step sets S

Generating function $\mathsf{Q}(t,x,y)$ is

- Algebraic in 4 cases (Satifies algebraic equation)
- D-finite in 19 further cases (Satifies linear differential equation)
- D-algebraic in 9 further cases (Satifies algebraic differential equation)

Remaining 47 cases are not D-algebraic [Dreyfus, Hardouin, Roques, Singer, 2020].

Solutions for quadrant walks

(using Jacobi theta functions)

[Kurkova, Raschel, 2012] + [Bernardi, Bousquet-Mélou, Raschel, 2017]

Quadrant walks solution

Recall: $S \subset \{-1,0,1\}^2$ and $P_S(x,y) = \sum_{(i,j)\in S} x^{i+1}y^{j+1}$.

The g.f. $\mathsf{Q}_S(t,x,y) \equiv \mathsf{Q}(x,y)$ is characterised by
$K(x,y)\mathsf{Q}(x,y) = R(x,y)$, where

$$K(x,y) = xy - tP_S(x,y)$$
$$R(x,y) = xy - tP_S(0,y)\mathsf{Q}(0,y) - tP_S(x,0)\mathsf{Q}(x,0) + tP_S(0,0)\mathsf{Q}(0,0).$$

Plan: Step 1: Fix $t \in [0,1/9)$. All series converge for $|x|,|y| < 1$.
Step 2: Find functions $X(z)$, $Y(z)$ satisfying $K(X(z),Y(z)) = 0$, as then $R(X(z),Y(z)) = 0$. **Step 3:** Consider functional equations with variable z. **Step 4:** Solve the new equations

QUADRANT WALKS: PARAMETRISING KERNEL

In general*: For fixed t (small), $K(x,y)=0$ is parameterised by

$$X(z)=c_1\frac{\vartheta(z-\alpha_1)\vartheta(\gamma_1-z-\alpha_1)}{\vartheta(z-\beta_1)\vartheta(\gamma_1-z-\beta_1)}\quad\text{and}\quad Y(z)=c_2\frac{\vartheta(z-\alpha_2)\vartheta(\gamma_2-z-\alpha_2)}{\vartheta(z-\beta_2)\vartheta(\gamma_2-z-\beta_2)},$$

where

$$\vartheta(z)\equiv\vartheta(z,\tau):=\sum_{n=-\infty}^{\infty}(-1)^n e^{\left(\frac{2n+1}{2}\right)^2 i\pi\tau+(2n+1)iz},$$

and $\tau, c_1, \alpha_1, \beta_1, \gamma_1, c_2, \alpha_2, \beta_2, \gamma_2 \in \mathbb{C}$ depend only on t.

Properties of $X(z)$:

- $X(z)=X(z+\pi)=X(z+\pi\tau)=X(\gamma_1-z)$
- Zeros at α_1 and $\gamma_1-\alpha_1$, poles at β_1 and $\gamma_1-\beta_1$

*We are ignoring the five semi-directed cases 

Quadrant walks solution

Let

$$X(z) = c_1 \frac{\vartheta(z-\alpha_1)\vartheta(\gamma_1 - z - \alpha_1)}{\vartheta(z-\beta_1)\vartheta(\gamma_1 - z - \beta_1)} \quad \text{and} \quad Y(z) = c_2 \frac{\vartheta(z-\alpha_2)\vartheta(\gamma_2 - z - \alpha_2)}{\vartheta(z-\beta_2)\vartheta(\gamma_2 - z - \beta_2)}.$$

Equation to solve for $\mathsf{Q}(x,y)$:

$$\mathsf{Q}(x,y)K(x,y) = R(x,y),$$

where

$$R(x,y) = xy - tP_S(0,y)\mathsf{Q}(0,y) - tP_S(x,0)\mathsf{Q}(x,0) + c.$$

and $K(X(z),Y(z)) = 0$.

QUADRANT WALKS SOLUTION

Let

$$X(z) = c_1 \frac{\vartheta(z-\alpha_1)\vartheta(\gamma_1 - z - \alpha_1)}{\vartheta(z-\beta_1)\vartheta(\gamma_1 - z - \beta_1)} \quad \text{and} \quad Y(z) = c_2 \frac{\vartheta(z-\alpha_2)\vartheta(\gamma_2 - z - \alpha_2)}{\vartheta(z-\beta_2)\vartheta(\gamma_2 - z - \beta_2)}.$$

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where

$$R(x,y) = xy - tP_S(0,y)\mathsf{Q}(0,y) - tP_S(x,0)\mathsf{Q}(x,0) + c.$$

and $K(X(z), Y(z)) = 0$.

Equation to solve for $\mathsf{Q}(x,0)$ and $\mathsf{Q}(0,y)$:

$$R(X(z), Y(z)) = 0.$$

QUADRANT WALKS SOLUTION

Let

$$X(z) = c_1 \frac{\vartheta(z-\alpha_1)\vartheta(\gamma_1 - z - \alpha_1)}{\vartheta(z-\beta_1)\vartheta(\gamma_1 - z - \beta_1)} \quad \text{and} \quad Y(z) = c_2 \frac{\vartheta(z-\alpha_2)\vartheta(\gamma_2 - z - \alpha_2)}{\vartheta(z-\beta_2)\vartheta(\gamma_2 - z - \beta_2)}.$$

Equation to solve for $\mathsf{Q}(x,0)$ and $\mathsf{Q}(0,y)$:

$$X(z)Y(z) - tP_S(0,Y(z))\mathsf{Q}(0,Y(z)) - tP_S(X(z),0)\mathsf{Q}(X(z),0) + c = 0$$

QUADRANT WALKS SOLUTION

Let

$$X(z) = c_1 \frac{\vartheta(z-\alpha_1)\vartheta(\gamma_1 - z - \alpha_1)}{\vartheta(z-\beta_1)\vartheta(\gamma_1 - z - \beta_1)} \quad \text{and} \quad Y(z) = c_2 \frac{\vartheta(z-\alpha_2)\vartheta(\gamma_2 - z - \alpha_2)}{\vartheta(z-\beta_2)\vartheta(\gamma_2 - z - \beta_2)}.$$

Equation to solve for $\mathsf{Q}(x,0)$ and $\mathsf{Q}(0,y)$:

$$X(z)Y(z) - tP_S(0,Y(z))\mathsf{Q}(0,Y(z)) - tP_S(X(z),0)\mathsf{Q}(X(z),0) + c = 0$$

Write $\mathsf{Q}_1(z) := \mathsf{Q}(X(z),0)$ and $\mathsf{Q}_2(z) := \mathsf{Q}(0,Y(z))$.

Equation to solve for $\mathsf{Q}_1(z)$ and $\mathsf{Q}_2(z)$:

$$X(z)Y(z) - tP_S(0,Y(z))\mathsf{Q}_2(z) - tP_S(X(z),0)\mathsf{Q}_1(z) + c = 0.$$

QUADRANT WALKS SOLUTION

Let

$$X(z) = c_1 \frac{\vartheta(z-\alpha_1)\vartheta(\gamma_1 - z - \alpha_1)}{\vartheta(z-\beta_1)\vartheta(\gamma_1 - z - \beta_1)} \quad \text{and} \quad Y(z) = c_2 \frac{\vartheta(z-\alpha_2)\vartheta(\gamma_2 - z - \alpha_2)}{\vartheta(z-\beta_2)\vartheta(\gamma_2 - z - \beta_2)}.$$

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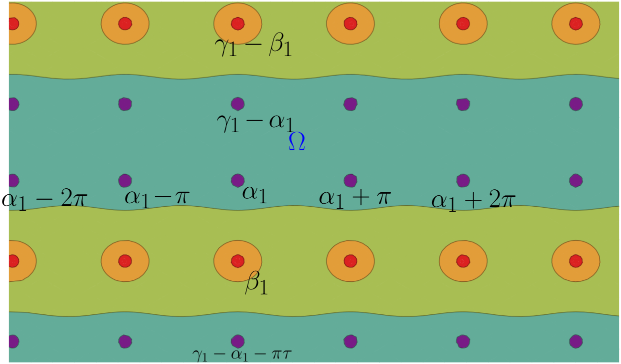
Equation to solve for $\mathsf{Q}_1(z)$ and $\mathsf{Q}_2(z)$:

$$X(z)Y(z) - tP_S(0,Y(z))\mathsf{Q}_2(z) - tP_S(X(z),0)\mathsf{Q}_1(z) + c = 0.$$

What information are we missing??

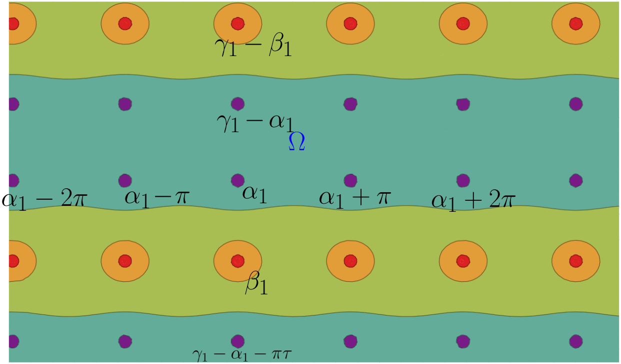
UNDERSTANDING $\mathsf{Q}(X(z), 0)$

Plot of $\left\{z : |X(z)| \in \left[0, \frac{1}{3}\right), \left(\frac{1}{3}, 1\right), (1,3), (3,9), (9,\infty]\right\}$.



For $z \in \Omega$, $|X(z)| < 1 \Rightarrow \mathsf{Q}_1(z) = \mathsf{Q}(X(z), 0)$ is well defined.

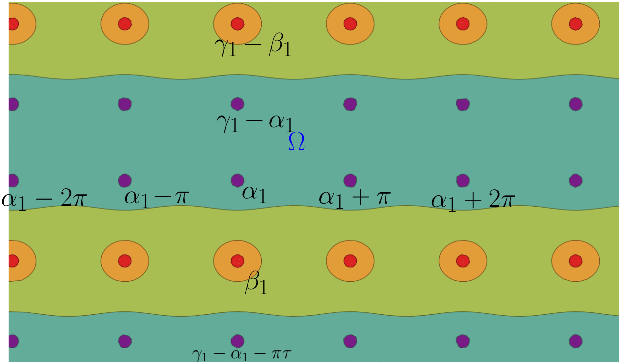
Plot of $\left\{z : |X(z)| \in \left[0, \frac{1}{3}\right), \left(\frac{1}{3}, 1\right), (1,3), (3,9), (9,\infty]\right\}$.



For $z \in \Omega$, $|X(z)| < 1 \Rightarrow \mathsf{Q}_1(z) = \mathsf{Q}(X(z),0)$ is well defined.
Moreover, $\mathsf{Q}_1(z) = \mathsf{Q}(X(z),0) = \mathsf{Q}(X(\gamma_1 - z),0) = \mathsf{Q}_1(\gamma_1 - z)$.

UNDERSTANDING $\mathsf{Q}(X(z), 0)$

Plot of $\left\{ z : |X(z)| \in \left[0, \frac{1}{3}\right), \left(\frac{1}{3}, 1\right), (1,3), (3,9), (9,\infty] \right\}$.



For $z \in \Omega$, $|X(z)| < 1 \Rightarrow \mathsf{Q}_1(z) = \mathsf{Q}(X(z), 0)$ is well defined.
Moreover, $\mathsf{Q}_1(z) = \mathsf{Q}(X(z), 0) = \mathsf{Q}(X(\gamma_1 - z), 0) = \mathsf{Q}_1(\gamma_1 - z)$.
Similarly, $\mathsf{Q}_2(z) = \mathsf{Q}_2(\gamma_2 - z)$.

Quadrant walks solution

Equation to solve for $\mathsf{Q}_1(z)$ and $\mathsf{Q}_2(z)$:

$$X(z)Y(z) - tP_S(0, Y(z))\mathsf{Q}_2(z) - tP_S(X(z), 0)\mathsf{Q}_1(z) + c = 0,$$

assuming $\mathsf{Q}_1(z) = \mathsf{Q}_1(\gamma_1 - z)$ and $\mathsf{Q}_2(z) = \mathsf{Q}_2(\gamma_2 - z)$.

QUADRANT WALKS SOLUTION

Equation to solve for $\mathsf{Q}_1(z)$ and $\mathsf{Q}_2(z)$:

$$X(z)Y(z) - tP_S(0, Y(z))\mathsf{Q}_2(z) - tP_S(X(z), 0)\mathsf{Q}_1(z) + c = 0,$$

assuming $\mathsf{Q}_1(z) = \mathsf{Q}_1(\gamma_1 - z)$ and $\mathsf{Q}_2(z) = \mathsf{Q}_2(\gamma_2 - z)$.

Simplify further: define

$$A(z) := tP_S(0, Y(z))\mathsf{Q}(0, Y(z))$$

and

$$B(z) := tP_S(X(z), 0)\mathsf{Q}(X(z), 0) - c.$$

Equation to solve for $A(z)$ and $B(z)$:

$$X(z)Y(z) = A(z) + B(z),$$

where $A(z) = A(\gamma_1 - z)$ and $B(z) = B(\gamma_2 - z)$.

Solution in general

Equation to solve for $A(z)$ and $B(z)$:

$$X(z)Y(z) = A(z) + B(z)$$

where

- $X(z)$ and $A(z)$ are fixed under $z \to \gamma_1 - z$.
- $Y(z)$ and $B(z)$ are fixed under $z \to \gamma_2 - z$.

Solution in D-finite cases ($\gamma_1 - \gamma_2 \in \pi\tau\mathbb{Q}$)

Equation to solve for $A(z)$ and $B(z)$:

$$X(z)Y(z) = A(z) + B(z)$$

where

- $X(z)$ and $A(z)$ are fixed under $z \to \gamma_1 - z$.
- $Y(z)$ and $B(z)$ are fixed under $z \to \gamma_2 - z$.

SOLUTION IN D-FINITE CASES ($\gamma_1 - \gamma_2 \in \pi\tau\mathbb{Q}$)

Equation to solve for $A(z)$ and $B(z)$:

$$X(z)Y(z) = A(z) + B(z)$$

$$(X(z) - X(\gamma_2 - z))Y(z) = A(z) - A(\gamma_2 - z)$$

where

- $X(z)$ and $A(z)$ are fixed under $z \to \gamma_1 - z$.
- $Y(z)$ and $B(z)$ are fixed under $z \to \gamma_2 - z$.

Solution in D-finite cases ($\gamma_1 - \gamma_2 \in \pi\tau\mathbb{Q}$)

Equation to solve for $A(z)$ and $B(z)$:

$$X(z)Y(z) = A(z) + B(z)$$

$$(X(z) - X(\gamma_2 - z))Y(z) = A(z) - A(\gamma_1 - \gamma_2 + z)$$

where

- $X(z)$ and $A(z)$ are fixed under $z \to \gamma_1 - z$.
- $Y(z)$ and $B(z)$ are fixed under $z \to \gamma_2 - z$.

SOLUTION IN D-FINITE CASES ($\gamma_1 - \gamma_2 \in \pi\tau\mathbb{Q}$)

Equation to solve for $A(z)$:

$$(X(z) - X(\gamma_2 - z))Y(z) = A(z) - A(\gamma_1 - \gamma_2 + z)$$

where

- $X(z)$ and $A(z)$ are fixed under $z \to \gamma_1 - z$.
- $Y(z)$ and $B(z)$ are fixed under $z \to \gamma_2 - z$.

SOLUTION IN D-FINITE CASES ($\gamma_1 - \gamma_2 \in \pi\tau\mathbb{Q}$)

Equation to solve for $A(z)$:

$$(X(z) - X(\gamma_2 - z))Y(z) = A(z) - A(\gamma_1 - \gamma_2 + z)$$

where $X(z)$ and $Y(z)$ have π and $\pi\tau$ as periods.

Solution in D-finite cases ($\gamma_1 - \gamma_2 \in \pi\tau\mathbb{Q}$)

Equation to solve for $A(z)$:

$$(X(z) - X(\gamma_2 - z))Y(z) = A(z) - A(\gamma_1 - \gamma_2 + z)$$

where $X(z)$ and $Y(z)$ have π and $\pi\tau$ as periods.

Since $\gamma_1 - \gamma_2 \in \pi\tau\mathbb{Q}$, we get:

$$F(z) = A(z) - A(n\pi\tau + z),$$

where $F(z)$ has periods π and $\pi\tau$.

Solution in D-finite cases ($\gamma_1 - \gamma_2 \in \pi\tau\mathbb{Q}$)

Equation to solve for $A(z)$:

$$(X(z) - X(\gamma_2 - z))Y(z) = A(z) - A(\gamma_1 - \gamma_2 + z)$$

where $X(z)$ and $Y(z)$ have π and $\pi\tau$ as periods.

Since $\gamma_1 - \gamma_2 \in \pi\tau\mathbb{Q}$, we get:

$$F(z) = A(z) - A(n\pi\tau + z),$$

where $F(z)$ has periods π and $\pi\tau$.
$\rightarrow \quad U(z) = \frac{A(z)}{F(z)}$ satisfies $1 = U(z) - U(z + n\pi\tau)$.
$\rightarrow$ The following all have π and $n\pi\tau$ as periods:

$$U'(z),\ F(z),\ X(z) \text{ and } Y(z),$$

so they are algebraically related using: *Meromorphic functions sharing two (independent) periods are algebraically related*.
It follows that $A(z) = U(z)F(z)$ is D-finite in $X(z)$.

Solution in general

Equation to solve for $A(z)$ and $B(z)$:

$$X(z)Y(z) = A(z) + B(z)$$

where

- $X(z)$ and $A(z)$ are fixed under $z \to \gamma_1 - z$.
- $Y(z)$ and $B(z)$ are fixed under $z \to \gamma_2 - z$.

Solution in D-algebraic (non-D-finite) cases

Equation to solve for $A(z)$ and $B(z)$:

$$X(z)Y(z) = A(z) + B(z)$$

where

- $X(z)$ and $A(z)$ are fixed under $z \to \gamma_1 - z$.
- $Y(z)$ and $B(z)$ are fixed under $z \to \gamma_2 - z$.

In D-algebraic (non D-finite) cases:

- $X(z)Y(z)$ splits as $X(z)Y(z) = R_1(X(z)) + R_2(Y(z))$, for explicit rational functions R_1, R_2.

SOLUTION IN D-ALGEBRAIC (NON-D-FINITE) CASES

Equation to solve for $A(z)$ and $B(z)$:

$$R_1(X(z)) + R_2(Y(z)) = A(z) + B(z)$$

where

- $X(z)$ and $A(z)$ are fixed under $z \to \gamma_1 - z$.
- $Y(z)$ and $B(z)$ are fixed under $z \to \gamma_2 - z$.

In D-algebraic (non D-finite) cases:

- $X(z)Y(z)$ splits as $X(z)Y(z) = R_1(X(z)) + R_2(Y(z))$, for explicit rational functions R_1, R_2.

Solution in D-algebraic (non-D-finite) cases

Equation to solve for $A(z)$ and $B(z)$:

$$I(z) = R_1(X(z)) - A(z) = B(z) - R_2(Y(z))$$

where

- $X(z)$ and $A(z)$ are fixed under $z \to \gamma_1 - z$.
- $Y(z)$ and $B(z)$ are fixed under $z \to \gamma_2 - z$.

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In D-algebraic (non D-finite) cases:

- $X(z)Y(z)$ splits as $X(z)Y(z) = R_1(X(z)) + R_2(Y(z))$, for explicit rational functions R_1, R_2.
- $I(z)$ is fixed under $z \to \gamma_1 - z$ and $z \to \gamma_2 - z$, so it has $\gamma_1 - \gamma_2$ and π as periods.
- We can then solve for $I(z)$.

Solution in D-algebraic (non-D-finite) cases

Equation to solve for $A(z)$ and $B(z)$:

$$I(z) = R_1(X(z)) - A(z) = B(z) - R_2(Y(z))$$

where

- $X(z)$ and $A(z)$ are fixed under $z \to \gamma_1 - z$.
- $Y(z)$ and $B(z)$ are fixed under $z \to \gamma_2 - z$.

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Algebraic cases: $\gamma_1 - \gamma_2 \in \pi\tau\mathbb{Q}$, so $I(z)$ and $X(z)$ share the period $\pi\tau n \Rightarrow$ everything is agebraic in $x \equiv X(z)$

SOLUTION IN D-ALGEBRAIC (NON-D-FINITE) CASES

Equation to solve for $A(z)$ and $B(z)$:

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Algebraic cases: $\gamma_1 - \gamma_2 \in \pi\tau\mathbb{Q}$, so $I(z)$ and $X(z)$ share the period $\pi\tau n \Rightarrow$ everything is agebraic in $x \equiv X(z)$

In general (non D-algebraic cases): [Kurkova, Raschel]

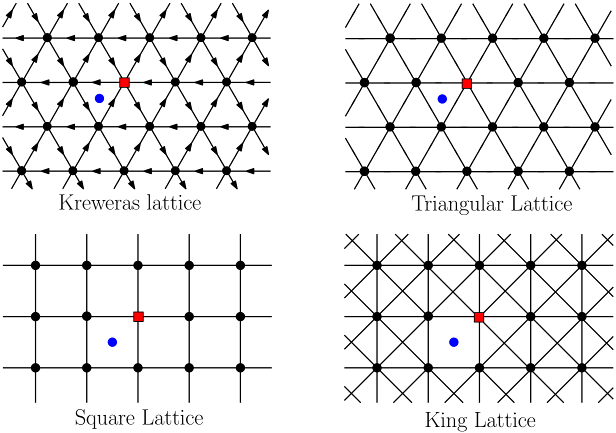
Same idea, but $R_1(X(z))$ and $R_2(X(z))$ are given by integrals.

Part 2: Walks by winding angle

LATTICE WALKS BY WINDING ANGLE

The model: count walks starting at ■ by end point and winding angle around ●.

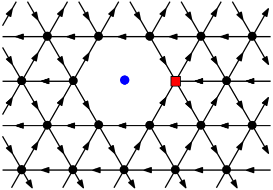
Cell-centred lattices:



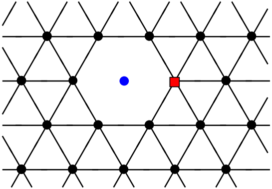
Lattice walks by winding angle

The model: count walks starting at ■ by end point and winding angle around ●.

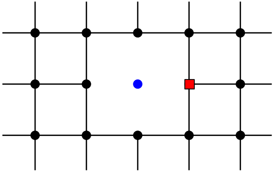
Vertex-centred lattices:



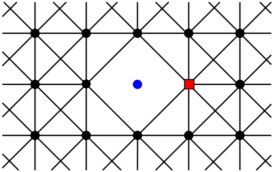
Kreweras lattice



Triangular Lattice



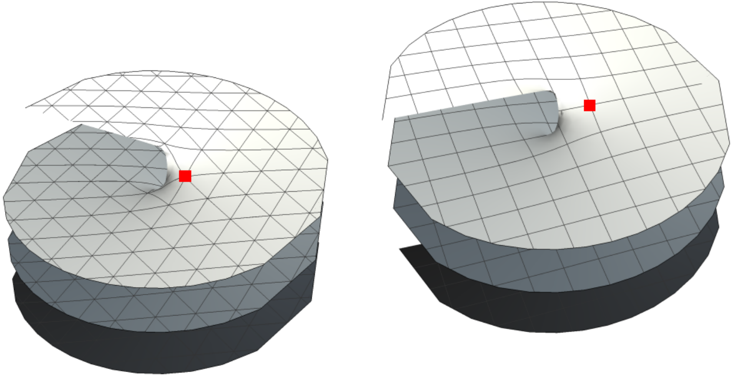
Square Lattice



King Lattice

Lattice walks by winding angle

The model: count walks starting at ■ (by end point).



Left: Cell-centred triangular lattice
Right: Vertex-centred square lattice

Square lattice walks by winding angle

[Timothy Budd, 2017]: enumeration of square lattice walks (starting and ending on an axis or diagonal) by winding angle

- **Method:** Matrices counting paths, eigenvalue decomposition etc.
- **Solution:** Jacobi theta function expressions
- **Corollaries:**
  - Square lattice walks in cones (eg. Gessel walks)
  - Loops around the origin (without a fixed starting point)
  - Algebraicity results, asymptotic results, etc.

Square lattice walks by winding angle

[Timothy Budd, 2017]: enumeration of square lattice walks (starting and ending on an axis or diagonal) by winding angle

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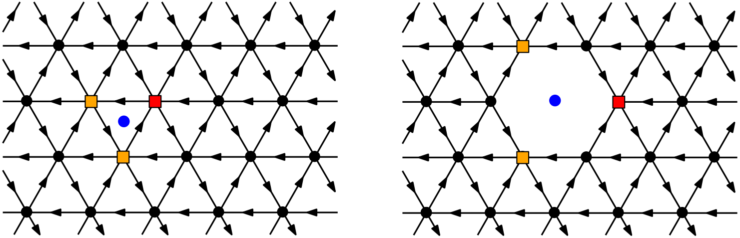
- **Corollaries:**
  - Square lattice walks in cones (eg. Gessel walks)
  - Loops around the origin (without a fixed starting point)
  - Algebraicity results, asymptotic results, etc.

This work:

- Completely different method
- Slightly different set of results
- Extension to three other lattices

PREVIEW: KREWERAS ALMOST-EXCURSIONS



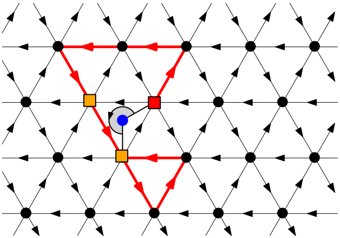
Cell-centred Kreweras lattice   Vertex-centred Kreweras lattice

On each lattice: count walks ■ → (■ or ■). Walks with length n and winding angle $\frac{2\pi k}{3}$ contribute $t^n s^k$.

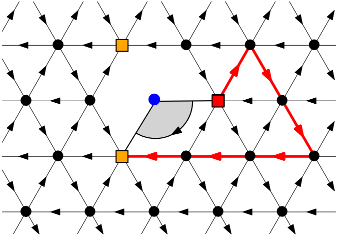
Cell-centred: $E(t,s) = 1 + st + \left(s^2 + s^{-1}\right) t^2 + \dots$

Vertex-centred: $\tilde{E}(t,s) = 1 + \left(s^{-1} + 4 + s\right) t^3 + \dots$

PREVIEW: KREWERAS ALMOST-EXCURSIONS



Cell-centred Kreweras lattice



Vertex-centred Kreweras lattice

On each lattice: count walks ■ $\rightarrow$ (■ or ■). Walks with length n and winding angle $\frac{2\pi k}{3}$ contribute $t^n s^k$.

Cell-centred: $E(t,s) = 1 + st + \left(s^2 + s^{-1}\right) t^2 + \ldots$

Vertex-centred: $\tilde{E}(t,s) = 1 + \left(s^{-1} + 4 + s\right) t^3 + \ldots$

Preview: Kreweras almost-excursions

Define $T_k(u,q) = \sum_{n=0}^{\infty} (-1)^n (2n+1)^k q^{n(n+1)/2} (u^{n+1} - (-1)^k u^{-n})$

$$= (u \pm 1) - 3^k q(u^2 \pm u^{-1}) + 5^k q^3 (u^3 \pm u^{-2}) + O(q^6).$$

Let $q(t) \equiv q = t^3 + 15t^6 + 279t^9 + \cdots$ satisfy

$$t = q^{1/3} \frac{T_1(1,q^3)}{4T_0(q,q^3) + 6T_1(q,q^3)}.$$

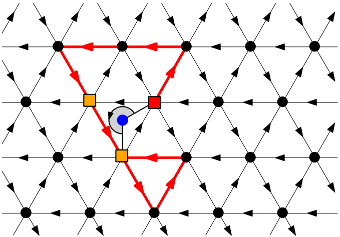
The gf for cell-centred Kreweras-lattice almost-excursions is:

$$E(t,s) = \frac{s}{(1-s^3)t} \left(s - q^{-1/3} \frac{T_1(q^2,q^3)}{T_1(1,q^3)} - q^{-1/3} \frac{T_0(q,q^3) T_1(sq^{-2/3},q)}{T_1(1,q^3) T_0(sq^{-2/3},q)} \right).$$

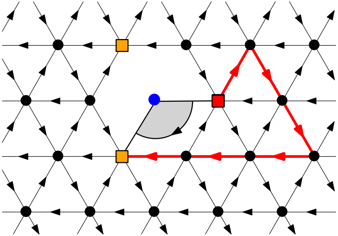
The gf for vertex-centred Kreweras-lattice almost-excursions is:

$$\tilde{E}(t,s) = \frac{s(1-s)q^{-\frac{2}{3}}}{t(1-s^3)} \frac{T_0(q,q^3)^2}{T_1(1,q^3)^2} \left(\frac{T_1(q,q^3)^2}{T_0(q,q^3)^2} - \frac{T_2(q,q^3)}{T_0(q,q^3)} - \frac{T_2(s,q)}{2T_0(s,q)} + \frac{T_3(1,q)}{6T_1(1,q)} + \frac{T_3(1,q^3)}{3T_1(1,q^3)} \right).$$

Part 2a: Functional equations for Kreweras walks by winding angle

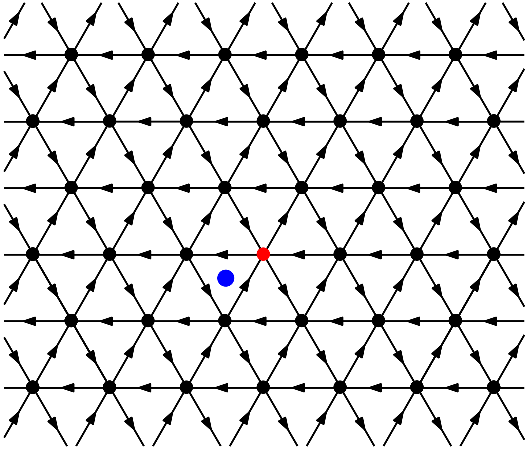


Cell-centred Kreweras lattice



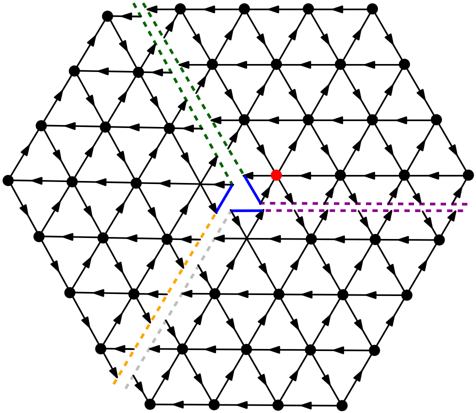
Vertex-centred Kreweras lattice

The model: Count walks starting at the red point by end point and number of times winding around the blue point.



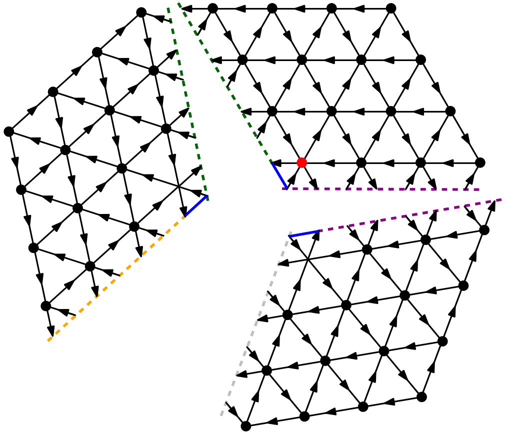
KREWERAS WALKS BY WINDING NUMBER

The model: Count walks starting at the red point by end point.



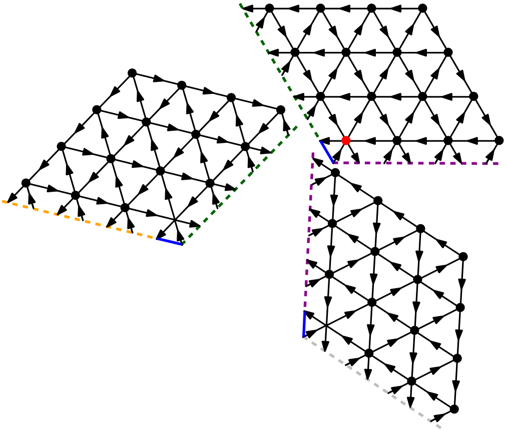
KREWERAS WALKS BY WINDING NUMBER

The model: Count walks starting at the red point by end point.



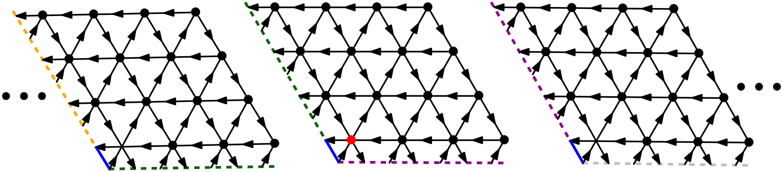
Kreweras walks by winding number

The model: Count walks starting at the red point by end point.



KREWERAS WALKS BY WINDING NUMBER

The model: Count walks starting at the red point by end point.



Kreweras walks by winding number

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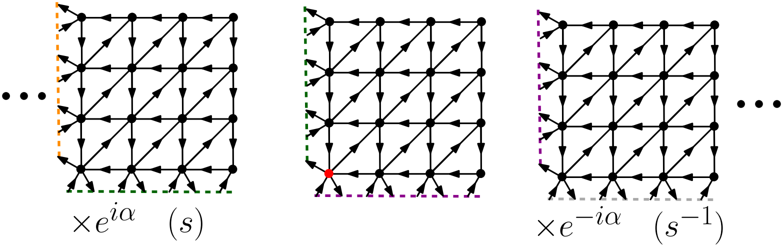
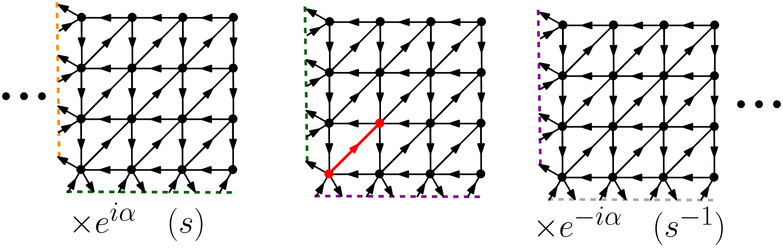


Definition: $Q(t, \alpha, x, y) \equiv Q(x, y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} e^{i\alpha n(p)}$

Note: $Q(0, 0) = E(t, e^{i\alpha})$

The model: Count walks starting at the red point by end point.



This example contributes txy.

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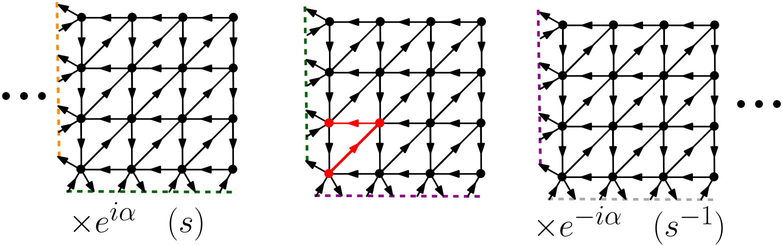
Note: $Q(0,0) = E(t, e^{i\alpha})$

KREWERAS WALKS BY WINDING NUMBER

The model: Count walks starting at the red point by end point.



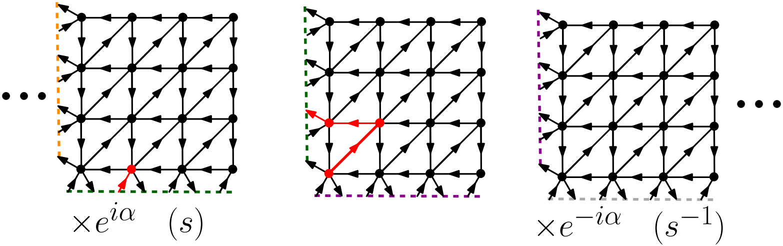
This example contributes $t^2 y$.

Definition: $Q(t, \alpha, x, y) \equiv Q(x, y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} e^{i\alpha n(p)}$

Note: $Q(0, 0) = E(t, e^{i\alpha})$

KREWERAS WALKS BY WINDING NUMBER

The model: Count walks starting at the red point by end point.

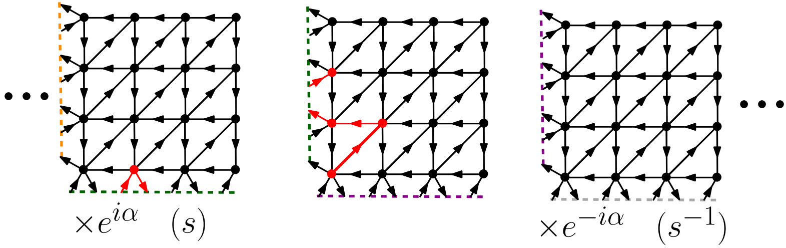


This example contributes $t^3 x e^{i\alpha}$.

Definition: $Q(t,\alpha,x,y) \equiv Q(x,y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} e^{i\alpha n(p)}$

Note: $Q(0,0) = E(t, e^{i\alpha})$

The model: Count walks starting at the red point by end point.



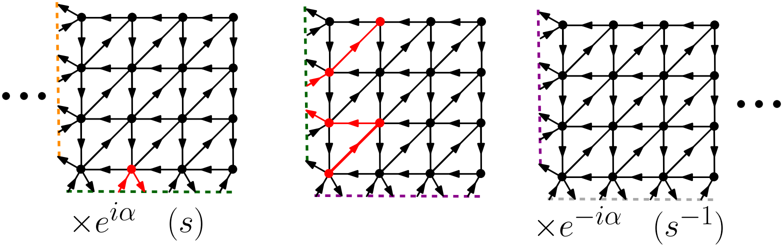
This example contributes $t^4 y^2$.

Definition: $Q(t,\alpha,x,y) \equiv Q(x,y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} e^{i\alpha n(p)}$

Note: $Q(0,0) = E(t, e^{i\alpha})$

KREWERAS WALKS BY WINDING NUMBER

The model: Count walks starting at the red point by end point.

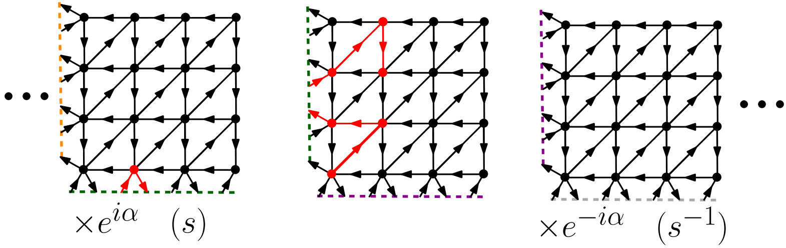


This example contributes t^5xy^3.

Definition: $Q(t,\alpha,x,y) \equiv Q(x,y) = \sum_{\text{paths } p} t^{|p|}x^{x(p)}y^{y(p)}e^{i\alpha n(p)}$

Note: $Q(0,0) = E(t,e^{i\alpha})$

The model: Count walks starting at the red point by end point.

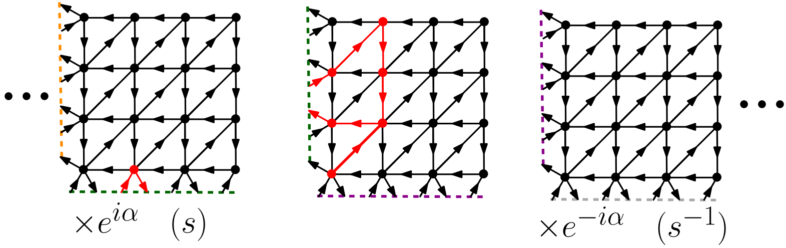


This example contributes t^6xy^2.

Definition: $Q(t,\alpha,x,y) \equiv Q(x,y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} e^{i\alpha n(p)}$

Note: $Q(0,0) = E(t, e^{i\alpha})$

The model: Count walks starting at the red point by end point.



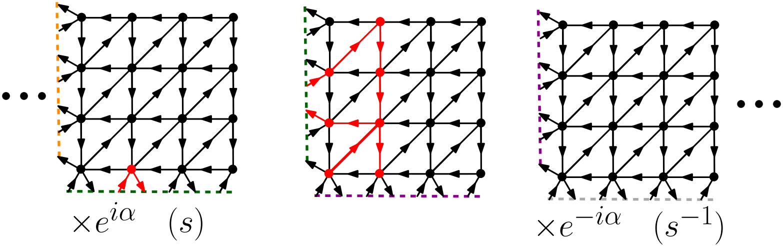
This example contributes t^7xy.

Definition: $Q(t,\alpha,x,y) \equiv Q(x,y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} e^{i\alpha n(p)}$

Note: $Q(0,0) = E(t, e^{i\alpha})$

KREWERAS WALKS BY WINDING NUMBER

The model: Count walks starting at the red point by end point.



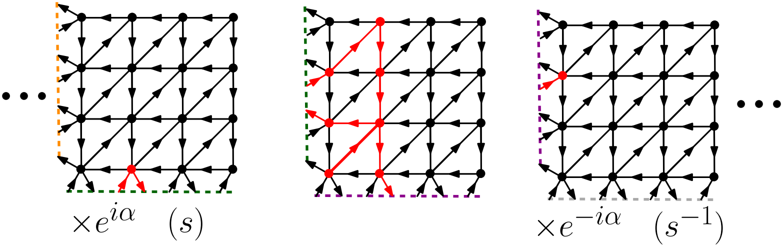
This example contributes t^8x.

Definition: $Q(t,\alpha,x,y) \equiv Q(x,y) = \sum_{\text{paths } p} t^{|p|}x^{x(p)}y^{y(p)}e^{i\alpha n(p)}$

Note: $Q(0,0) = E(t,e^{i\alpha})$

Kreweras walks by winding number

The model: Count walks starting at the red point by end point.

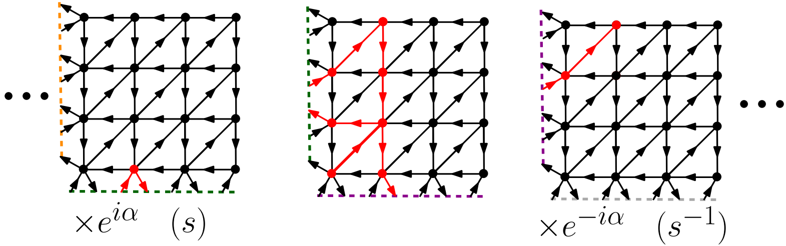


This example contributes $t^9 y^2 e^{-i\alpha}$.

Definition: $Q(t, \alpha, x, y) \equiv Q(x, y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} e^{i\alpha n(p)}$

Note: $Q(0, 0) = E(t, e^{i\alpha})$

The model: Count walks starting at the red point by end point.



This example contributes $t^{10}xy^3e^{-i\alpha}$.

Definition: $Q(t,\alpha,x,y) \equiv Q(x,y) = \sum_{\text{paths } p} t^{|p|}x^{x(p)}y^{y(p)}e^{i\alpha n(p)}$

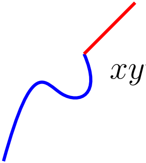
Note: $Q(0,0) = E(t, e^{i\alpha})$

Functional equation

Recursion → functional equation: separate by *type* of final step.

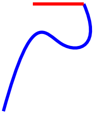
$$Q(x,y) = 1$$

$$+$$



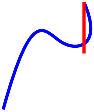
$$xytQ(x,y)$$

$$+$$



$$\frac{t}{x}(Q(x,y) - Q(0,y))$$

$$+$$



$$\frac{t}{y}(Q(x,y) - Q(x,0))$$

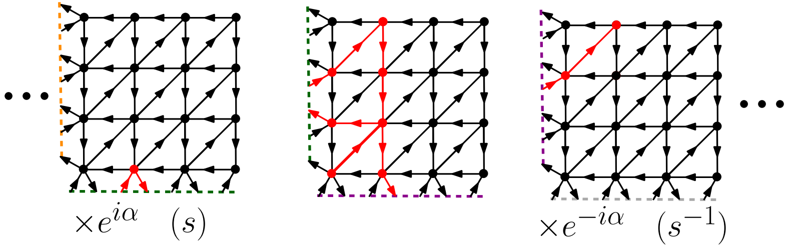
$$+ e^{i\alpha}tQ(0,x)$$

(Final step goes through left wall)

$$+ e^{-i\alpha}tyQ(y,0)$$

(Final step goes through bottom wall)

The model: Count walks starting at the red point by end point.



Definition: $Q(t,\alpha,x,y) \equiv Q(x,y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} e^{i\alpha n(p)}$.

Characterised by:

$$Q(x,y) = 1 + txyQ(x,y) + t\frac{Q(x,y)-Q(0,y)}{x} + t\frac{Q(x,y)-Q(x,0)}{y}$$
$$+e^{i\alpha}tQ(0,x) + e^{-i\alpha}tyQ(y,0).$$

Part 2b: Solution (using theta functions)

Equation to solve:

$$Q(x,y) = 1 + txyQ(x,y) + t\frac{Q(x,y) - Q(0,y)}{x} + t\frac{Q(x,y) - Q(x,0)}{y} + e^{i\alpha}tQ(0,x) + e^{-i\alpha}tyQ(y,0).$$

Equation to solve:

$$Q(x,y) = 1 + txyQ(x,y) + t\frac{Q(x,y) - Q(0,y)}{x} + t\frac{Q(x,y) - Q(x,0)}{y} + e^{i\alpha}tQ(0,x) + e^{-i\alpha}tyQ(y,0).$$

Solution:

Step 1: Fix $t \in [0, 1/3), \alpha \in \mathbb{R}$. All series converge for $|x|, |y| < 1$.

Equation to solve:

$$Q(x,y) = 1 + txyQ(x,y) + t\frac{Q(x,y) - Q(0,y)}{x} + t\frac{Q(x,y) - Q(x,0)}{y}$$
$$+ e^{i\alpha}tQ(0,x) + e^{-i\alpha}tyQ(y,0).$$

Solution:

Step 1: Fix $t \in [0, 1/3), \alpha \in \mathbb{R}$. All series converge for $|x|, |y| < 1$.

Step 2: Write equation as $K(x,y)Q(x,y) = R(x,y)$, where

$$K(x,y) = 1 - txy - t/y - t/x$$
$$R(x,y) = 1 - \frac{t}{x}Q(0,y) - \frac{t}{y}Q(x,0) + e^{i\alpha}tQ(0,x) + e^{-i\alpha}tyQ(y,0).$$

SOLUTION TO KREWERAS WALKS BY WINDING NUMBER

Equation to solve:

$$Q(x,y) = 1 + txyQ(x,y) + t\frac{Q(x,y)-Q(0,y)}{x} + t\frac{Q(x,y)-Q(x,0)}{y} + e^{i\alpha}tQ(0,x) + e^{-i\alpha}tyQ(y,0).$$

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Step 3: Consider the curve $K(x,y) = 0$ (Then $R(x,y) = 0$).

Solution to Kreweras walks by winding number

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Step 3: Consider the curve $K(x,y) = 0$ (Then $R(x,y) = 0$).

Parameterisation involves the Jacobi theta function $\vartheta(z,\tau)$.

So far: Similar to [Kurkova, Raschel 12] and [Bernardi, Bousquet-Mélou, Raschel 17] for quadrant models.

Equation to solve:

$$K(x,y)Q(x,y) = R(x,y),$$

where

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Define

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}}\vartheta(z,3\tau)\vartheta(z-\pi\tau,3\tau)}{\vartheta(z+\pi\tau,3\tau)\vartheta(z-2\pi\tau,3\tau)}.$$

Then $K(X(z),X(z+\pi\tau)) = 0$.

Equation to solve:

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Then $K(X(z), X(z+\pi\tau)) = 0$. Hence $R(X(z), X(z+\pi\tau)) = 0$ (assuming $|X(z)| \leq 1$ and $|X(z+\pi\tau)| \leq 1$).

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New equation to solve:

$$R(X(z),X(z+\pi\tau)) = 0,$$

SOLUTION TO KREWERAS WALKS BY WINDING NUMBER

Equation to solve:

$$K(x,y)Q(x,y) = R(x,y),$$

where

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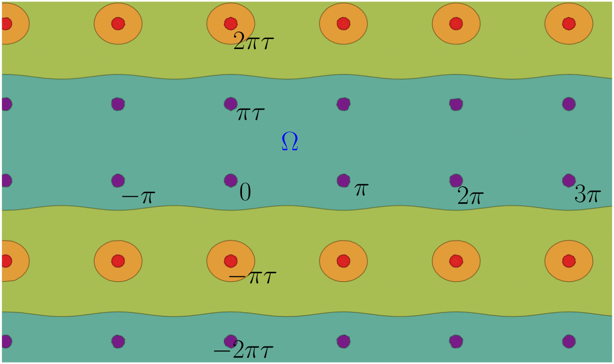
$$R(X(z),X(z+\pi\tau)) = 0,$$

Solution to Kreweras walks by winding number

Plot of $\left\{z : |X(z)| \in \left[0, \frac{1}{3}\right), \left(\frac{1}{3}, 1\right), (1,3), (3,9), (9,\infty]\right\}$.



For $z \in \Omega$, $|X(z)| < 1 \Rightarrow Q(X(z),0)$ and $Q(0,X(z))$ are well defined.

Solution to Kreweras walks by winding number

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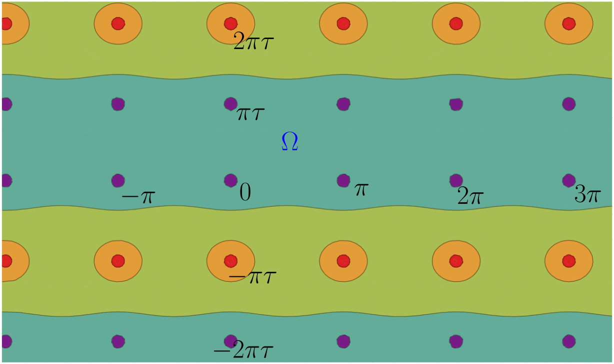

For $z \in \Omega$, $\quad |X(z)| < 1 \Rightarrow Q(X(z), 0)$ and $Q(0, X(z))$ are well defined.
Near $\mathrm{Re}(z) = 0$, we have $z \in \Omega$ and $z + \pi\tau \in \Omega$.

SOLUTION TO KREWERAS WALKS BY WINDING NUMBER

Equation to solve: (near $\mathrm{Re}(z) = 0$)

$$R(X(z), X(z + \pi\tau)) = 0$$

where

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}} \vartheta(z, 3\tau)\vartheta(z - \pi\tau, 3\tau)}{\vartheta(z + \pi\tau, 3\tau)\vartheta(z - 2\pi\tau, 3\tau)}.$$

$$R(x, y) = 1 - \frac{t}{x}Q(0, y) - \frac{t}{y}Q(x, 0) + e^{i\alpha}tQ(0, x) + e^{-i\alpha}tyQ(y, 0).$$

Equation to solve: (near $\mathrm{Re}(z) = 0$)

$$1 = \frac{t}{X(z)} Q(0, X(z+\pi\tau)) + \frac{t}{X(z+\pi\tau)} Q(X(z), 0) - e^{i\alpha} t Q(0, X(z)) - e^{-i\alpha} t X(z+\pi\tau) Q(X(z+\pi\tau), 0),$$

where

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}} \vartheta(z, 3\tau) \vartheta(z - \pi\tau, 3\tau)}{\vartheta(z+\pi\tau, 3\tau) \vartheta(z - 2\pi\tau, 3\tau)}.$$

SOLUTION TO KREWERAS WALKS BY WINDING NUMBER

Equation to solve: (near $\mathrm{Re}(z) = 0$)

$$1 = \frac{t}{X(z)} Q(0, X(z + \pi\tau)) + \frac{t}{X(z + \pi\tau)} Q(X(z), 0) - e^{i\alpha} t Q(0, X(z)) - e^{-i\alpha} t X(z + \pi\tau) Q(X(z + \pi\tau), 0),$$

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For z near 0, define

$$L(z) = \frac{t}{X(z + \pi\tau)} Q(X(z), 0) - e^{i\alpha} t Q(0, X(z)).$$

Both $L(z)$ and $L(z + \pi\tau)$ converge.

Equation to solve: (near $\mathrm{Re}(z)=0$)

$$1 = \frac{t}{X(z)}Q(0,X(z+\pi\tau)) + \frac{t}{X(z+\pi\tau)}Q(X(z),0)$$
$$-e^{i\alpha}tQ(0,X(z)) - e^{-i\alpha}tX(z+\pi\tau)Q(X(z+\pi\tau),0),$$

where

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Equation to solve: (near $\mathrm{Re}(z) = 0$)

$$1 = \frac{t}{X(z)}Q(0, X(z+\pi\tau)) + L(z) - e^{-i\alpha}tX(z+\pi\tau)Q(X(z+\pi\tau), 0),$$

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SOLUTION TO KREWERAS WALKS BY WINDING NUMBER

Equation to solve: (near $\mathrm{Re}(z) = 0$)

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Solution to Kreweras walks by winding number

Equation to solve: (near $\mathrm{Re}(z)=0$)

$$1 = \frac{t}{X(z)}Q(0, X(z+\pi\tau)) + L(z) - \frac{e^{-i\alpha}t}{X(z)X(z+2\pi\tau)}Q(X(z+\pi\tau), 0),$$

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Solution to Kreweras walks by winding number

Equation to solve: (near $\mathrm{Re}(z) = 0$)

$$1 = -\frac{e^{-i\alpha}}{X(z)} L(z + \pi\tau) + L(z).$$

where

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We can solve this exactly:

$$L(z) = -\frac{e^{3i\alpha}}{1-e^{3i\alpha}}\left(1+\frac{e^{-i\alpha}}{X(z)}+e^{-2i\alpha}X(z-\pi\tau)\right)$$
$$-\frac{e^{i\alpha+\frac{5i\pi\tau}{3}}\vartheta(\pi\tau,3\tau)\vartheta'(0,\tau)}{(1-e^{3i\alpha})\vartheta(\frac{\alpha}{2}-\frac{2\pi\tau}{3},\tau)\vartheta'(0,3\tau)}\frac{\vartheta(z-2\pi\tau,3\tau)\vartheta(z-\frac{\alpha}{2}+\frac{2\pi\tau}{3},\tau)}{\vartheta(z,\tau)\vartheta(z,3\tau)}$$

Solution to Kreweras walks by winding number

Equation to solve: (near $\mathrm{Re}(z) = 0$)

$$1 = -\frac{e^{-i\alpha}}{X(z)} L(z + \pi\tau) + L(z).$$

where

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}} \vartheta(z, 3\tau) \vartheta(z - \pi\tau, 3\tau)}{\vartheta(z + \pi\tau, 3\tau) \vartheta(z - 2\pi\tau, 3\tau)}.$$

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We can extract $E(t, e^{i\alpha}) = Q(0, 0)$...

Recall: τ is determined by

$$t = e^{-\frac{\pi\tau i}{3}} \frac{\vartheta'(0,3\tau)}{4i\vartheta(\pi\tau,3\tau) + 6\vartheta'(\pi\tau,3\tau)}.$$

The gf $E(t,e^{i\alpha}) = Q(0,0) \equiv Q(t,\alpha,0,0)$ is given by:

$$E(t,e^{i\alpha}) = \frac{e^{i\alpha}}{t(1-e^{3i\alpha})}\left(e^{i\alpha} - e^{\frac{4\pi\tau i}{3}}\frac{\vartheta'(2\pi\tau,3\tau)}{\vartheta'(0,3\tau)} - e^{\frac{\pi\tau i}{3}}\frac{\vartheta(\pi\tau,3\tau)\vartheta'(\frac{\alpha}{2}-\frac{2\pi\tau}{3},\tau)}{\vartheta'(0,3\tau)\vartheta(\frac{\alpha}{2}-\frac{2\pi\tau}{3},\tau)}\right).$$

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Equivalently:

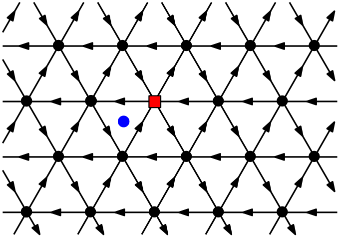
Let $q(t) \equiv q = t^3 + 15t^6 + 279t^9 + \cdots$ satisfy

$$t = q^{1/3}\frac{T_1(1,q^3)}{4T_0(q,q^3)+6T_1(q,q^3)}.$$

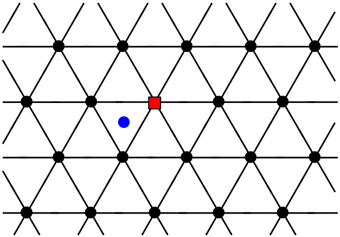
The gf for cell-centred Kreweras-lattice almost-excursions is:

$$E(t,s) = \frac{s}{(1-s^3)t}\left(s - q^{-1/3}\frac{T_1(q^2,q^3)}{T_1(1,q^3)} - q^{-1/3}\frac{T_0(q,q^3)T_1(sq^{-2/3},q)}{T_1(1,q^3)T_0(sq^{-2/3},q)}\right).$$

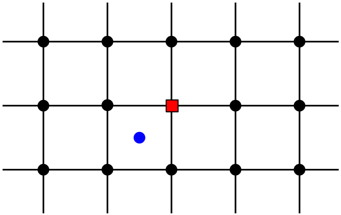
Part 2c: Winding on other lattices



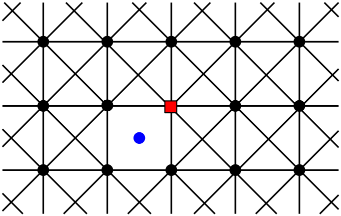
Kreweras lattice



Triangular Lattice



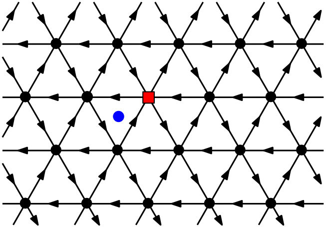
Square Lattice



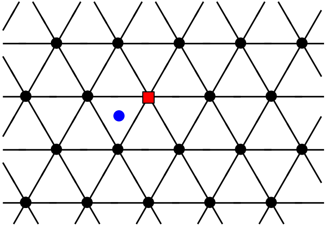
King Lattice

CELL-CENTRED LATTICES

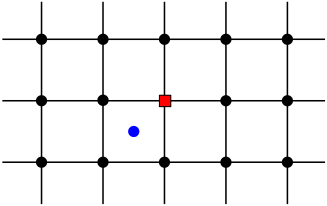
Important property: Decomposable into congruent sectors



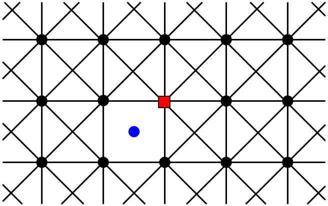
Kreweras lattice



Triangular Lattice

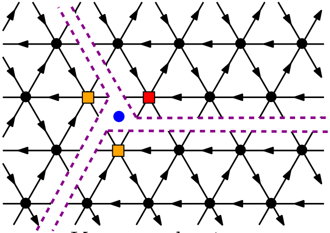


Square Lattice

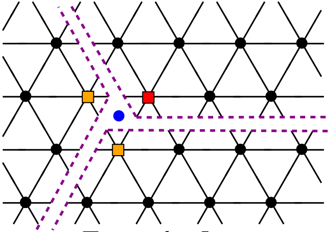


King Lattice

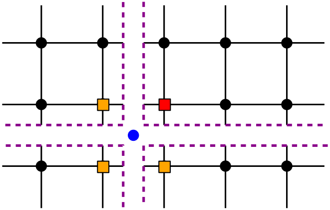
Important property: Decomposable into congruent sectors



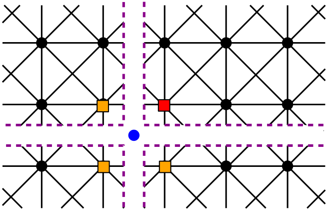
Kreweras lattice



Triangular Lattice

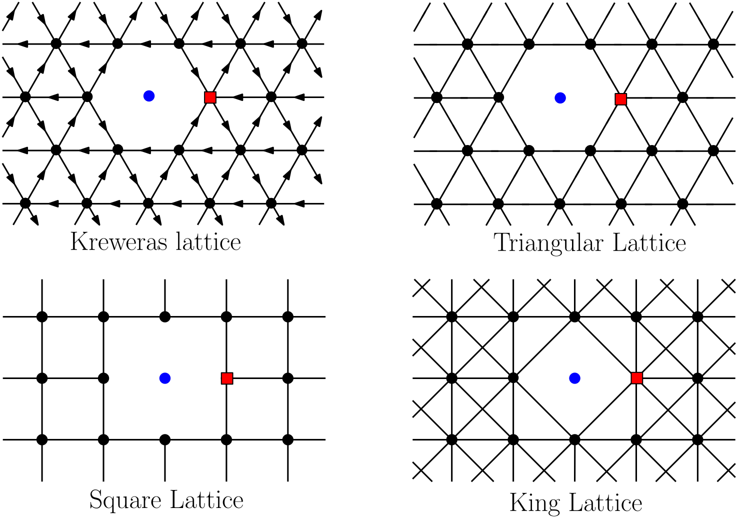


Square Lattice

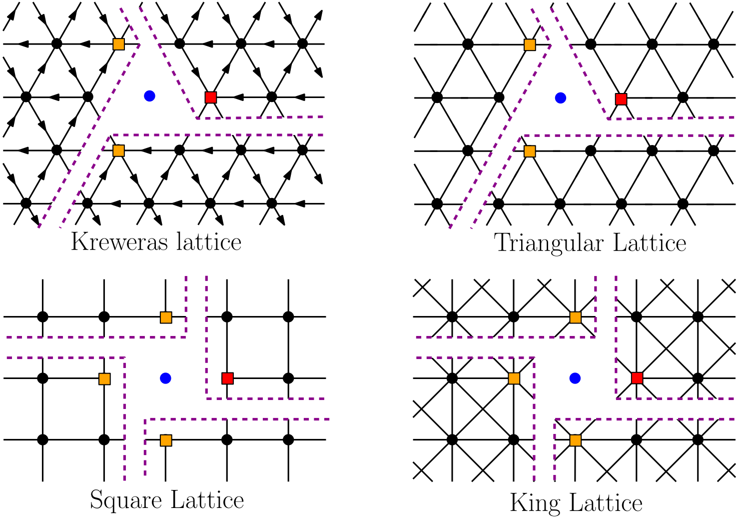


King Lattice

Decompose into rotationally congruent sectors



Decompose into rotationally congruent sectors



Recall: Kreweras almost-excursions

Define $T_k(u,q) = \sum_{n=0}^{\infty}(-1)^n(2n+1)^k q^{n(n+1)/2}(u^{n+1} - (-1)^k u^{-n})$

$= (u \pm 1) - 3^k q(u^2 \pm u^{-1}) + 5^k q^3(u^3 \pm u^{-2}) + O(q^6).$

Let $q(t) \equiv q = t^3 + 15t^6 + 279t^9 + \cdots$ satisfy

$$t = q^{1/3}\frac{T_1(1,q^3)}{4T_0(q,q^3) + 6T_1(q,q^3)}.$$

The gf for cell-centred Kreweras-lattice almost-excursions is:

$$E(t,s) = \frac{s}{(1-s^3)t}\left(s - q^{-1/3}\frac{T_1(q^2,q^3)}{T_1(1,q^3)} - q^{-1/3}\frac{T_0(q,q^3)T_1(sq^{-2/3},q)}{T_1(1,q^3)T_0(sq^{-2/3},q)}\right).$$

The gf for vertex-centred Kreweras-lattice almost-excursions is:

$$\tilde{E}(t,s) = \frac{s(1-s)q^{-\frac{2}{3}}}{t(1-s^3)}\frac{T_0(q,q^3)^2}{T_1(1,q^3)^2}\left(\frac{T_1(q,q^3)^2}{T_0(q,q^3)^2} - \frac{T_2(q,q^3)}{T_0(q,q^3)} - \frac{T_2(s,q)}{2T_0(s,q)} + \frac{T_3(1,q)}{6T_1(1,q)} + \frac{T_3(1,q^3)}{3T_1(1,q^3)}\right).$$

SQUARE LATTICE ALMOST-EXCURSIONS

Define
$$T_k(u,q) = \sum_{n=0}^{\infty}(-1)^n(2n+1)^k q^{n(n+1)/2}(u^{n+1}-(-1)^k u^{-n})$$
$$= (u \pm 1) - 3^k q(u^2 \pm u^{-1}) + 5^k q^3(u^3 \pm u^{-2}) + O(q^6).$$

Let $q(t) \equiv q = t + 4t^3 + 34t^5 + 360t^7 + \cdots$ satisfy

$$t = \frac{qT_0(q^2,q^8)T_1(1,q^8)}{2T_0(q^4,q^8)(T_0(q^2,q^8)+2T_1(q^2,q^8))}.$$

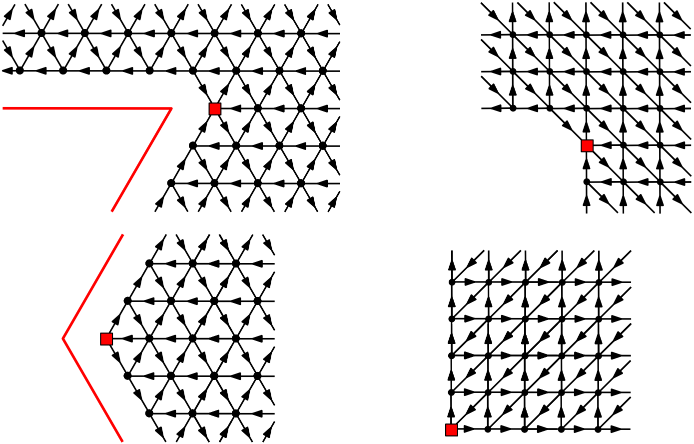
The gf for cell-centred Square-lattice almost-excursions is:

$$\frac{s^2}{(1-s^4)t}\left(s - s^{-1} + \frac{T_0(q^4,q^8)}{qT_1(1,q^8)} - \frac{T_0(q^4,q^8)T_1(s^{-1}q,q^2)}{qT_1(1,q^8)T_0(s^{-1}q,q^2)}\right).$$

The gf for vertex-centred Square-lattice almost-excursions is:

$$\frac{sT_0(q^4,q^8)}{qt(1+s^2)T_1(1,q^8)}\left(1 + \frac{2T_1(q^2,q^8)}{T_0(q^2,q^8)} + \frac{(1-s)T_1(s^{-1},q^2)}{(1+s)T_0(s^{-1},q^2)}\right).$$

Part 3: Walks in cones



Walks in cones with small steps

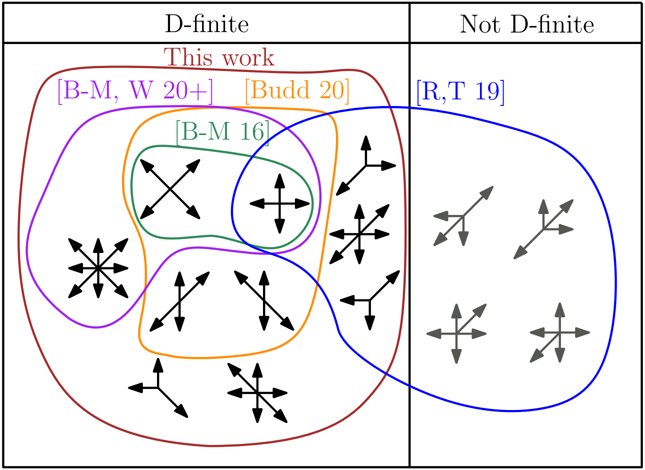
- **Quarter plane walks:** Completely classified into rational, algebraic, D-finite, D-algebraic cases.

  [Mishna, Rechnitzer 09], [Bousquet-Mélou, Mishna 10], [Bostan, Kauers 10], [Fayolle, Raschel 10], [Kurkova, Raschel 12], [Melczer, Mishna 13], [Bostan, Raschel, Salvy 14], [Bernardi, Bousquet-Mélou, Raschel 17], [Dreyfus, Hardouin, Roques, Singer 18]
- **Half plane walks:** Easy
- **Three quarter plane walks:** Active area of research (Previously) solved in 6-12 of the 74 non-trivial cases

  [Bousquet-Mélou 16], [Raschel-Trotignon 19], [Budd 20], [Bousquet-Mélou, Wallner 20+]
- **Walks on the slit plane $\mathbb{C} \setminus \mathbb{R}_{<0}$:** solved in all cases

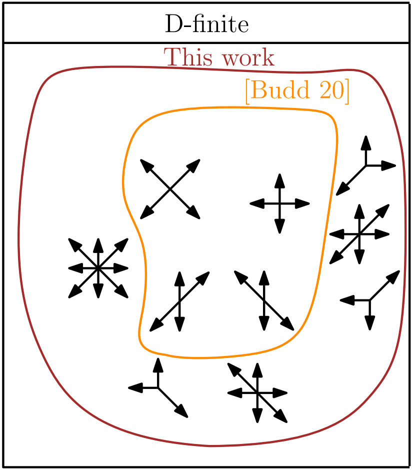
  [Bousquet-Mélou, 01], [Bousquet-Mélou, Schaeffer, 02]

Walks in the 3/4-plane: solved cases



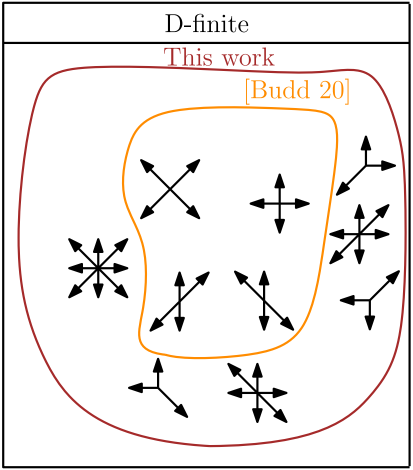
[Bousquet-Mélou 16],[Raschel, Trotignon 19], [Budd 20], [Bousquet-Mélou, Wallner 20+]

WALKS IN THE 5/4-PLANE: SOLVED CASES



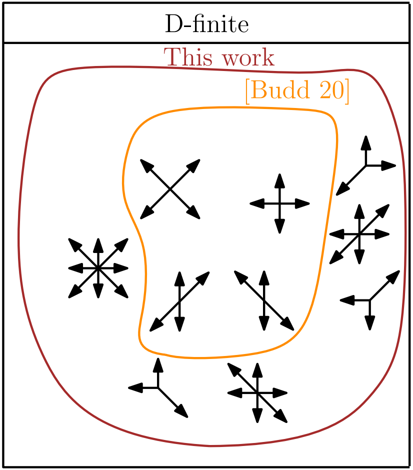
[Budd 20]

Walks in the 6/4-plane: solved cases

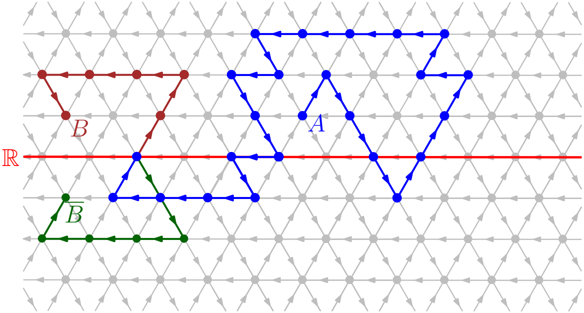


[Budd 20]

Walks in the 7/4-plane: solved cases



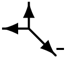
[Budd 20]

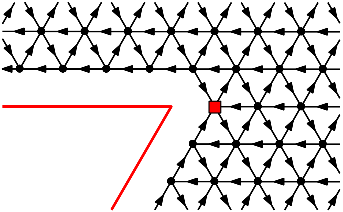


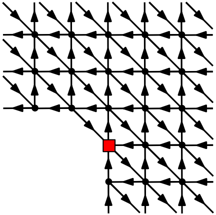
In the upper half plane: Use reflection principle

$$\begin{aligned} &\#(\text{Walks from } A \text{ to } B \text{ above } \mathbb{R}) \\ =\ &\#(\text{Walks from } A \text{ to } B) - \#(\text{Walks from } A \text{ to } B \text{ through } \mathbb{R}) \\ =\ &\#(\text{Walks from } A \text{ to } B) - \#(\text{Walks from } A \text{ to } \overline{B}) \end{aligned}$$

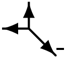
Counting Kreweras excursions in 5/6-plane

New model: -excursions avoiding a quadrant.

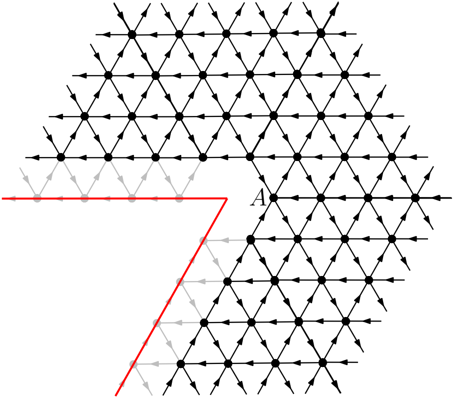




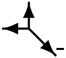
Counting Kreweras excursions in 5/6-plane

New model:  -excursions avoiding a quadrant.

First step: Transform to half plane



COUNTING KREWERAS EXCURSIONS IN 5/6-PLANE

New model: -excursions avoiding a quadrant.

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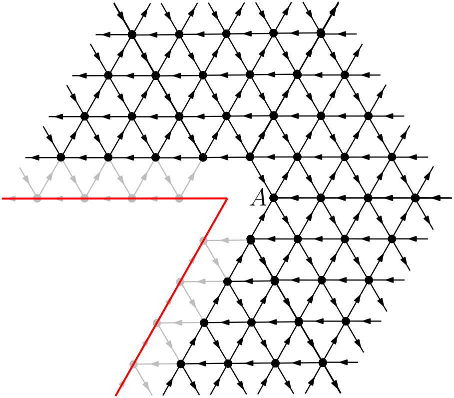
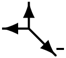
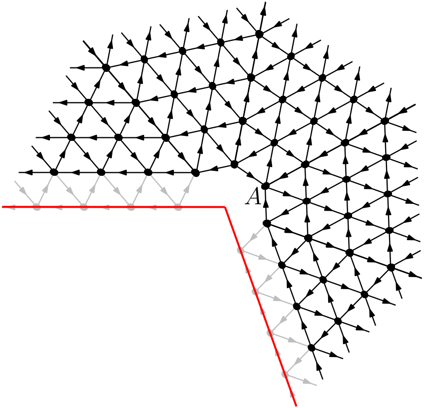
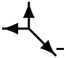


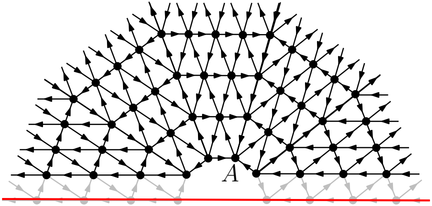
Counting Kreweras excursions in 5/6-plane

New model: -excursions avoiding a quadrant.
First step: Transform to half plane

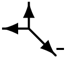


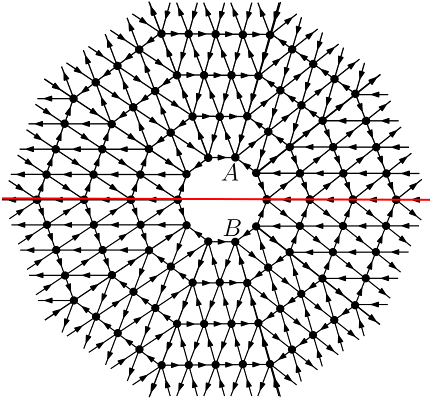
COUNTING KREWERAS EXCURSIONS IN 5/6-PLANE

New model: -excursions avoiding a quadrant.
First step: Transform to half plane

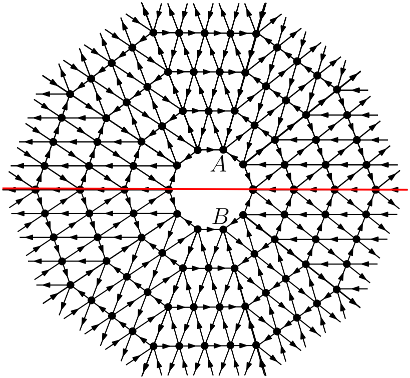


COUNTING KREWERAS EXCURSIONS IN 5/6-PLANE

New model: -excursions avoiding a quadrant.
First step: Transform to half plane → whole (punctured) plane



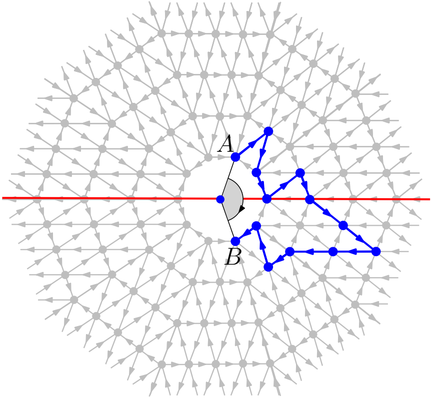
Counting Kreweras excursions in 5/6-plane

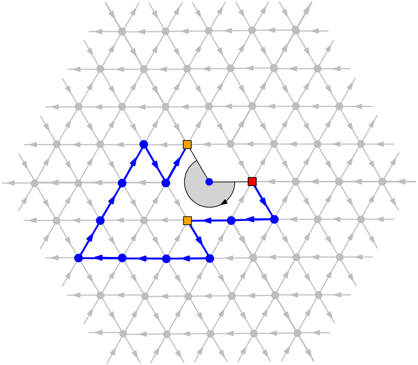


$$\#(\text{Kreweras excursions in 5/6-plane})$$
$$= \#(\text{Walks } A \to A \text{ in upper half plane})$$
$$= \#(\text{Walks } A \to A) - \#(\text{Walks } A \to B)$$

COUNTING KREWERAS EXCURSIONS IN 5/6-PLANE

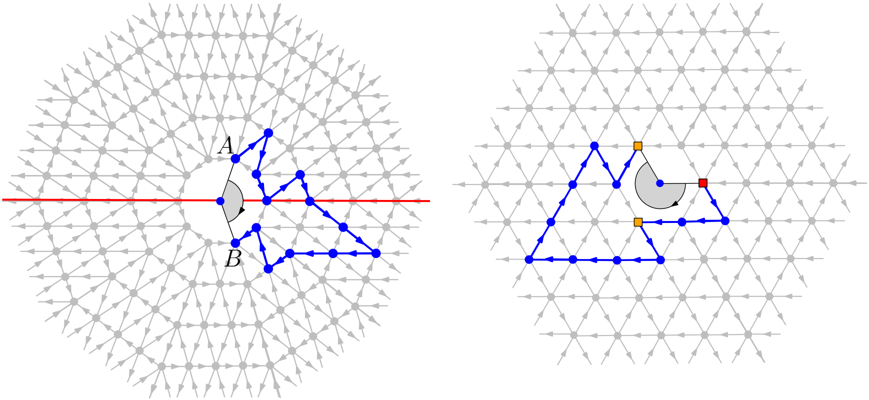
Walks $A \to B$ with winding angle β
$\equiv$ Kreweras almost-excursions with winding angle $\frac{5\beta}{3}$.





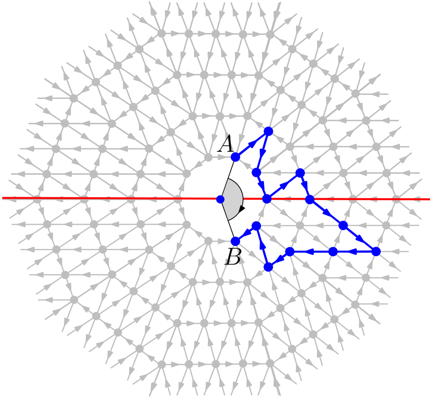
Walks $A \to B$ with winding angle $2\pi k - \frac{4\pi}{5}$

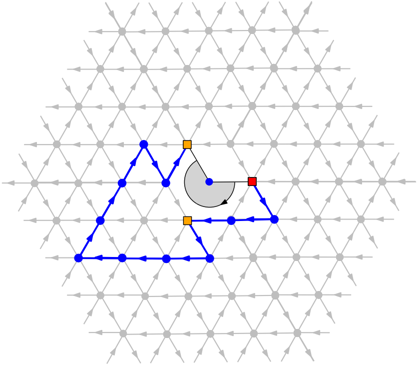
$\equiv$ Kreweras almost-excursions with winding angle $\frac{5}{3}\left(2\pi k - \frac{4\pi}{5}\right)$.



Counting Kreweras excursions in 5/6-plane

Walks $A \to B$ with winding angle $2\pi k - \frac{4\pi}{5}$
$\equiv$ Kreweras almost-excursions with winding angle $\frac{10\pi k}{3} - \frac{4\pi}{3}$.



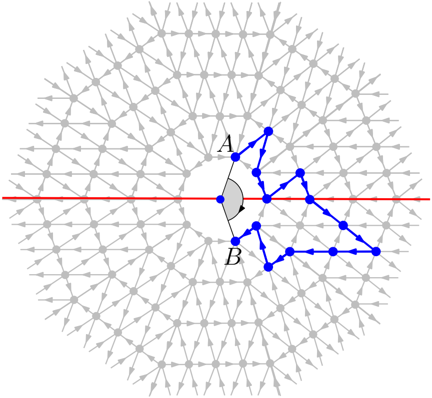


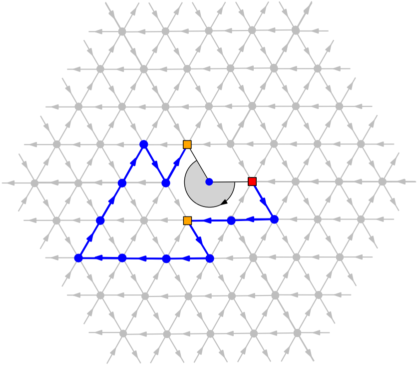
COUNTING KREWERAS EXCURSIONS IN 5/6-PLANE

Walks $A \to B$ with winding angle $2\pi k - \frac{4\pi}{5}$

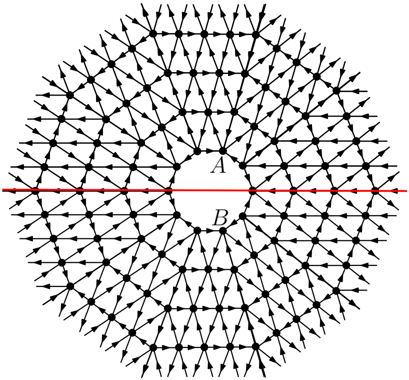
$\equiv$ Kreweras almost-excursions with winding angle $\frac{10\pi k}{3} - \frac{4\pi}{3}$.

Counted by: $s^{5k-2}\tilde{E}(t,s)$



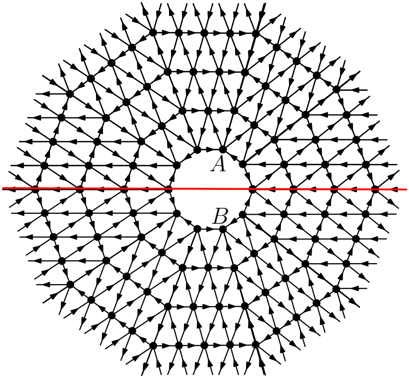


COUNTING KREWERAS EXCURSIONS IN 5/6-PLANE



$$
\begin{aligned}
&\#(\text{Kreweras excursions in } 5/6\text{-plane}) \\
&= \#(\text{Walks } A \to A \text{ in upper half plane}) \\
&= \#(\text{Walks } A \to A) - \#(\text{Walks } A \to B) \\
&= \left(\sum_{k\in\mathbb{Z}} [s^{5k}]\tilde{E}(t,s)\right) - \left(\sum_{k\in\mathbb{Z}} [s^{5k-3}]\tilde{E}(t,s)\right) \\
&= \frac{1}{5}\sum_{j=1}^{4}\left(1 - e^{\frac{4\pi ij}{5}}\right)\tilde{E}\left(t, e^{\frac{2\pi i}{5}}\right)
\end{aligned}
$$

Counting Kreweras excursions in 5/6-plane



$$\begin{aligned}
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&= \#(\text{Walks } A \to A \text{ in upper half plane}) \\
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&= \frac{1}{5}\sum_{j=1}^{4}\left(1 - e^{\frac{4\pi ij}{5}}\right)\tilde{E}\left(t, e^{\frac{2\pi i}{5}}\right)
\end{aligned}$$

More generally: The gf $C_{k,r}(t)$ for excursions in the $k/6$-plane is

$$C_{k,r}(t) = \frac{1}{k}\sum_{j=1}^{k-1}\left(1 - e^{\frac{2\pi ijr}{k}}\right)\tilde{E}\left(t, e^{\frac{2\pi ij}{k}}\right).$$

Final comments

Functional equation theta solution method

Project: develop this method of solving functional equations.

Problems solved so far:

- Quadrant walks [Kurkova, Raschel, 2012] + [Bernardi, Bousquet-Mélou, Raschel, 2017]
- Some walks avoiding a quadrant [Raschel, Trotignon, 2019]
- Some walks by winding number [E.P., 2020+]
- Six vertex model on 4-valent maps [Kostov, 2000], [E.P., Zinn-Justin, 2020+], [Bousquet-Mélou, E.P., 2020+]
- Properly coloured triangulations [E.P., 2020+]

To do:

- Solve more problems.
- Streamline the method.
- Convert techniques to world of formal power series.
- find a good name for the method.

Thank you!

BONUS SLIDE: PROPERLY COLOURED TRIANGULATIONS

To solve:

$$T(x,y)K(x,y) = R(x,y),$$

where

$$K(x,y) = 1 - xytT(1,y) - \frac{xt}{y} - \frac{x^2yt}{1-x}$$
$$R(x,y) = x(s-1) - \frac{xt}{y}T(x,0) + x^2yt\frac{T(1,y)}{x-1}.$$

Want to parametrise $K(x,y)=0$, as then $R(x,y)=0$.

Guess: There is some pair $X(z)$, $Y(z)$ satisfying

- $K(X(z),Y(z)) = 0$ and therefore $R(X(z),Y(z)) = 0$.
- $X(z+\pi) = X(z)$ and $Y(z+\pi) = Y(z)$.
- $X(-z) = X(z)$ and $Y(\pi\tau - z) = Y(z)$.

Guess: Solve under this assumption then check the solution.

Kernel not explicit, but method still works.

Bonus slide: another winding angle

For self-avoiding walks, a different parameter is sometimes called the winding angle (e.g. in work of Duminil-Copin and Smirnov). I'll call it the turning angle.

Definition: Imagining a walker taking the walk, the turning angle is the total anti-clockwise angle they turn during the walk.
Relation to winding angle: The turning angle is the winding angle of the walk minus the winding angle of the reversed walk.

Bonus slide: another turning angle

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ASYMPTOTICS OF $\tilde{E}(t, e^{i\alpha})$ AND $C_{k,r}(t)$

Fix $\alpha \in (0, \pi) \setminus \{\frac{2\pi}{3}\}$.
Writing $\hat{\tau} = -\frac{1}{3\tau}$ and $\hat{q} = e^{2\pi i\hat{\tau}}$, the dominant singularity $t = 1/3$ corresponds to $\hat{q} = 0$.

ASYMPTOTICS OF $\tilde{E}(t, e^{i\alpha})$ AND $C_{k,r}(t)$

Fix $\alpha \in (0, \pi) \setminus \{\frac{2\pi}{3}\}$.
Writing $\hat{\tau} = -\frac{1}{3\tau}$ and $\hat{q} = e^{2\pi i\hat{\tau}}$, the dominant singularity $t = 1/3$ corresponds to $\hat{q} = 0$.
Series in $\hat{q}$:

$$t = \frac{1}{3} - 3\hat{q} + 18\hat{q}^2 + O(\hat{q}^3)$$

$$t\tilde{E}(t, e^{i\alpha}) = a_0 + a_1\hat{q} - \frac{27\alpha e^{i\alpha}}{2\pi(1 + e^{i\alpha} + e^{2i\alpha})}\hat{q}^{\frac{3\alpha}{2\pi}} + o\left(\hat{q}^{\frac{3\alpha}{2\pi}}\right),$$

$\rightarrow \tilde{E}(t, e^{i\alpha})$ as a series in $(1 - 3t)$,

Asymptotics of $\tilde{E}(t, e^{i\alpha})$ and $C_{k,r}(t)$

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$\rightarrow \tilde{E}(t, e^{i\alpha})$ as a series in $(1 - 3t)$, $\rightarrow$

$$[t^n]\tilde{E}(t, e^{i\alpha}) \sim -\frac{3^{5-\frac{3\alpha}{\pi}} e^{\alpha i}\alpha}{2\pi(1 + e^{\alpha i} + e^{2\alpha i})\Gamma\left(-\frac{3\alpha}{2\pi}\right)} n^{-\frac{3\alpha}{2\pi}-1} 3^n,$$

$$[t^n]C_{k,r}(t) \sim -\frac{2 \cdot 3^{5-\frac{6}{k}} \sin^2\left(\frac{r\pi}{k}\right)}{\pi k^2 \left(1 + 2\cos\left(\frac{2\pi}{k}\right)\right)\Gamma\left(-\frac{3}{k}\right)} n^{-1-\frac{3}{k}} 3^n.$$

Asymptotics of $\tilde{E}(t, e^{i\alpha})$ and $C_{k,r}(t)$

Fix $\alpha \in (0, \pi) \setminus \{\frac{2\pi}{3}\}$.
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Series in $\hat{q}$:

$$t = \frac{1}{3} - 3\hat{q} + 18\hat{q}^2 + O(\hat{q}^3)$$

$$t\tilde{E}(t, e^{i\alpha}) = a_0 + a_1\hat{q} - \frac{27\alpha e^{i\alpha}}{2\pi(1 + e^{i\alpha} + e^{2i\alpha})}\hat{q}^{\frac{3\alpha}{2\pi}} + o\left(\hat{q}^{\frac{3\alpha}{2\pi}}\right),$$

$\to \tilde{E}(t, e^{i\alpha})$ as a series in $(1 - 3t)$, $\to$

$$[t^n]\tilde{E}(t, e^{i\alpha}) \sim -\frac{3^{5-\frac{3\alpha}{\pi}} e^{\alpha i}\alpha}{2\pi(1 + e^{\alpha i} + e^{2\alpha i})\Gamma\left(-\frac{3\alpha}{2\pi}\right)} n^{-\frac{3\alpha}{2\pi}-1} 3^n,$$

$$[t^n]C_{k,r}(t) \sim -\frac{2 \cdot 3^{5-\frac{6}{k}} \sin^2\left(\frac{r\pi}{k}\right)}{\pi k^2 \left(1 + 2\cos\left(\frac{2\pi}{k}\right)\right)\Gamma\left(-\frac{3}{k}\right)} n^{-1-\frac{3}{k}} 3^n.$$

Alternatively: Terms 3^n and $n^{-1-\frac{3}{k}}$ known [Denisov, Wachtel, 2015].

Analysis of solution: Algebraicity

Recall: $\vartheta(z,\tau)$ is differentially algebraic $\rightarrow$ so are $\tilde{E}(t,s)$ and $Q(t,\alpha,x,y)$.
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Recall: The gf for excursions in the $k/6$-plane is

$$C_{k,r}(t) = \frac{1}{k}\sum_{j=1}^{k-1}\left(1 - e^{\frac{2\pi ijr}{k}}\right)\tilde{E}\left(t, e^{\frac{2\pi ij}{k}}\right).$$

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Algebraic iff $3\nmid k$. (always D-finite).

BONUS SLIDE: PARAMETERIZATION OF $K(x, y) = 0$

Write $K(x, y) = A(x)y^2 + B(x)y + C(x)$, then

$$Y(x) = \frac{-B(x) \pm \sqrt{B(x)^2 - 4A(x)C(x)}}{2A(x)}$$

parameterizes $K(x, Y(x)) = 0$. Typically, $Y_+(x)$ is meromorphic on:

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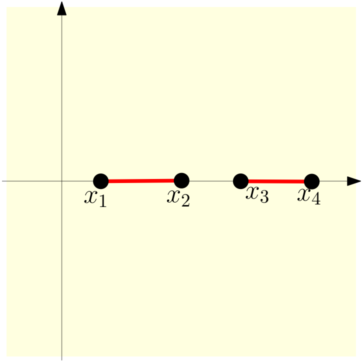

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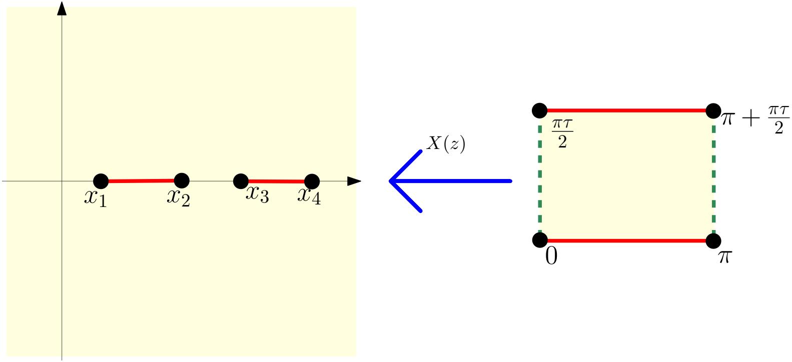

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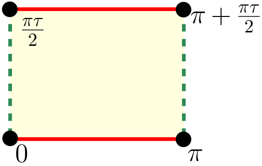

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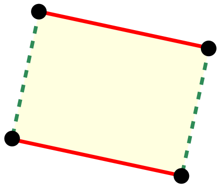

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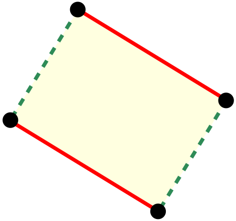

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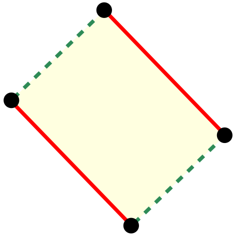

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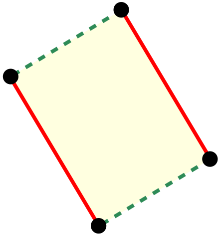

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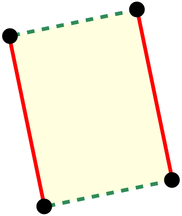

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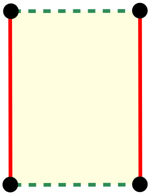

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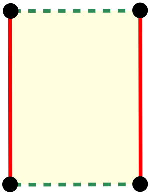

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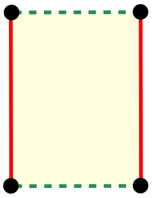

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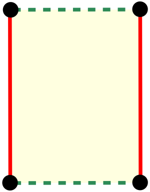

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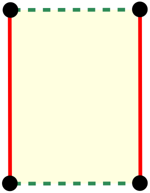

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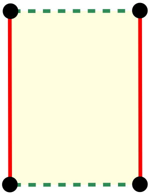

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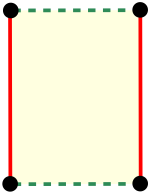

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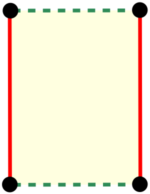

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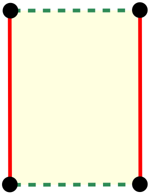

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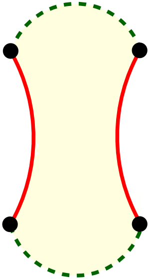

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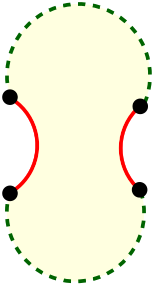

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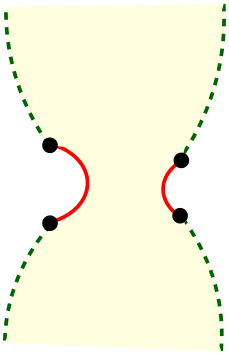

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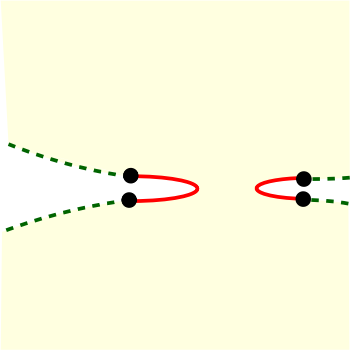

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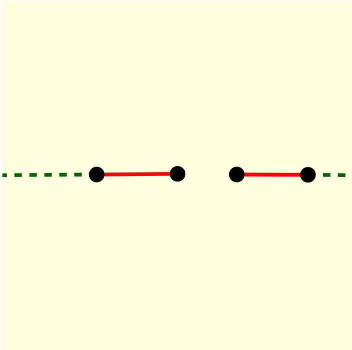

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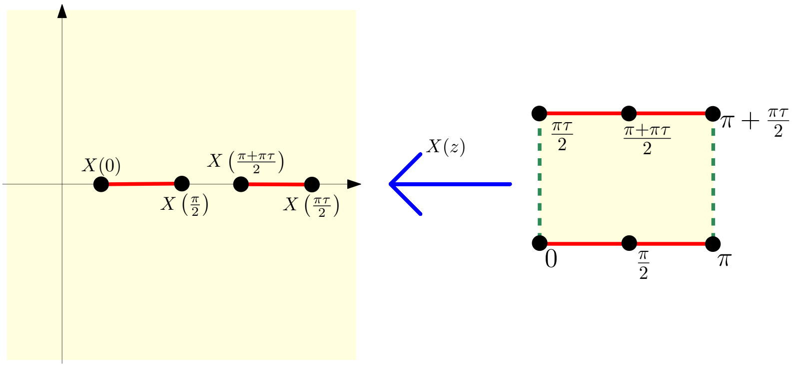

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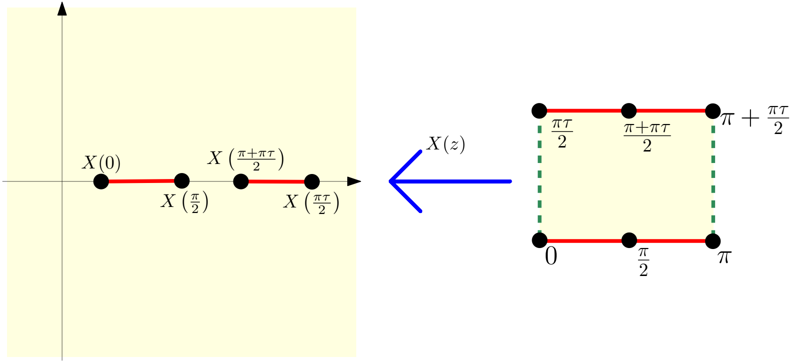
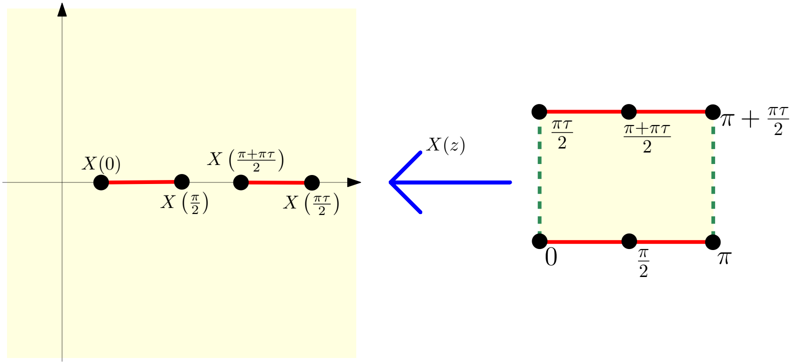


By symmetry, for $r \in \mathbb{R}$:

- $X(r) = X(\pi - r) = X(-r)$
- $X(\frac{\pi\tau}{2} + r) = X(\frac{\pi\tau}{2} - r)$

BONUS SLIDE: PARAMETERIZATION OF $K(x, y) = 0$



For $z \in \mathbb{C}$:

- $X(z) = X(\pi - z) = X(-z)$
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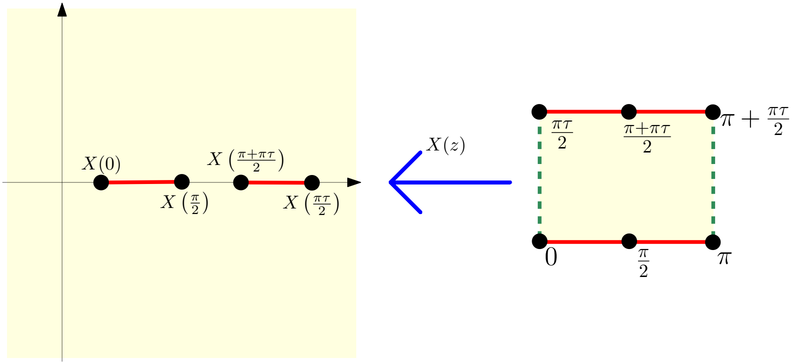

For $z \in \mathbb{C}$:

- $X(z) = X(\pi - z) = X(-z) = X(\pi\tau + z)$
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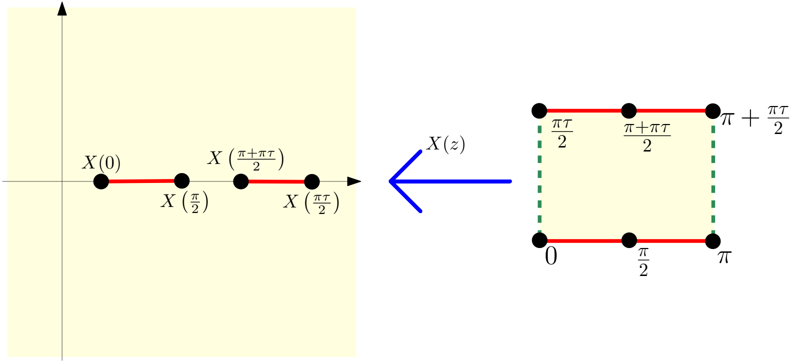
BONUS SLIDE: PARAMETERIZATION OF $K(x, y) = 0$

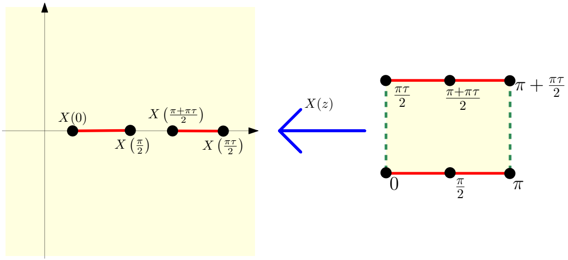


For $z \in \mathbb{C}$:

- $X(z) = X(\pi - z) = X(-z) = X(\pi\tau + z)$

$$X(z) = c\frac{\vartheta(z-\alpha)\vartheta(z+\alpha)}{\vartheta(z-\beta)\vartheta(z+\beta)}$$

Bonus slide: Parameterization of $K(x,y)=0$



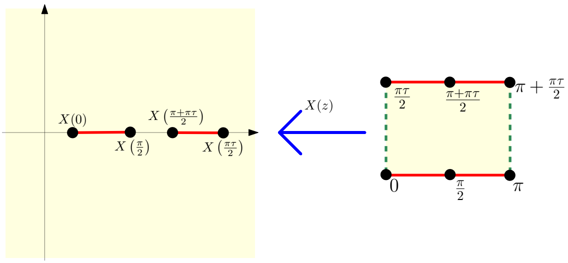
Recall:

$$y(x) = \frac{-B(x) \pm \sqrt{B(x)^2 - 4A(x)C(x)}}{2A(x)}.$$

Consider $Y(z) = y(X(z))$. By symmetry, for $r \in \mathbb{R}$:

- $X(r) = X(-r)$, so $Y(r) + Y(-r) = -\frac{B(X(r))}{A(X(r))}$.
- Similarly, $Y\left(\frac{\pi\tau}{2} + r\right) + Y\left(\frac{\pi\tau}{2} - r\right) = -\frac{B\left(X\left(\frac{\pi\tau}{2} + r\right)\right)}{A\left(X\left(\frac{\pi\tau}{2} + r\right)\right)}$.

BONUS SLIDE: PARAMETERIZATION OF $K(x, y) = 0$



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Consider $Y(z) = y(X(z))$. For $z \in \mathbb{C}$:

- $Y(z) + Y(-z) = -\frac{B(X(z))}{A(X(z))}$.
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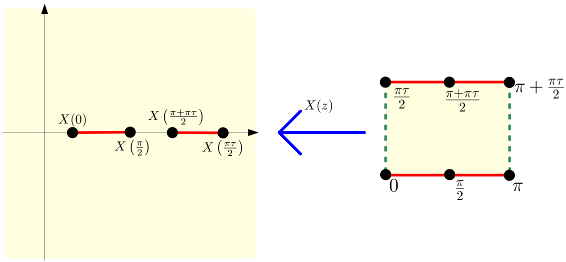
Bonus slide: Parameterization of $K(x, y) = 0$



For $z \in \mathbb{C}$:

- $Y(z) + Y(-z) = -\frac{B(X(z))}{A(X(z))}$.
- $Y(z) + Y(\pi\tau - z) = -\frac{B(X(z))}{A(X(z))}$.

So $Y(z) = Y(z + \pi\tau) = Y(z + \pi)$

$$\Rightarrow Y(z) = c\frac{\vartheta(z-\gamma)\vartheta(z-\delta)}{\vartheta(z-\epsilon)\vartheta(z-\gamma-\delta+\epsilon)}.$$

BONUS SLIDE: PARAMETERIZATION OF $K(x, y) = 0$

Equation characterising $Q(x, y) \equiv Q(t, x, y)$ for quadrant walks:

$$K(x, y)Q(x, y) + R(x, y) = 0.$$

$K(x, y) = 0$ is parameterised by

$$X(z) = c_1 \frac{\vartheta(z - \alpha_1)\vartheta(z - \beta_1)}{\vartheta(z - \gamma_1)\vartheta(z - \delta_1)} \quad \text{and} \quad Y(z) = c_2 \frac{\vartheta(z - \alpha_2)\vartheta(z - \beta_2)}{\vartheta(z - \gamma_2)\vartheta(z - \delta_2)},$$

where the constants satisfy $\alpha_j + \beta_j = \gamma_j + \delta_j$ for $j = 1, 2$.
So, $R(X(z), Y(z)) = 0$.

BONUS SLIDE: PARAMETERIZATION OF $K(x,y)=0$

In general: $K(x,y)=0$ is parameterised by

$$X(z)=c_1\frac{\vartheta(z-\alpha_1)\vartheta(z-\beta_1)}{\vartheta(z-\gamma_1)\vartheta(z-\delta_1)}\quad\text{and}\quad Y(z)=c_2\frac{\vartheta(z-\alpha_2)\vartheta(z-\beta_2)}{\vartheta(z-\gamma_2)\vartheta(z-\delta_2)},$$

with $\alpha_j+\beta_j=\gamma_j+\delta_j$ for $j=1,2$.

BONUS SLIDE: PARAMETERIZATION OF $K(x, y) = 0$

For Kreweras paths:

$$Q(x,y) = 1+xytQ(x,y)+\frac{t}{x}\left(Q(x,y) - Q(0,y)\right)+\frac{t}{y}\left(Q(x,y) - Q(x,0)\right).$$

Then $K(x, y) = xy - tx^2y^2 - tx - ty = 0$ is parameterised by

$$X(z) = c_1\frac{\vartheta(z-\alpha_1)\vartheta(z-\beta_1)}{\vartheta(z-\gamma_1)\vartheta(z-\delta_1)} \quad \text{and} \quad Y(z) = c_2\frac{\vartheta(z-\alpha_2)\vartheta(z-\beta_2)}{\vartheta(z-\gamma_2)\vartheta(z-\delta_2)},$$

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with $\alpha_j + \beta_j = \gamma_j + \delta_j$ for $j = 1, 2$.

- $K(0,0) = 0$, so WLOG $\alpha_1 = \alpha_2 = 0$.

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- $K(0,0)=0$, so WLOG $\alpha_1=\alpha_2=0$.
- as $x\to 0$, we have $y(x)\sim -x$ or $y(x)\sim -\frac{1}{x^2}$, so $Y(z)$ has a double pole at $z=\beta_1$.

BONUS SLIDE: PARAMETERIZATION OF $K(x, y) = 0$

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$$Q(x,y) = 1 + xytQ(x,y) + \frac{t}{x}\left(Q(x,y) - Q(0,y)\right) + \frac{t}{y}\left(Q(x,y) - Q(x,0)\right).$$

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with $\alpha_j + \beta_j = \gamma_j + \delta_j$ for $j = 1, 2$.

- $K(0,0) = 0$, so WLOG $\alpha_1 = \alpha_2 = 0$.
- as $x \to 0$, we have $y(x) \sim -x$ or $y(x) \sim -\frac{1}{x^2}$, so $Y(z)$ has a double pole at $z = \beta_1$.

BONUS SLIDE: PARAMETERIZATION OF $K(x, y) = 0$

For Kreweras paths:

$$Q(x,y) = 1 + xytQ(x,y) + \frac{t}{x}\left(Q(x,y) - Q(0,y)\right) + \frac{t}{y}\left(Q(x,y) - Q(x,0)\right).$$

Then $K(x,y) = xy - tx^2y^2 - tx - ty = 0$ is parameterised by

$$X(z) = c_1 \frac{\vartheta(z)\vartheta(z-\beta_1)}{\vartheta(z-\gamma_1)\vartheta(z-\delta_1)} \quad \text{and} \quad Y(z) = c_2 \frac{\vartheta(z)\vartheta(z-2\beta_1)}{\vartheta(z-\beta_1)^2},$$

with $\alpha_j + \beta_j = \gamma_j + \delta_j$ for $j = 1, 2$.

- $K(0,0) = 0$, so WLOG $\alpha_1 = \alpha_2 = 0$.
- as $x \to 0$, we have $y(x) \sim -x$ or $y(x) \sim -\frac{1}{x^2}$, so $Y(z)$ has a double pole at $z = \beta_1$.

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BONUS SLIDE: PARAMETERIZATION OF $K(x, y) = 0$

Then $K(x, y) = xy - tx^2y^2 - tx - ty = 0$ is parameterised by

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}}\vartheta(z, 3\tau)\vartheta(z - \pi\tau, 3\tau)}{\vartheta(z + \pi\tau, 3\tau)\vartheta(z - 2\pi\tau, 3\tau)} \quad \text{and} \quad Y(z) = X(z + \pi\tau),$$

where

$$t = \frac{1}{X(z)Y(z) + X(z)^{-1} + Y(z)^{-1}}.$$

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where

$$t = e^{-\frac{\pi\tau i}{3}}\frac{\vartheta'(0,3\tau)}{4i\vartheta(\pi\tau,3\tau)+6\vartheta'(\pi\tau,3\tau)}.$$

Part 4: Analysis of solutions