Shift spaces defined by lexicographic inequalities and unique double base expansions

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β-expansions (with digits 0 and 1), $\beta > 1$

$$\pi_\beta(i_1 i_2 \cdots) := \sum_{k=1}^{\infty} \frac{i_k}{\beta^k} \in \Big[0, \frac{1}{\beta-1}\Big], \quad i_1 i_2 \cdots \in \{0,1\}^\infty$$

$$X_\beta := \pi_\beta(\{0,1\}^\infty), \quad X_\beta = \frac{1}{\beta} X_\beta \cup \frac{1}{\beta}(1 + X_\beta)$$

$$X_\beta = \big[0, \tfrac{1}{\beta-1}\big] \iff \big[0, \tfrac{1}{\beta-1}\big] = \big[0, \tfrac{1}{\beta(\beta-1)}\big] \cup \big[\tfrac{1}{\beta}, \tfrac{1}{\beta-1}\big] \iff \beta \le 2$$

unique β-expansions

$$U_\beta := \big\{\mathbf{u} \in \{0,1\}^\infty : \pi_\beta(\mathbf{u}) \neq \pi_\beta(\mathbf{v}) \text{ for all } \mathbf{v} \neq \mathbf{u}\big\}$$

$$= \big\{i_1 i_2 \cdots \in \{0,1\}^\infty : \pi_\beta(i_n i_{n+1} \cdots) \notin \big[\tfrac{1}{\beta}, \tfrac{1}{\beta(\beta-1)}\big] \ \forall n \ge 1\big\}$$

$$\overset{\beta \le 2}{=} \big\{i_1 i_2 \cdots \in \{0,1\}^\infty : i_n i_{n+1} \cdots <_{\mathrm{lex}} \mathbf{a}_\beta \text{ or } i_n i_{n+1} \cdots >_{\mathrm{lex}} \mathbf{b}_\beta \ \forall n \ge 1\big\}$$

$\mathbf{a}_\beta$ quasi-greedy β-expansion of $\frac{1}{\beta} = \pi_\beta(1\overline{0})$
(lex. largest β-expansion of $\frac{1}{\beta}$ not ending with $\overline{0} = 00\cdots$),
$\mathbf{b}_\beta$ quasi-lazy β-expansion of $\frac{1}{\beta(\beta-1)} = \pi_\beta(0\overline{1})$
(lex. smallest β-expansion of $\frac{1}{\beta(\beta-1)}$ not ending with $\overline{1} = 11\cdots$)
Example: $\beta = \frac{1+\sqrt{5}}{2}$ $(\beta^2 = \beta+1)$, $\mathbf{a}_\beta = \overline{01} = 0101\cdots$, $\mathbf{b}_\beta = \overline{10}$

Unique β-expansions, $\beta \in (1,2]$

$$\begin{aligned} U_\beta &= \big\{ i_1 i_2 \cdots \in \{0,1\}^\infty \ : \ i_n i_{n+1} \cdots <_{\mathrm{lex}} \mathbf{a}_\beta \text{ or } i_n i_{n+1} \cdots >_{\mathrm{lex}} \mathbf{b}_\beta \ \forall n \geq 1 \big\} \\ &= \big\{ i_1 i_2 \cdots \in \{0,1\}^\infty \ : \ i_n i_{n+1} \cdots <_{\mathrm{lex}} \mathbf{a}_\beta \text{ when } i_n = 0, \\ &\qquad\qquad\qquad\qquad\qquad\quad\ i_n i_{n+1} \cdots >_{\mathrm{lex}} \mathbf{b}_\beta \text{ when } i_n = 1 \big\} \end{aligned}$$

$\mathbf{a}_\beta$ quasi-greedy β-exp. of $\pi_\beta(1\overline{0})$, $\mathbf{b}_\beta$ quasi-lazy β-exp. of $\pi_\beta(0\overline{1})$
symmetry: $\mathbf{b}_\beta$ obtained from $\mathbf{a}_\beta$ by exchanging 0 and 1

Daróczy–Kátai'93, Glendinning–Sidorov'01:

$$U_\beta \neq \{\overline{0}, \overline{1}\} \iff \overline{01} \in U_\beta \iff \mathbf{a}_\beta >_{\mathrm{lex}} \overline{01} \iff \beta^2 > \beta + 1 \iff \beta > \tfrac{1+\sqrt{5}}{2}$$

$$\begin{aligned} U_\beta \text{ uncountable } &\iff \mathbf{a}_\beta \geq 01101001\cdots \quad \text{(Thue–Morse sequence)} \\ &\iff \beta \geq 1.78723\cdots \quad \text{(Komornik–Loreti constant)} \end{aligned}$$

Bernoulli convolutions, $\beta \in (1,2]$

$$\pi_\beta : \{0,1\}^\infty \to \left[0, \tfrac{1}{\beta-1}\right], \quad i_1 i_2 \cdots \mapsto \sum_{k=1}^{\infty} \frac{i_k}{\beta^k}$$

$$\nu_\beta = m \circ \pi_\beta^{-1}, \quad \text{with } m \text{ the } (\tfrac{1}{2}, \tfrac{1}{2}) \text{ Bernoulli measure on } \{0,1\}^\infty$$

Jessen–Wintner'35:
ν_β either absolutely continuous w.r.t. Lebesgue or singular

Erdős'39:
singular for Pisot numbers $\beta \in (1,2]$
(algebraic integers $\beta > 1$ s.t. $|\alpha| < 1$ for all conjugates $\alpha \neq \beta$)

Garsia'63, Solomyak'95, Hochman'14, ...

Shmerkin'14:
set of $\beta \in (1,2]$ with singular ν_β has zero Hausdorff dimension

survey by Gouëzel'17/18 (also on dimension of ν_β)

(β_0, β_1)-expansions, $\beta_0, \beta_1 > 1$

$$\pi_{\beta_0,\beta_1}(i_1 i_2 \cdots) := \sum_{k=1}^{\infty} \frac{i_k}{\beta_{i_1}\beta_{i_2}\cdots\beta_{i_k}} \in \left[0, \frac{1}{\beta_1 - 1}\right], \quad i_1 i_2 \cdots \in \{0,1\}^\infty$$

$$X_{\beta_0,\beta_1} := \pi_{\beta_0,\beta_1}(\{0,1\}^\infty), \quad X_{\beta_0,\beta_1} = \frac{1}{\beta_0} X_{\beta_0,\beta_1} \cup \frac{1}{\beta_1}(1 + X_{\beta_0,\beta_1})$$

$$X_{\beta_0,\beta_1} = \left[0, \tfrac{1}{\beta_1 - 1}\right] \iff \left[0, \tfrac{1}{\beta_1-1}\right] = \left[0, \tfrac{1}{\beta_0(\beta_1-1)}\right] \cup \left[\tfrac{1}{\beta_1}, \tfrac{1}{\beta_1-1}\right]$$

$$\iff \tfrac{1}{\beta_0(\beta_1-1)} \ge \tfrac{1}{\beta_1} \iff \beta_0 + \beta_1 \ge \beta_0\beta_1 \iff \tfrac{1}{\beta_0} + \tfrac{1}{\beta_1} \ge 1$$

unique (β_0, β_1)-expansions

$$U_{\beta_0,\beta_1} := \left\{\mathbf{u} \in \{0,1\}^\infty : \pi_{\beta_0,\beta_1}(\mathbf{u}) \ne \pi_{\beta_0,\beta_1}(\mathbf{v}) \text{ for all } \mathbf{v} \ne \mathbf{u}\right\}$$

$$= \left\{i_1 i_2 \cdots \in \{0,1\}^\infty : \pi_{\beta_0,\beta_1}(i_n i_{n+1} \cdots) \notin \left[\tfrac{1}{\beta_1}, \tfrac{1}{\beta_0(\beta_1-1)}\right] \; \forall n \ge 1\right\}$$

$$\overset{\beta_0+\beta_1\ge\beta_0\beta_1}{=} \left\{i_1 i_2 \cdots \in \{0,1\}^\infty : i_n i_{n+1} \cdots < \mathbf{a}_{\beta_0,\beta_1} \text{ or } i_n i_{n+1} \cdots > \mathbf{b}_{\beta_0,\beta_1} \; \forall n \ge 1\right\}$$

$\mathbf{a}_{\beta_0,\beta_1}$ quasi-greedy (β_0, β_1)-expansion of $\frac{1}{\beta_1} = \pi_{\beta_0,\beta_1}(1\overline{0})$,
$\mathbf{b}_{\beta_0,\beta_1}$ quasi-lazy (β_0, β_1)-expansion of $\frac{1}{\beta_0(\beta_1-1)} = \pi_{\beta_0,\beta_1}(0\overline{1})$

Example: $\mathbf{a}_{1.5,2} = \overline{01} = 0101\cdots$, $\mathbf{b}_{1.5,2} = 10\overline{01} = 100101\cdots$

Subshift with a (lexicographic) hole

$$\Omega_{\mathbf{a},\mathbf{b}} := \left\{\mathbf{u} \in \{0,1\}^\infty \,:\, \Sigma^n(\mathbf{u}) \notin (\mathbf{a},\mathbf{b}) \;\; \forall n \geq 0\right\}$$
$$\mathbf{a} \in 0\{0,1\}^\infty,\; \mathbf{b} \in 1\{0,1\}^\infty, \Sigma \text{ shift map,}$$
$$(\mathbf{a},\mathbf{b}) = \{\mathbf{u} \in \{0,1\}^\infty : \mathbf{a} <_{\text{lex}} \mathbf{u} <_{\text{lex}} \mathbf{b}\}$$

$$\Omega_{\mathbf{a},\mathbf{b}} = \left\{i_1 i_2 \cdots \in \{0,1\}^\infty \,:\, i_n i_{n+1} \cdots \leq_{\text{lex}} \mathbf{a} \text{ if } i_n = 0, \; i_n i_{n+1} \cdots \geq_{\text{lex}} \mathbf{b} \text{ if } i_n = 1\right\}$$
$$U_{\beta_0,\beta_1} = \Omega_{\mathbf{a}_{\beta_0,\beta_1},\mathbf{b}_{\beta_0,\beta_1}} \setminus \{0,1\}^*\{\mathbf{a}_{\beta_0,\beta_1},\mathbf{b}_{\beta_0,\beta_1}\}$$

Examples:

$$\Omega_{00\overline{1},11\overline{0}} = \{\overline{0},\overline{1}\}$$
$$\Omega_{0\overline{1},1\overline{0}} = \{0,1\}^\infty$$
$$\Omega_{\overline{01},\overline{10}} = 0^*\overline{01} \cup 1^*\overline{10} \cup \{\overline{0},\overline{1}\}$$

Subshift contained in a (lexicographic) interval

$$\Theta_{\mathbf{a},\mathbf{b}} := \left\{\mathbf{u} \in \{0,1\}^\infty \,:\, \Sigma^n(\mathbf{u}) \in [\mathbf{a},\mathbf{b}] \;\; \forall n \geq 0\right\}$$
$$\Omega_{0\mathbf{b},1\mathbf{a}} = 0^*\Theta_{\mathbf{a},\mathbf{b}} \cup 1^*\Theta_{\mathbf{a},\mathbf{b}} \cup \{\overline{0},\overline{1}\}$$

Thue–Morse–Sturmian substitutions, S-adic words

$$L: \begin{array}{l} 0 \mapsto 0 \\ 1 \mapsto 10 \end{array} \qquad M: \begin{array}{l} 0 \mapsto 01 \\ 1 \mapsto 10 \end{array} \qquad R: \begin{array}{l} 0 \mapsto 01 \\ 1 \mapsto 1 \end{array}$$

limit word (or S-adic word)

$$\boldsymbol{\sigma}(\mathbf{u}) := \lim_{n\to\infty} \sigma_1\sigma_2\cdots\sigma_n(\mathbf{u}),$$

$$\boldsymbol{\sigma} = (\sigma_n)_{n\geq 1} \in \{L, M, R\}^\infty, \ \mathbf{u} \in \{0,1\}^\infty$$

exists because $L(i), M(i), R(i)$ start with i for $i \in \{0,1\}$

Examples:
$\overline{M}(\overline{0}) = \overline{M}(\overline{01}) = \overline{M}(\overline{0110}) = \overline{M}(\overline{01101001}) = 0110100110010110\cdots$
$LM\overline{R}(\overline{0}) = LM\overline{R}(0\overline{1}) = LM(0\overline{1}) = L(01\overline{10}) = 01\overline{010}$

$\boldsymbol{\sigma}$ primitive: $\forall k \geq 1\, \exists\, n \geq k\ \forall i,j \in \{0,1\}$: $\sigma_k\cdots\sigma_n(i)$ contains j
$\boldsymbol{\sigma} \in \{L, M, R\}^\infty$ primitive if and only if $\boldsymbol{\sigma}$ does not end with $\overline{L}$ or $\overline{R}$

Theorem (Labarca–Moreira '06, Glendinning–Sidorov '15, St–Komornik–Zou)

Let $\mathbf{a} \in 0\{0,1\}^\infty$, $\mathbf{b} \in 1\{0,1\}^\infty$.

(i) $\Omega_{\mathbf{a},\mathbf{b}} \neq \{\overline{0}, \overline{1}\}$ *if and only if* $\mathbf{a} = 0\overline{1}$, *or* $\mathbf{b} = 1\overline{0}$, *or*
$\mathbf{a} \geq \sigma(\overline{0})$, $\mathbf{b} \leq \sigma(\overline{1})$ *for some* $\sigma \in \{L, R\}^* M$, *or*
$\mathbf{a} = \boldsymbol{\sigma}(\overline{0})$, $\mathbf{b} = \boldsymbol{\sigma}(\overline{1})$ *for some primitive* $\boldsymbol{\sigma} \in \{L, R\}^\infty$.

(ii) $\Omega_{\mathbf{a},\mathbf{b}} = \{\overline{0}, \overline{1}\}$ *if and only if*
$\mathbf{a} < \sigma(\overline{0})$, $\mathbf{b} \geq \sigma(1\overline{0})$ *for some* $\sigma \in \{L, R\}^* M$, *or*
$\mathbf{a} \leq \sigma(0\overline{1})$, $\mathbf{b} > \sigma(\overline{1})$ *for some* $\sigma \in \{L, R\}^* M$.

(iii) $\Omega_{\mathbf{a},\mathbf{b}}$ *is uncountable with positive entropy if and only if*
$\mathbf{a} \geq \sigma(\overline{0})$, $\mathbf{b} < \sigma(1\overline{0})$ *for some* $\sigma \in \{L, M, R\}^* M$, *or*
$\mathbf{a} > \sigma(0\overline{1})$, $\mathbf{b} \leq \sigma(\overline{1})$ *for some* $\sigma \in \{L, M, R\}^* M$.

(iv) $\Omega_{\mathbf{a},\mathbf{b}}$ *is uncountable with zero entropy if and only if*
$\mathbf{a} = \boldsymbol{\sigma}(\overline{0})$, $\mathbf{b} = \boldsymbol{\sigma}(\overline{1})$ *for some primitive* $\boldsymbol{\sigma} \in \{L, M, R\}^\infty$.

(v) $\Omega_{\mathbf{a},\mathbf{b}}$ *is countable if and only if*
$\mathbf{a} \leq \sigma(01\overline{0})$, $\mathbf{b} \geq \sigma(1\overline{0})$ *for some* $\sigma \in \{L, M, R\}^*$, *or*
$\mathbf{a} \leq \sigma(0\overline{1})$, $\mathbf{b} \geq \sigma(10\overline{1})$ *for some* $\sigma \in \{L, M, R\}^*$.

Elements of the proof

- σ order-preserving for all $\sigma \in \{L, M, R\}^*$

$$L: 0 \mapsto 0 \quad M: 0 \mapsto 01 \quad R: 0 \mapsto 01$$
$$ 1 \mapsto 10 \quad 1 \mapsto 10 \quad 1 \mapsto 1$$

- $\mathbf{u} \in \Omega_{\sigma(\mathbf{a}),\sigma(\mathbf{b})} \setminus \{\bar{0}, \bar{1}\},\ \sigma \in \{L, M, R\} \implies \mathbf{u} \in \{0,1\}^* \sigma(\Omega_{\mathbf{a},\mathbf{b}})$

$$\Omega_{\mathbf{a},\mathbf{b}} = \left\{\mathbf{u} \in \{0,1\}^\infty : \Sigma^n(\mathbf{u}) \notin (\mathbf{a}, \mathbf{b}) \ \ \forall n \geq 0\right\}$$

$$\mathbf{u} \in \Omega_{L(0\bar{1}),L(1\bar{0})} = \Omega_{\overline{01},1\bar{0}} \ \Rightarrow \ \mathbf{u} \in 1^*\{0,10\}^\infty \cup \{\bar{1}\}$$
$$\mathbf{u} \in \Omega_{R(0\bar{1}),R(1\bar{0})} = \Omega_{0\bar{1},\overline{10}} \ \Rightarrow \ \mathbf{u} \in 0^*\{1,01\}^\infty \cup \{\bar{0}\}$$
$$\mathbf{u} \in \Omega_{M(0\bar{1}),M(1\bar{0})} = \Omega_{01\overline{10},10\overline{01}} \ \Rightarrow \ \mathbf{u} \in 0^*\{01,10\}^\infty \cup 1^*\{01,10\}^\infty \cup \{\bar{0}, \bar{1}\}$$

- $\Omega_{M(\bar{0}),M(\bar{1})} = 0^*\overline{01} \cup 1^*\overline{10} \cup \{\bar{0}, \bar{1}\} \neq \{\bar{0}, \bar{1}\}$

- $\Omega_{M(\bar{0}),\mathbf{b}}$, $\mathbf{b} < M(1\bar{0})$, contains $\{0(01)^k, 0(01)^{k+1}\}^\infty$ for some $k \geq 0$,
  $\Omega_{\mathbf{a},M(\bar{1})}$, $\mathbf{a} > M(0\bar{1})$, contains $\{1(10)^k, 1(10)^{k+1}\}^\infty$ for some $k \geq 0$
  $\Rightarrow$ uncountable with positive entropy

Partitions

$$\big(\sigma(01\bar{0}),\sigma(0\bar{1})\big) = \underbrace{\big(\sigma(01\bar{0}),\sigma(\overline{01})\big)}_{(\sigma L(01\bar{0}),\sigma L(0\bar{1}))} \cup \underbrace{\big[\sigma(\overline{01}),\sigma(01\overline{10})\big]}_{[\sigma M(\bar{0}),\sigma M(0\bar{1})]} \cup \underbrace{\big(\sigma(01\overline{10}),\sigma(0\bar{1})\big)}_{(\sigma R(01\bar{0}),\sigma R(0\bar{1}))}$$

$$(01\bar{0},0\bar{1}) = \bigcup_{\sigma\in\{L,R\}^* M} \big[\sigma(\bar{0}),\sigma(0\bar{1})\big] \;\cup\; \big\{\boldsymbol{\sigma}(\bar{0}) : \boldsymbol{\sigma} \in \{L,R\}^\infty \text{ primitive}\big\}$$

$$\big(\sigma(01\bar{0}),\sigma(0\bar{1})\big) = \underbrace{\big(\sigma(01\bar{0}),\sigma(\overline{01})\big)}_{(\sigma L(01\bar{0}),\sigma L(0\bar{1}))} \cup \underbrace{\big[\sigma(\overline{01}),\sigma(0110\overline{01})\big]}_{[\sigma M(\bar{0}),\sigma M(01\bar{0})]}$$

$$\cup \underbrace{\big(\sigma(0110\overline{01}),\sigma(01\overline{10})\big)}_{(\sigma M(01\bar{0}),\sigma M(0\bar{1}))} \cup \Big\{\underbrace{\sigma(01\overline{10})}_{\sigma M(0\bar{1})}\Big\} \cup \underbrace{\big(\sigma(01\overline{10}),\sigma(0\bar{1})\big)}_{(\sigma R(01\bar{0}),\sigma R(0\bar{1}))}$$

$$(01\bar{0},0\bar{1}) = \bigcup_{\sigma\in\{L,M,R\}^* M} \Big(\big[\sigma(\bar{0}),\sigma(01\bar{0})\big] \cup \big\{\sigma(0\bar{1})\big\}\Big) \cup \{\boldsymbol{\sigma}(\bar{0}) : \boldsymbol{\sigma} \in \{L,M,R\}^\infty \text{ primitive}\}$$

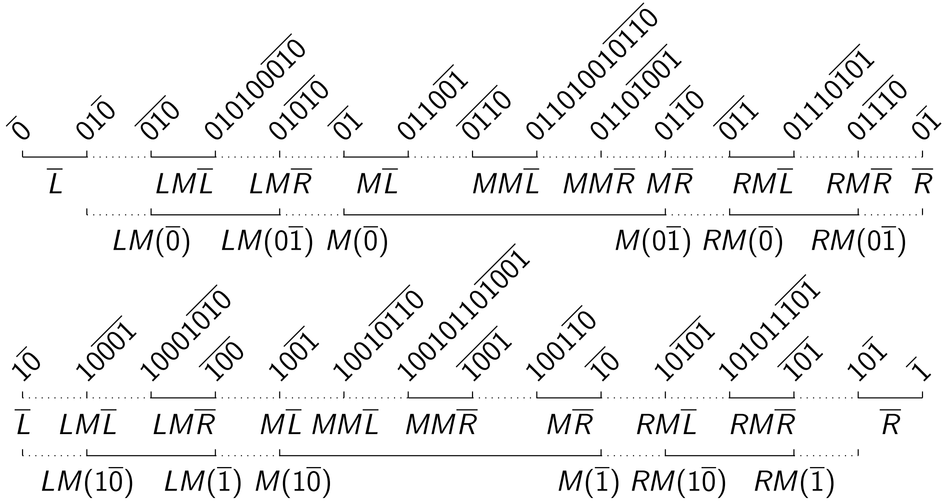
$$[\bar{0},0\bar{1}] = \bigcup_{\boldsymbol{\sigma}\in\{L,M,R\}^\infty\setminus\{L,M,R\}^*\{L\bar{R},R\bar{L}\}} [\boldsymbol{\sigma}(\bar{0}),\boldsymbol{\sigma}(0\bar{1})]$$

$$\boldsymbol{\sigma} <_{\mathrm{lex}} \boldsymbol{\tau} \Rightarrow \boldsymbol{\sigma}(0\bar{1}) <_{\mathrm{lex}} \boldsymbol{\tau}(\bar{0}) \quad (\boldsymbol{\sigma},\boldsymbol{\tau} \in \{L,M,R\}^\infty \setminus \{L,M,R\}^*\{L\bar{R},R\bar{L}\})$$

$$s : \{0,1\}^\infty \to \{L,M,R\}^\infty \setminus \{L,M,R\}^*\{L\overline{R}, R\overline{L}\},$$

$$\mathbf{u} \mapsto \boldsymbol{\sigma} \quad \text{if } \mathbf{u} \in [\boldsymbol{\sigma}(\overline{0}), \boldsymbol{\sigma}(0\overline{1})] \cup [\boldsymbol{\sigma}(1\overline{0}), \boldsymbol{\sigma}(\overline{1})],$$

montonically increasing on $0\{0,1\}^\infty$, $1\{0,1\}^\infty$



Theorem (reformulation)

Let $\mathbf{a} \in 0\{0,1\}^\infty$, $\mathbf{b} \in 1\{0,1\}^\infty$.

(i) $\Omega_{\mathbf{a},\mathbf{b}} \neq \{\bar{0}, \bar{1}\}$ *if and only if* $\mathbf{a} = 0\bar{1}$, *or* $\mathbf{b} = 1\bar{0}$, *or*
$\mathbf{a} \geq \sigma(\bar{0})$, $\mathbf{b} \leq \sigma(\bar{1})$ *for some* $\sigma \in \{L, R\}^* M$, *or*
$\mathbf{a} = \boldsymbol{\sigma}(\bar{0})$, $\mathbf{b} = \boldsymbol{\sigma}(\bar{1})$ *for some primitive* $\boldsymbol{\sigma} \in \{L, R\}^\infty$.

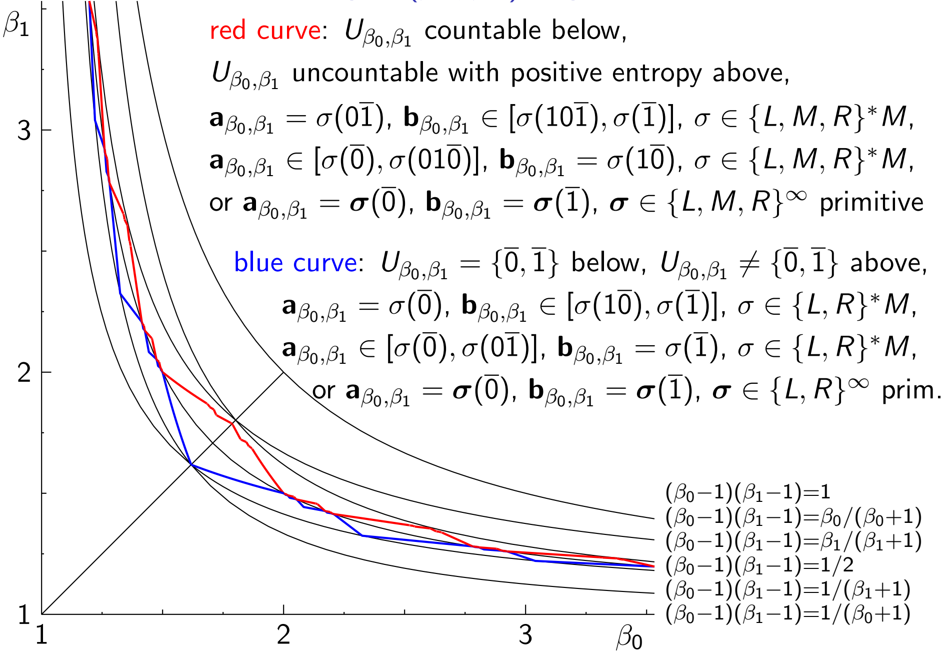
(ii) $\Omega_{\mathbf{a},\mathbf{b}} = \{\bar{0}, \bar{1}\}$ *if and only if*
$\mathbf{a} < \sigma(\bar{0})$, $\mathbf{b} \geq \sigma(1\bar{0})$ *for some* $\sigma \in \{L, R\}^* M$, *or*
$\mathbf{a} \leq \sigma(0\bar{1})$, $\mathbf{b} > \sigma(\bar{1})$ *for some* $\sigma \in \{L, R\}^* M$.

(iii') $\Omega_{\mathbf{a},\mathbf{b}}$ *is uncountable with positive entropy if and only if* $s(\mathbf{a}) > s(\mathbf{b})$.

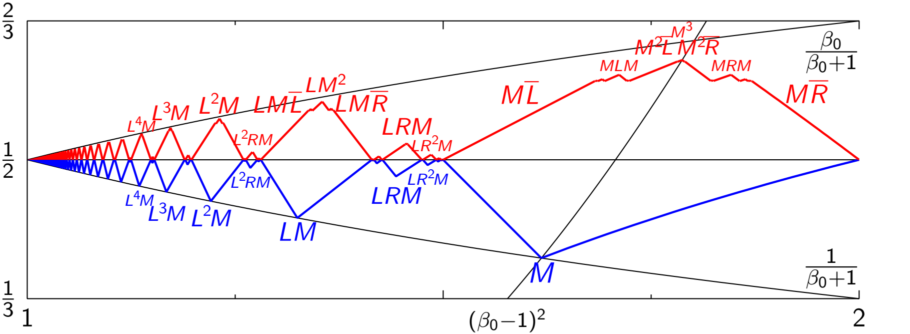
(iv') $\Omega_{\mathbf{a},\mathbf{b}}$ *is uncountable with zero entropy if and only if* $s(\mathbf{a}) = s(\mathbf{b})$ *is primitive.*

(v') $\Omega_{\mathbf{a},\mathbf{b}}$ *is countable if and only if* $s(\mathbf{a}) < s(\mathbf{b})$ *or* $s(\mathbf{a}) = s(\mathbf{b})$ *ends with* $\overline{L}$ *or* $\overline{R}$.

Critical curves for unique (β_0, β_1)-expansions



Critical values $(\beta_0, (\beta_0-1)(\beta_1-1))$ for unique β-expansions



red curve: U_{β_0,β_1} countable below, U_{β_0,β_1} positive entropy above,
$\mathbf{a}_{\beta_0,\beta_1} = \sigma(0\overline{1})$, $\mathbf{b}_{\beta_0,\beta_1} \in [\sigma(10\overline{1}), \sigma(\overline{1})]$, $\sigma \in \{L, M, R\}^* M$,
$\mathbf{a}_{\beta_0,\beta_1} \in [\sigma(\overline{0}), \sigma(01\overline{0})]$, $\mathbf{b}_{\beta_0,\beta_1} = \sigma(1\overline{0})$, $\sigma \in \{L, M, R\}^* M$,
or $\mathbf{a}_{\beta_0,\beta_1} = \boldsymbol{\sigma}(\overline{0})$, $\mathbf{b}_{\beta_0,\beta_1} = \boldsymbol{\sigma}(\overline{1})$, $\boldsymbol{\sigma} \in \{L, M, R\}^\infty$ primitive

blue curve: $U_{\beta_0,\beta_1} = \{\overline{0}, \overline{1}\}$ below, $U_{\beta_0,\beta_1} \neq \{\overline{0}, \overline{1}\}$ above,
$\mathbf{a}_{\beta_0,\beta_1} = \sigma(\overline{0})$, $\mathbf{b}_{\beta_0,\beta_1} \in [\sigma(1\overline{0}), \sigma(\overline{1})]$, $\sigma \in \{L, R\}^* M$,
$\mathbf{a}_{\beta_0,\beta_1} \in [\sigma(\overline{0}), \sigma(0\overline{1})]$, $\mathbf{b}_{\beta_0,\beta_1} = \sigma(\overline{1})$, $\sigma \in \{L, R\}^* M$,
or $\mathbf{a}_{\beta_0,\beta_1} = \boldsymbol{\sigma}(\overline{0})$, $\mathbf{b}_{\beta_0,\beta_1} = \boldsymbol{\sigma}(\overline{1})$, $\boldsymbol{\sigma} \in \{L, R\}^\infty$ primitive

Unique β-expansions with digits $0, 1, 2$

$$X_\beta = \frac{1}{\beta} X_\beta \cup \frac{1}{\beta}(1 + X_\beta) \cup \frac{1}{\beta}(2 + X_\beta)$$

$$\beta \leq 3: \ \left[0, \tfrac{2}{\beta-1}\right] = \left[0, \tfrac{2}{\beta(\beta-1)}\right] \cup \left[\tfrac{1}{\beta}, \tfrac{1}{\beta} + \tfrac{2}{\beta(\beta-1)}\right] \cup \left[\tfrac{2}{\beta}, \tfrac{2}{\beta-1}\right]$$

$$U_\beta(\{0,1,2\}) = \left\{i_1 i_2 \cdots \in \{0,1,2\}^\infty : i_n i_{n+1} \cdots \notin [\mathbf{a}_\beta, \mathbf{b}_\beta] \cup [\mathbf{c}_\beta, \mathbf{d}_\beta] \ \forall n \geq 1\right\}$$

$$= \left\{i_1 i_2 \cdots \in \{0,1,2\}^\infty : \pi_\beta(i_n i_{n+1} \cdots) \notin \left[\tfrac{1}{\beta}, \tfrac{2}{\beta(\beta-1)}\right] \cup \left[\tfrac{2}{\beta}, \tfrac{1}{\beta} + \tfrac{2}{\beta(\beta-1)}\right] \forall n \geq 1\right\}$$

$\mathbf{a}_\beta$ quasi-greedy exp. of $\pi_\beta(1\overline{0})$, $\mathbf{b}_\beta$ quasi-lazy exp. of $\pi_\beta(0\overline{2})$,
$\mathbf{c}_\beta$ quasi-greedy exp. of $\pi_\beta(2\overline{0})$, $\mathbf{d}_\beta$ quasi-lazy exp. of $\pi_\beta(1\overline{2})$,
$\mathbf{a}_\beta$ equal to $\mathbf{c}_\beta$ up to first letter, $\mathbf{b}_\beta$ equal to $\mathbf{d}_\beta$ up to first letter,
symmetry: $\mathbf{c}_\beta, \mathbf{d}_\beta$ obtained from $\mathbf{b}_\beta, \mathbf{a}_\beta$ by exchanging 0 and 2

$$U_\beta(\{0,1,2\}) \neq \{\overline{0}, \overline{2}\} \Leftrightarrow \overline{1} \in U_\beta(\{0,1,2\}) \Leftrightarrow \tfrac{2}{\beta(\beta-1)} < \tfrac{1}{\beta-1} < \tfrac{2}{\beta} \Leftrightarrow \beta > 2$$

Allouche '83, Komornik–Loreti '02, Allouche–Frougny '09:

$$U_\beta(\{0,1,2\}) \text{ uncountable } \Longleftrightarrow \mathbf{a}_\beta \geq \iota(0\overline{N}(\overline{2})) = 0210201210120210\cdots$$

$$N: \ 0 \mapsto 01_2 \quad 1_0 \mapsto 02 \quad 1_2 \mapsto 20 \quad 2 \mapsto 21_0$$

$$\iota: \ 0 \mapsto 0 \quad 1_0 \mapsto 1 \quad 1_2 \mapsto 1 \quad 2 \mapsto 2$$

Subshifts with two holes

$$\Omega_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}} := \big\{\mathbf{u} \in \{0,1,2\}^\infty \,:\, \Sigma^n(\mathbf{u}) \notin (\mathbf{a},\mathbf{b}) \cup (\mathbf{c},\mathbf{d}) \ \ \forall n \ge 0\big\}$$
$$(\mathbf{a} \in 0\{0,1,2\}^\infty,\ \mathbf{b} \in 1\{0,1,2\}^\infty,\ \mathbf{c} \in 1\{0,1,2\}^\infty,\ \mathbf{d} \in 2\{0,1,2\}^\infty)$$

$$U_\beta(\{0,1,2\}) = \Omega_{\mathbf{a}_\beta,\mathbf{b}_\beta,\mathbf{c}_\beta,\mathbf{d}_\beta} \setminus \{0,1,2\}^*\{\mathbf{a}_\beta,\mathbf{b}_\beta,\mathbf{c}_\beta,\mathbf{d}_\beta\}$$
$$\Sigma\mathbf{a}_\beta = \Sigma\mathbf{c}_\beta,\ \Sigma\mathbf{b}_\beta = \Sigma\mathbf{d}_\beta,\ \mathbf{d}_\beta = F(\mathbf{a}_\beta),\ \mathbf{c}_\beta = F(\mathbf{b}_\beta)$$

(F exchanges letters 0 and 2)

$$\Omega_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}} \text{ uncountable},\ \Sigma\mathbf{a} = \Sigma\mathbf{c},\ \Sigma\mathbf{b} = \Sigma\mathbf{d},\ \mathbf{d} = F(\mathbf{a}),\ \mathbf{c} = F(\mathbf{b})$$
$$\iff \mathbf{a} \ge \iota(0\overline{N}(\overline{2})) = \iota\overline{N'N''}(\overline{0}) = 0210201210120210\cdots$$

$$\begin{array}{rlrlrl} \iota: & 0 \mapsto 0 & N': & 0 \mapsto 02 & N'': & 0 \mapsto 01_0 \\ & 1_0 \mapsto 1 & & 1_0 \mapsto 1_0 0 & & 1_0 \mapsto 1_0 1_2 \\ & 1_2 \mapsto 1 & & 1_2 \mapsto 1_2 2 & & 1_2 \mapsto 1_2 1_0 \\ & 2 \mapsto 2 & & 2 \mapsto 20 & & 2 \mapsto 21_2 \end{array}$$

Conditions for non-triviality and uncountability of $\Omega_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}$ when $\Sigma\mathbf{a} = \Sigma\mathbf{c}$, $\Sigma\mathbf{b} = \Sigma\mathbf{d}$ OR $\mathbf{d} = F(\mathbf{a})$, $\mathbf{c} = F(\mathbf{b})$, general $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}$?