Ramification

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Outline

Primitive recursion

A first order structure for unary numbers

- $\mathbf{Nat} = \langle 0, \mathbf{suc} \rangle$

is a set of data objects defined by

- O is a number of **Nat**
- if n is a number, then $\mathbf{suc}(n)$ is a number of **Nat**

Semantics

$$[\![\mathbf{Nat}]\!] = \mathbb{N}$$

Primitive recursion over **Nat**

$$f(0, \overline{x}) = g(\overline{x})$$
$$f(\mathbf{suc}(n), \overline{x}) = h(n, \overline{x}, f(n, \overline{x}))$$

Primitive recursion on Words

A first order structure for binary words

$$\textbf{Word} = \langle \epsilon, \mathbf{0}, \mathbf{1} \rangle$$

is a set of data objects defined by

- ϵ is a word of **Word**
- if u is a word, then $\mathbf{0}(u)$ and $\mathbf{1}(u)$ are words of **Word**

Semantics

$$[\![\textbf{Word}]\!] = \{0, 1\}^*$$

Primitive recursive functions on Words

The class of primitive recursive functions over **Word** contains

- Constructors of **Word**

$$x \mapsto \epsilon \qquad x \mapsto \mathbf{0}(x) \qquad x \mapsto \mathbf{1}(x)$$

- Projections : $\pi_i(x_1, \ldots, x_n) = x_i$

is closed under

- Composition

$$f(\overline{x}) = h(g_1(\overline{x}), \ldots, g_k(\overline{x}))$$

  where $\overline{x} = x_1, \ldots, x_n$
- and Primitive recursion over **Word**

$$\begin{aligned} f(\epsilon, \overline{x}) &= g(\overline{x}) \\ f(\mathbf{0}(w), \overline{x}) &= h_0(w, \overline{x}, f(w, \overline{x})) \\ f(\mathbf{1}(w), \overline{x}) &= h_1(w, \overline{x}, f(w, \overline{x})) \qquad \overline{x} = x_1, \ldots, x_n \end{aligned}$$

  - recurrence parameter : w
  - recursive call : $f(w, \overline{x})$

Word concatenation or addition

Example

$$
\begin{aligned}
\mathrm{add}(\epsilon, x) &= x \\
\mathrm{add}(\mathbf{0}(w), x) &= \mathbf{0}(\mathrm{add}(w, x)) \\
\mathrm{add}(\mathbf{1}(w), x) &= \mathbf{1}(\mathrm{add}(w, x))
\end{aligned}
$$

$$[\![\mathrm{add}]\!] : (\{0,1\}^*)^2 \mapsto \{0,1\}^*$$

$$\mathrm{add}(\mathbf{1}(\mathbf{0}(\epsilon)), v) = \mathbf{1}(\mathbf{0}(v))$$

Primitive recursion on an arbitrary structure Σ

A first order structure $\Sigma = \langle a_1, \ldots, a_n, b_1, \ldots, b_m \rangle$
Primitive recursion over Σ

$$
\begin{aligned}
f(a_i, \overline{x}) &= g_i(\overline{x}) && i = 1, n \\
f(b_j(w_1, \ldots, w_n), \overline{x}) &= h_j(\overline{w}, \overline{x}, f(w_1, \overline{x}), \ldots \\
&\qquad \ldots, f(w_n, \overline{x})) && j = 1, m
\end{aligned}
$$

Theorem

The class of primitive recursive functions over Σ is exactly the set of primitive recursive functions over natural numbers.

Bounded recursion

- G_0 is the class containing zero, suc., projections and closed under composition and bounded recursion:

$$
\begin{aligned}
f(0,\overline{x}) &= g(\overline{x}) \\
f(\mathbf{suc}(t),\overline{x}) &= h(t,\overline{x},f(t,\overline{x})) \\
f(t,\overline{x}) &\leq k(t,\overline{x}) \qquad \text{for } k \text{ is in } G_0
\end{aligned}
$$

Grzegorczyk Hierarchy

$$E_0(x,y) = x + y \qquad E_1(x) = x^2 + 1$$
$$E_{n+2}(0) = 2 \qquad E_{n+2}(x+1) = E_{n+1}(E_{n+2}(x))$$

- G_{n+1} is the class containing zero, **suc**, projections, E_n and closed under composition and bounded recursion:

$$\begin{aligned} f(0,\overline{x}) &= g(\overline{x}) \\ f(\mathbf{suc}(t),\overline{x}) &= h(t,\overline{x},f(t,\overline{x})) \\ f(t,\overline{x}) &\leq k(t,\overline{x}) \qquad \text{for } k \text{ is in } G_{n+1} \end{aligned}$$

Theorem

The union $\cup_n G_n$ is the class of P.R. functions.

- G_3 is the class of elementary functions (Kalmar)

PTIME

Definition

PTIME is the set of functions which are computed in polynomial time with a Turing machine.

- PTIME computationally tractable problems, *Cook's thesis*
- All reasonable formalizations of the intuitive notion of tractable computability are equivalent within a polynomial bounded overhead
- Polynomial-time Turing machines computability capture all tractable functions.

Bounded recursion overs words

The class $\mathcal{L}$ contains

- Constructors of **Word**

$$x \mapsto \epsilon \qquad x \mapsto \mathbf{0}(x) \qquad x \mapsto \mathbf{1}(x)$$

- Projections : $\pi_i(x_1, \ldots, x_n) = x_i$
- The *smash* function $x \# y = 2^{|x| \cdot |y|}$ where $|x|$ is the length of x.

and is closed under

- composition
- and bounded recursion

$$\begin{aligned} f(\epsilon, \overline{x}) &= g(\overline{x}) \\ f(\mathbf{0}(t), \overline{x}) &= h(t, \overline{x}, f(t, \overline{x})) \\ f(\mathbf{1}(t), \overline{x}) &= h(t, \overline{x}, f(t, \overline{x})) \\ f(t, \overline{x}) &\leq k(t, \overline{x}) \qquad k \text{ is in } \mathcal{L} \end{aligned}$$

Cobham's Characterization of Ptime

$$
\begin{aligned}
f(\epsilon, \overline{x}) &= g(\overline{x}) \\
f(\mathbf{0}(t), \overline{x}) &= h(t, \overline{x}, f(t, \overline{x})) \\
f(\mathbf{1}(t), \overline{x}) &= h(t, \overline{x}, f(t, \overline{x})) \\
f(t, x) &\leq k(t, \overline{x}) \qquad\qquad k \text{ is in } \mathcal{L}
\end{aligned}
$$

Based on Ritchies's work:

Theorem (Cobham (65))

The class $\mathcal{L}$ is exactly the class of PTIME *of functions which are computable in Polynomial time.*

Bounded recursion as a complexity model

- A lof of characterizations of complexity classes follow the Cobham's idea. See the survey of Clote.
- Polynomial resource bound is inside the $\mathcal{L}$'s formalization
- Not intrinsic : Separate resources from algorithms
- Applications
  - Difficult to show that a program is PTIME
  - Difficult to extract complexity bounds

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P.R. Global functions

- Interpret $[\![\mathbf{Nat}]\!] = \{0, \ldots, n\}$
- See f is a primitive recursive schema
- Define $[\![f]\!]_n$ as the interpretation of f over $\{0, \ldots, n\}$ where

$$\mathbf{suc}(m) = \begin{cases} m+1 & \text{if } m < n \\ n & \text{if } n = m \end{cases}$$

A global function F is defined from a primitive recursive schema f

$$F(n, \overline{x}) = [\![f]\!]_n(\overline{x}) \qquad\qquad x_i \leq n$$

Theorem (Gurevich)

The set of global functions is exactly the set of LOGSPACE *functions*

P.R. Global functions and PTIME

- Sazonov and Gurevich characterize PTIME using the Herbrand-Gödel equations over finite structures.
- Jones characterizes PTIME using cons-free while language.

reference

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Domains with two colors

Two first order structures for binary words

$$\mathbf{Word} = \langle \epsilon, \mathbf{0}, \mathbf{1} \rangle \qquad \text{Normal}$$

$$\mathbf{Word} = \langle \epsilon, \mathbf{0}, \mathbf{1} \rangle \qquad \text{Safe}$$

Functions over domains with colors :

$$[\![f]\!] : [\![\mathbf{Word}]\!]^p \times [\![\mathbf{Word}]\!]^q \to [\![\mathbf{Word}]\!]$$

$$\overline{x}, \overline{y} \to f(\overline{x}; \overline{y})$$

Note the semicolon ; separates arguments

Safe Composition and Recursion

- safe composition

$$f(\overline{x};\overline{y}) = g(h_1(\overline{x};);h_2(\overline{x};\overline{y}))$$

- safe recursion

$$\begin{aligned} f(\epsilon,\overline{x};\overline{y}) &= g(\overline{x};\overline{y}) \\ f(\mathbf{0}(z),\overline{x};\overline{y}) &= h_0(z,\overline{x};f(z,\overline{x};\overline{y}),\overline{y}) \\ f(\mathbf{1}(z),\overline{x};\overline{y}) &= h_1(z,\overline{x};f(z,\overline{x};\overline{y}),\overline{y}) \end{aligned}$$

Recursive calls are safe !!

Data ramification

Data Ramification implies

$$\mathbf{Word} > \mathbf{Word}$$

because

$$f(x;) = g(;I(x;)) \qquad \text{Safe comp}$$
$$I(x;) = x \qquad \text{projection}$$

But the converse does not hold !!

Safe Recursive functions

The class $\mathcal{B}$ of *safe recursive functions* contains
Safe basic functions

- Constructors : $x \mapsto \epsilon$, $x \mapsto \mathbf{0}(; x)$, and $x \mapsto \mathbf{1}(; x)$
- Predecessor : $p(; \epsilon) = \epsilon$, $p(; \mathbf{i}(; x)) = x$
- Conditional : $C(; x, y, z) = \begin{cases} y & \text{if } x = \mathbf{0}(x') \\ z & \text{otherwise} \end{cases}$
- Projections : $\pi_i(x_1, \ldots, x_n; x_{n+1}, \ldots, x_{n+m}) = x_i$

and is closed

- safe composition
- safe recursion

Examples

Concatenation or addition :

$$\begin{aligned} \mathtt{add}(\epsilon; x) &= x \\ \mathtt{add}(\mathbf{0}(w); x) &= \mathbf{0}(; \mathtt{add}(w; x)) \\ \mathtt{add}(\mathbf{1}(w); x) &= \mathbf{1}(; \mathtt{add}(w; x)) \end{aligned}$$

Multiplication by iterating addition

$$\begin{aligned} \mathtt{mul}(\epsilon, y;) &= \epsilon \\ \mathtt{mul}(\mathbf{0}(v), y;) &= \mathtt{add}(y; \mathtt{mul}(v, y;)) \\ \mathtt{mul}(\mathbf{1}(v), y;) &= \mathtt{add}(y; \mathtt{mul}(v, y;)) \end{aligned}$$

Theorem (Bellantoni-Cook)

The set PTIME *of functions which are computable in polynomial time is exactly the class* $\mathcal{B}$ *of safe recursive functions.*

- *Simmons (88)* was the first to suggest this data-separation for primitive recursion.

Exponential is not safe !

Double length function is safe

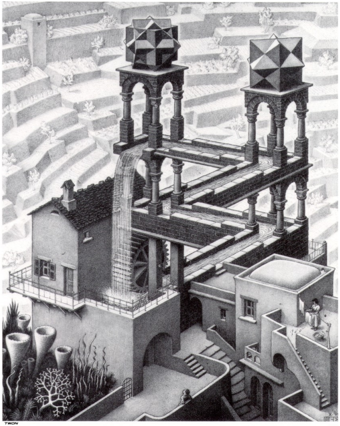
$$\begin{aligned}\mathtt{double}(\epsilon;) &= \epsilon\\ \mathtt{double}(\mathbf{0}(w);) &= \mathbf{1}(;\mathbf{1}(;\mathtt{double}(w;)))\\ \mathtt{double}(\mathbf{1}(w);) &= \mathbf{1}(;\mathbf{1}(;\mathtt{double}(w;)))\end{aligned}$$

Exponential by doubling is not safe

$$\begin{aligned}\mathtt{exp}(\epsilon;) &= \mathbf{1}(\epsilon)\\ \mathtt{exp}(\mathbf{0}(v);) &= \mathtt{double}(\mathtt{exp}(v;))\\ \mathtt{exp}(\mathbf{1}(v);) &= \mathtt{double}(\mathtt{exp}(v;))\end{aligned}$$

The recursive call $\mathtt{exp}(v;)$ should be safe. But `double` requires a normal argument.

A well-known Escher drawing to make a break



$$f : \mathbf{Word}(n+1) \rightarrow \mathbf{Word}(n)$$

Analysis of the energy of arguments in recursive definitions

Domains is stratified by tiers

Word(k) Domain of words of tier k

$\mathbf{0}, \mathbf{1} : \mathbf{Word}(k) \rightarrow \mathbf{Word}(k)$ and $\epsilon : \mathbf{Word}(k)$

Word(1) Recursion

$\mathbf{0}, \mathbf{1} : \mathbf{Word}(1) \rightarrow \mathbf{Word}(1)$ and $\epsilon : \mathbf{Word}(1)$

Word(0) No recursion - flat domain

$\mathbf{0}, \mathbf{1} : \mathbf{Word}(0) \rightarrow \mathbf{Word}(0)$ and $\epsilon : \mathbf{Word}(0)$

Refer to the same set of words, $[\![\mathbf{Word}(k)]\!] = \{0, 1\}^*$

Tiered recursion

$$f(\epsilon, y) = g(y)$$
$$f(\mathbf{0}(x), y) = h_0(x, y, f(x, y))$$
$$f(\mathbf{1}(x), y) = h_1(x, y, f(x, y))$$

$$g : \mathbf{Word}(m) \to \mathbf{Word}(n)$$
$$h_i : \mathbf{Word}(n+1) \to \mathbf{Word}(m) \to \mathbf{Word}(n) \to \mathbf{Word}(n)$$
$$f : \mathbf{Word}(n+1), \mathbf{Word}(m) \to \mathbf{Word}(n)$$

Tier of Recurrence param. $>$ Tier of the recursive calls

Now the inputs and outputs have colors.

Keep that in mind !!!

Addition and Multiplication

Concatenation or addition :

$$\begin{aligned} \mathtt{add}(\epsilon, x) &= x \\ \mathtt{add}(\mathbf{0}(w), x) &= \mathbf{0}(\mathtt{add}(w, x)) \\ \mathtt{add}(\mathbf{1}(w), x) &= \mathbf{1}(\mathtt{add}(w, x)) \end{aligned}$$

$$\mathtt{add} : \mathbf{Word}(n+1) \to \mathbf{Word}(n) \to \mathbf{Word}(n)$$

Multiplication by iterating addition

$$\begin{aligned} \mathtt{mul}(\epsilon, y) &= \epsilon \\ \mathtt{mul}(\mathbf{0}(v), y) &= \mathtt{add}(y, \mathtt{mul}(v, y)) \\ \mathtt{mul}(\mathbf{1}(v), y) &= \mathtt{add}(y, \mathtt{mul}(v, y)) \end{aligned}$$

$$\mathtt{mul} : \mathbf{Word}(n+1) \to \mathbf{Word}(n+1) \to \mathbf{Word}(n)$$

Down casting

$$\texttt{cast}(\epsilon) = \epsilon$$
$$\texttt{cast}(\mathbf{0}(w)) = \mathbf{0}(\texttt{cast}(w))$$
$$\texttt{cast}(\mathbf{1}(w)) = \mathbf{1}(\texttt{cast}(w))$$

$$\texttt{cast} : \textbf{Word}(n+1) \to \textbf{Word}(n)$$

But up-casting is forbidden !
$\Rightarrow$ strict data ramification

$$\ldots > \textbf{Word}(k+1) > \textbf{Word}(k) > \ldots > \textbf{Word}(1) > \textbf{Word}(0)$$

Flat recurrence

Special case of tiered recursion where there is no recursive call

$$f(\epsilon, \overline{y}) = g(\overline{y})$$

$$g : \mathbf{Word}(n) \to \mathbf{Word}(n)$$

$$f(\mathbf{0}(x), \overline{y}) = h_0(x, \overline{y})$$

$$h_i : \mathbf{Word}(n) \to \mathbf{Word}(n) \to \mathbf{Word}(n)$$

$$f(\mathbf{1}(x)), \overline{y}) = h_1(x, \overline{y})$$

$$f : \mathbf{Word}(n) \to \mathbf{Word}(n) \to \mathbf{Word}(n)$$

$$\mathtt{pred}(\epsilon) = \epsilon \qquad \mathtt{pred}(\mathbf{0}(x), y) = x$$

$$\mathtt{pred}(\mathbf{1}(x), y) = x$$

$$\mathtt{cond}(\epsilon, y, z, w) = w \qquad \mathtt{cond}(\mathbf{0}(x), y, z, w) = y$$

$$\mathtt{cond}(\mathbf{1}(x), y, z, w) = z$$

Simultaneous tiered recursion

$$
\begin{aligned}
f_0(\epsilon,\overline{y}) = g_0(\overline{y}) \quad \ldots \quad f_p(\epsilon,\overline{y}) = g_p(\overline{y})\\
f_0(\mathbf{0}(x),\overline{y}) = h_0(x,\overline{y},f_0(x,\overline{y}),\ldots,f_p(x,\overline{y}))\\
f_0(\mathbf{1}(x),\overline{y}) = h'_0(x,\overline{y},f_0(x,\overline{y}),\ldots,f_p(x,\overline{y}))\\
\ldots\\
f_p(\mathbf{0}(x),\overline{y}) = h_0(x,\overline{y},f_0(x,\overline{y}),\ldots,f_p(x,\overline{y}))\\
f_p(\mathbf{1}(x),\overline{y}) = h'_0(x,\overline{y},f_0(x,\overline{y}),\ldots,f_p(x,\overline{y}))
\end{aligned}
$$

where

$$
\begin{aligned}
g_i : \mathbf{Word}(i) \rightarrow \mathbf{Word}(j)\\
h_i, h'_i : \mathbf{Word}(k+1) \rightarrow \mathbf{Word}(i) \rightarrow \mathbf{Word}(j)^p \rightarrow \mathbf{Word}(j)\\
f_i : \mathbf{Word}(k+1) \rightarrow \mathbf{Word}(i) \rightarrow \mathbf{Word}(j)
\end{aligned}
$$

Tiering condition implies

$$k+1 \geq i > j$$

Characterization of PTIME

Definition

The class $\mathbf{TRec}^*(\mathbf{Word})$ is the set of functions defined by simultaneous tiered recursion and explicit definitions (projections and composition well typed).

Theorem (Leivant 94)

The three sets are identical

- *The set* PTIME *of functions computable in polynomial time.*
- *The set* $\mathbf{TRec}^*(\boldsymbol{Word})$ *using any tiers*
- *The set* $\mathbf{TRec}^*(\boldsymbol{Word})$ *using 2 tiers only*

Proof.

We are going to sketch it shortly. $\square$

Another look at the exponential

Double the length

$$\begin{aligned}\mathtt{double}(\epsilon) &= \epsilon \\ \mathtt{double}(\mathbf{0}(w)) &= \mathbf{1}(\mathbf{1}(\mathtt{double}(w))) \\ \mathtt{double}(\mathbf{1}(w)) &= \mathbf{1}(\mathbf{1}(\mathtt{double}(w)))\end{aligned}$$

$$\mathtt{double} : \mathbf{Word}(n+1) \to \mathbf{Word}(n)$$

Exponential by doubling

$$\begin{aligned}\mathtt{exp}(\epsilon) &= \mathbf{1}(\epsilon) \\ \mathtt{exp}(\mathbf{0}(w)) &= \mathtt{double}(\mathtt{exp}(w)) \\ \mathtt{exp}(\mathbf{1}(w)) &= \mathtt{double}(\mathtt{exp}(w))\end{aligned}$$

$$\mathtt{exp} : \mathbf{Word}(k+1) \to \mathbf{Word}(k)$$

No solution, this definition is circular !!!

Tiered recursion captures PTIME

Take a Turing Machine M,

- q is a state
- u the left tape
- v the right tape
- the head is scanning the first letter of u

$$\begin{aligned}\texttt{state}(q,u,v) &= \text{next state}\\ \texttt{left}(q,u,v) &= \text{left side of the tape}\\ \texttt{right}(q,u,v) &= \text{right side of the tape}\end{aligned}$$

build by explicit definitions

$$\texttt{state}, \texttt{left}, \texttt{right} : \textbf{Word}(0)^3 \rightarrow \textbf{Word}(0)$$

Linear length Iteration

$$
\begin{aligned}
T_1^S(\epsilon, q, u, v) &= q \\
T_1^L(\epsilon, q, u, v) &= u \\
T_1^R(\epsilon, q, u, v) &= v \\
T_1^S(\mathbf{i}(t), q, u, v) &= \mathtt{state}(T_1^S(t, q, u, v), \\
&\qquad T_1^L(t, q, u, v), T_1^R(t, q, u, v)) \\
T_1^L(\mathbf{i}(t), q, u, v) &= \mathtt{left}(T_1^S(t, q, u, v), \\
&\qquad T_1^L(t, q, u, v), T_1^R(t, q, u, v)) \\
T_1^R(\mathbf{i}(t), q, u, v) &= \mathtt{right}(T_1^S(t, q, u, v), \\
&\qquad T_1^L(t, q, u, v), T_1^R(t, q, u, v))
\end{aligned}
$$

Linear length Iteration

$$T_1^S, T_1^L, T_1^R : \mathbf{Word}(1) \to \mathbf{Word}(0)^3 \to \mathbf{Word}(0)$$

$$\begin{aligned}
[\![T_1^S(t,q,u,v)]\!] &= \text{state after } t \text{ steps} \\
[\![T_1^L(t,q,u,v)]\!] &= \text{left tape after } t \text{ steps} \\
[\![T_1^R(t,q,u,v)]\!] &= \text{right tape after } t \text{ steps}
\end{aligned}$$

We make k nested simultaneous recursion to iterate n^k times

Polynomial length iteration

$$T^S_{k+1}(\epsilon, q, u, v) = q \quad T^L_{k+1}(\epsilon, q, u, v) = u \quad T^R_{k+1}(\epsilon, q, u, v) = v$$

$$\begin{aligned}
T^S_{k+1}(\mathbf{i}(t), \bar{t}, q, u, v) &= T^S_k(\bar{t}, T^S_k(\bar{t}, q, u, v,), \\
&\qquad T^L_k(\bar{t}, q, u, v), T^R_k(\bar{t}, q, u, v)) \\
T^L_{k+1}(\mathbf{i}(t), \bar{t}, q, u, v) &= T^L_k(\bar{t}, T^S_k(\bar{t}, q, u, v,), \\
&\qquad T^L_k(\bar{t}, q, u, v), T^R_k(\bar{t}, q, u, v)) \\
T^R_{k+1}(\mathbf{i}(t), \bar{t}, q, u, v) &= T^R_k(\bar{t}, T^S_k(\bar{t}, q, u, v,), \\
&\qquad T^L_k(\bar{t}, q, u, v), T^R_k(\bar{t}, q, u, v))
\end{aligned}$$

where $\bar{t} = t, t_k, \ldots, t_1$

$$T^S_{k+1}, T^L_{k+1}, T^R_{k+1} : \mathbf{Word}(1)^{k+1} \to \mathbf{Word}(0)^3 \to \mathbf{Word}(0)$$

Simulation of PTIME computation

Lemma

A polynomial time computable function $\phi : \{0,1\}^* \to \{0,1\}^*$ *is captured by a function in* **TRec***(***Word***) *using 2 tiers only*

Proof.

Suppose that Φ is computed by a TM in time n^k.

$$\phi(w) = T_k^R(w, \ldots, w, q_0, w, \epsilon)$$

where

$$[\![T_k^S(\overline{w}, q, u, v)]\!] = \text{state after } |w|^k \text{ steps}$$
$$[\![T_k^L(\overline{w}, q, u, v)]\!] = \text{left tape after } |w|^k \text{ steps}$$
$$[\![T_k^R(\overline{w}, q, u, v)]\!] = \text{right tape after } |w|^k \text{ steps}$$

$\square$

Computation of tiered recursion

Lemma

For any tier 1 arguments $u_1, \ldots, u_p$, and tier 0 arguments $v_1, \ldots, v_q$, the computation of $f(u_1, \ldots, u_p, v_1, \ldots, v_q)$ runs in time bounded by $c \times (\sum_{i=1,p} |u_i|)^k$.

Proof

$$
\begin{aligned}
f(\epsilon, y; z) &= g(y; z) \\
f(\mathbf{0}(x), y; z) &= h_0(x, y; z, f(x, y; z)) \\
f(\mathbf{1}(x), y; z) &= h_1(x, y; z, f(x, y; z))
\end{aligned}
$$

$$f : \mathbf{Word}(1), \mathbf{Word}(1), \mathbf{Word}(0) \to \mathbf{Word}(0)$$

- Ind. Hyp applies to g and h_i
- So, g runs within a time bounded by a polynomial in tier 1 inputs

$$Time(g(v; w)) \leq |v|^{k'}$$

- So, the run time of h_i is polynomial in tier 1 inputs

$$Time(h_i(u', v; w, w')) \leq (|u'| + |v|)^{k''}$$

- $f(u, v, w)$ is computed by iterating $|u|$ times h_i's

$$
\begin{aligned}
Time(f(u, v, w)) &\leq |u| \times (|u| + |v|)^{k''} + |v|^{k'} \\
&\leq (|u| + |v|)^{k' + k''}
\end{aligned}
$$

Church-numerals

$$\underline{n} = \lambda f : \alpha \to \alpha\, \lambda x : \alpha, f^{n}(x)$$

where $f^{0}(x) = x$ and $f^{k+1}(x) = f(f^{k}(x))$

$$\underline{n} : \mathbf{N}(\alpha) = (\alpha \to \alpha) \to (\alpha \to \alpha)$$

Successor :

$$\mathbf{suc} = \lambda n : \mathbf{N}(\alpha)\ \lambda f : \alpha \to \alpha\ \lambda x : \alpha,\ f(n\ f\ x)$$

$$\underline{0} = \lambda f\, \lambda x, x \qquad \underline{n+1} = (\mathbf{suc}\ \underline{n})$$

Church numeral arithmetic

$$(\texttt{add}\ \underline{n}\ \underline{m}) = \lambda f \lambda x, (\underline{n}\ f\ (\underline{m}\ f\ x))$$
$$\texttt{add} : \mathbf{N}(\alpha) \rightarrow \mathbf{N}(\alpha) \rightarrow \mathbf{N}(\alpha)$$
$$[\![\texttt{add}]\!](\underline{n}, \underline{m}) = \underline{n + m}$$

$$(\texttt{mul}\ \underline{n}\ \underline{m}) = \lambda f \lambda x, (\underline{n}\ (\underline{m}\ f)\ x)$$
$$\texttt{mul} : \mathbf{N}(\alpha) \rightarrow \mathbf{N}(\alpha) \rightarrow \mathbf{N}(\alpha)$$
$$[\![\texttt{mul}]\!](\underline{n}, \underline{m}) = \underline{n \times m}$$

But

$$(\texttt{exp}\ \underline{n}) = (\underline{n}\ \underline{2})$$
$$\texttt{exp} : \mathbf{N}(\alpha \rightarrow \alpha) \rightarrow \mathbf{N}(\alpha)$$
$$[\![\texttt{exp}]\!](\underline{n}) = \underline{2^n}$$

Extended polynomials

Definition

A function $\phi : \mathbb{N} \to \mathbb{N}$ is Church-representable if there is a λ-term F such that

$$(F \underline{n}) = \underline{\phi(n)}$$
$$F : \mathbf{N}(\alpha) \to \mathbf{N}(\alpha)$$

- Exponential is not Church-representable

Theorem (Schwichtenberg)

The set of Church-representable functions is the set of extended polynomials (= polynomials + test if $n = 0$).

Church representation of first order structures

$$\mathbf{W}(\alpha) = (\alpha \to \alpha) \to (\alpha \to \alpha) \to (\alpha \to \alpha)$$

$$\begin{aligned} \underline{\epsilon} &= \lambda f_0 \lambda f_1 \lambda x, x \\ \underline{1} &= \lambda u \lambda f_0 \lambda f_1 \lambda x, f_0(\underline{u} f_0 f_1 x) \\ \underline{0} &= \lambda u \lambda f_0 \lambda f_1 \lambda x, f_1(\underline{u} f_0 f_1 x) \end{aligned}$$

$$\underline{\mathbf{0}(u)} = (\underline{0}\ \underline{u}) \qquad \underline{\mathbf{1}(u)} = (\underline{1}\ \underline{u})$$

$$\mathbf{W}(\alpha) = (\alpha \to \alpha) \to (\alpha \to \alpha) \to (\alpha \to \alpha)$$
$$\underline{u} : \mathbf{W}(\alpha)$$

Böhm and Berarducci (85)

Two levels of data representations

Abstract level : $\lambda u \lambda f_0 \lambda f_1 \lambda x \, , f_0(f_0(f_1 x))$

Data level : $\mathbf{0}(\mathbf{0}(\mathbf{1}(\epsilon)))$

- Abstract level $\Rightarrow$ Data Level
- But the converse does not hold
- Data ramification principle !

λ-calculus over **Word***

Atomic types : **Word** and **Word***

Constructors :

$$\epsilon : \textbf{Word} \qquad \textbf{0}, \textbf{1} : \textbf{Word} \to \textbf{Word}$$

$$\textbf{nil} : \textbf{Word}^* \qquad \textbf{cons} : \textbf{Word} \to \textbf{Word}^* \to \textbf{Word}^*$$

λ-calculus over **Word***

Destructors:

$$\mathtt{pred}(\epsilon) = \epsilon \qquad \mathtt{pred}(\mathbf{0}(u)) = u \qquad \mathtt{pred}(\mathbf{1}(u)) = u$$

$$\mathtt{cond}(u, a, b, c) = \begin{cases} a & u = \epsilon \\ b & u = \mathbf{0}(u) \\ c & u = \mathbf{1}(u) \end{cases}$$

$$\mathrm{hd}(\mathbf{nil}) = \mathbf{nil} \qquad \mathrm{hd}(\mathbf{cons}(u, L)) = u$$

$$\mathrm{tl}(\mathbf{nil}) = \mathbf{nil} \qquad \mathrm{tl}(\mathbf{cons}(u, L)) = L$$

$$\mathtt{cond}(L, a, b) = \begin{cases} a & u = \mathbf{nil} \\ b & \text{otherwise} \end{cases}$$

Church representation of algebra terms

Definition

$\mathbf{1}\lambda^{\mathbf{p}}(\mathbf{Word}^*)$ is the class of terms of simply typed λ-calculus with constructors and destructors over $\mathbf{Word}^*$

Definition

A function $\Phi : \{0,1\}^* \to \{0,1\}^*$ is computed by a λ-term F of $\mathbf{1}\lambda^{\mathbf{p}}(\mathbf{Word}^*)$ if $F(\underline{w}) = \phi(w)$ where $\underline{f} : \mathbf{W}(\mathbf{Word}^*) \to \mathbf{Word}^*$.

Theorem (Leivant-Marion)

The set of functions computed by $\mathbf{1}\lambda^{\mathbf{p}}(\boldsymbol{Word}^*)$*-terms is exactly the set* PTIME *of polynomial time functions.*

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Parallel Register Machines (PRM)

1. a finite set $S = \{s_0, s_1, \ldots, s_k\}$ of *states*
2. a finite list $\Pi = \{\pi_1, \ldots, \pi_m\}$ of *registers*
3. and commands
   - [$\mathbf{Succ}(\pi = \mathbf{i}(\pi), s')$], [$\mathbf{Pred}(\pi = \mathbf{p}(\pi), s')$],
   - [$\mathbf{Branch}(\pi, s', s'')$],
   - [$\mathbf{Fork}_{\min}(s', s'')$], [$\mathbf{Fork}_{\max}(s', s'')$],
   - [**End**].

(PRM)

$$\texttt{eval}(0, s, \Pi) = \perp$$

$\texttt{cmd}(s) = \pi_j = \mathbf{i}(\pi_j)$ and the next state is s'

$$\texttt{eval}(t+1, s, \Pi) = \texttt{eval}(t, s', \{\pi \leftarrow \mathbf{i}(\pi)\}\Pi)$$

$\texttt{cmd}(s) = \texttt{pred}(\pi)$

$$\texttt{eval}(t+1, s, \Pi) = \texttt{eval}(t, s', \{\pi \leftarrow \mathbf{p}(\pi)\}\Pi)$$

$\texttt{cmd}(s) = \mathbf{Branch}(\pi, s', s'')$

$$\texttt{eval}(t+1, s, \Pi) = \begin{cases} \texttt{eval}(t, s', \Pi) & \text{if } \pi = \mathbf{0}(w) \\ \texttt{eval}(t, s'', \Pi) & \text{if } \pi = \mathbf{1}(w) \end{cases}$$

PRM

And the Fork

$\texttt{cmd}(s) = \textbf{Fork}_{\min}(s', s'')$

$$\texttt{eval}(t+1, s, \Pi) = \min_{\blacktriangleleft}(\texttt{eval}(t, s', \Pi), \texttt{eval}(t, s'', \Pi))$$

$\texttt{cmd}(s) = \textbf{Fork}_{\max}(s', s'')$

$$\texttt{eval}(t+1, s', \Pi) = \max_{\blacktriangleleft}(\texttt{eval}(t, s', \Pi), \texttt{eval}(t, s'', \Pi))$$

where $\blacktriangleleft$ is the lexicographic order on words.

$$\texttt{eval}(t+1, \textbf{End}, \Pi) = \Pi(\textsc{output})$$

PRM

A function $\phi : \{0,1\}^* \to \{0,1\}^*$ is PRM-computable in time $T : \mathbb{N} \to \mathbb{N}$ if there is a PRM M such that for each $(w_1, \cdots, w_k) \in \mathbb{W}^k$, we have

$$\mathtt{eval}(T(|w|), \mathrm{BEGIN}, F_0) = \phi(w)$$

Time-bound semantics

$$\mathtt{eval} : \mathbb{N} \times S \times \mathbb{W}^m \mapsto \mathbb{W}$$

Trade-off between time and space

Theorem

The following are equivalent

1. ϕ *is computable in polynomial space*
2. ϕ *is computable in non-deterministic polynomial space*
3. ϕ *is computable in polynomial time on Alternating Turing Machine*
4. ϕ *is computable in polynomial time on PRM*

Proof.

See Chandra, Kozen, Stockmeyer and Savitch □

Tiered recursion with substitutions

$$
\begin{aligned}
f(\epsilon, \overline{y}) &= g(\overline{y}) \\
f(\mathbf{0}(x), \overline{y}) &= h_0(x, \overline{y}, f(x, \sigma_1(\overline{y})), \ldots, f(x, \sigma_k(\overline{y}))) \\
f(\mathbf{1}(x), \overline{y}) &= h_1(x, \overline{y}, f(x, \sigma'_1(\overline{y})), \ldots, f(x, \sigma'_k(\overline{y})))
\end{aligned}
$$

$$\sigma_i, \sigma'_i : \mathbf{Word}(m) \to \mathbf{Word}(n)$$
$$g : \mathbf{Word}(m) \to \mathbf{Word}(n)$$
$$h_i : \mathbf{Word}(n+1) \to \mathbf{Word}(m) \to \mathbf{Word}(n) \to \mathbf{Word}(n)$$
$$\sigma_j, \sigma' j : \mathbf{Word}(m) \to \mathbf{Word}(m)$$
$$f : \mathbf{Word}(n+1), \mathbf{Word}(m) \to \mathbf{Word}(n)$$

Characterization of PSPACE

Definition

The class **Sub**(**Word**) is the set of functions defined by tiered recursion with substitutions and explicit definitions (projections and composition).

Theorem (LM95)

The three sets are identical

- *The set* PSPACE *of functions computable in polynomial space*
- *The set* **Sub**(***Word***) *using any tiers*
- *The set* **Sub**(***Word***) *using 3 tiers only*

Proof.

We are going to sketch it shortly. □

Simulating PSPACE

Lemma

A polynomial time PRM computable function $\phi : \{0,1\}^* \to \{0,1\}^*$ *is captured by a function in* $\mathbf{Sub}(\boldsymbol{Word})$ *using 3 tiers only*

Proof.

`eval` is defined by rec. with substitution of parameters:
$\mathtt{cmd}(s) = \mathbf{Fork}_{\min}(s', s'')$

$$\mathtt{eval}(t+1, s, \Pi) = \min(\mathtt{eval}(t, \delta_0(s), \Pi), \mathtt{eval}(t, \delta_1(s), \Pi))$$

where

$$\delta_0(s) = s'$$
$$\delta_1(s) = s''$$

$$\mathtt{eval} : \mathbf{Word}(1) \to \mathbf{Word}(0)^m \to \mathbf{Word}(0)$$

□

Simulating PSPACE

Define a polynomial time clock $T : \textbf{Word}(2) \to \textbf{Word}(1)$ by composing tiered addition and multiplication that we have already seen.

$$\phi(w) = \mathtt{eval}(T(|w|), \text{BEGIN}, F_0)$$

Computation in PSPACE

Lemma

A function ϕ in **Sub(*Word*)** *using 3 tiers only is computed in space $O(n^k) + m$ for some k*

- *n is the size of tier 2 and 1 arguments*
- *m is the size of tier 0 arguments*

Other tiered characterizations of low complexity classes

- $NC^1 \equiv$ A-LOG-TIME
  - *Bloch (94)*,
  - *Leivant-Marion (00)*
- NC^k
  - *Bonfante, Kähle, Marion, Oitavem (06)*,
- NC
  - *Leivant (98)*,
  - *Oitavem (04)*
- NP
  - *Bellantoni (94)*
- FPSPACE $\equiv$ A-POLY-TIME
  - *Leivant-Marion (95)*

Computing over a structure $\mathcal{K}$

A computational structure

$$\mathcal{K} = \langle \mathbb{K}, \{op_i\}_{i \in I}, rel_1 \ldots, rel_\ell, =, \mathbf{0}, \mathbf{1} \rangle$$

- A domain $\mathbb{K}$
- operators $\{op_i\}_{i \in I}$ over $\mathbb{K}$
- relations $rel_1, \ldots, rel_l$
- the equality over $\mathbb{K}$
- two particular constants $\mathbf{0}$ and $\mathbf{1}$

$\mathbb{K}^*$ denotes lists of elements of $\mathbb{K}$.

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Blum-Shub-Smale machine over $\mathcal{K}$

Similar to a Turing machine, with the properties:

- Its tape cells hold arbitrary elements of $\mathbb{K}$.
- It has Computation nodes for computing the operations $\{op_i\}_{i \in I}$ with unit cost.
- It has Branch nodes for computing the relations $rel_1, \ldots, rel_l$ with unit cost.
- It has Shift nodes for moving the head.
- Inputs and outputs are vectors in $\mathbb{K}^*$

A TM computes a function from $\mathbb{K}^*$ to $\mathbb{K}^*$.

$\mathbb{K}^*$ denotes lists of elements of $\mathbb{K}$.

Polynomial time functions over $\mathcal{K}$

Definition

A function $f : (\mathbb{K}^*)^n \rightarrow (\mathbb{K}^*)^m$ is in class $\mathrm{PTIME}_{\mathcal{K}}$
iff
f is computable in polynomial time.
That is,
there is a polynomial p and a BSS-TM M, such that

- M computes f
- M stops in $p(|\overline{w}|)$ steps on each input w of $\mathbb{K}^*$.

$|\overline{w}|$ is the length of the list $\overline{w} \in \mathbb{K}^*$.

Complexity Theory over $\mathcal{K}$

- Over the structure $\mathcal{B} = (\{0,1\}, =, 0, 1)$, we compute $f : \{0,1\}^* \rightarrow \{0,1\}^*$, corresponds to classical complexity and $\mathrm{PTIME}_{\mathcal{B}} = \mathrm{PTIME}$.
- Over the structure $\mathcal{R} = (\mathcal{R}, +, -, *, /, >, =, 0, 1)$ corresponds to the original Blum Shub and Smale (89) paper.

Safe Recursion over a structure $\mathcal{K}$

Two types of arguments,“normal”and “safe”

$$f(\overline{x}; \overline{y})$$

The set of safe recursive functions over $\mathcal{K}$ is the smallest set of functions containing basic safe functions

- structure operators and relations
- projections
- list destructors : **hd** and **tl**
- list constructor : **cons**
- Boolean selection : **if** $x = 1$ **then** y **else** z

Basic functions

- $\mathrm{hd}(;a.\overline{x}) = a,\ \mathrm{tl}(;a.\overline{x}) = \overline{x},\ \mathrm{cons}(;a.\overline{x},\overline{y}) = a.\overline{y}$
- Projections
- Application of operators and relations

$$\begin{aligned} \mathrm{Op}_\imath(;a_1.\overline{x_1},\ldots,a_{n_\imath}.\overline{x_{n_\imath}}) &= (op_\imath(a_1,\ldots,a_{n_\imath})).\overline{x_{n_\imath}} \\ \mathrm{Rel}_\imath(;a_1.\overline{x_1},\ldots,a_{n_\imath}.\overline{x_{n_\imath}}) &= \begin{cases} \mathbf{1} \text{ if } rel_\imath(a_1,\ldots,a_{n_\imath}) \\ \epsilon \text{ otherwise} \end{cases} \end{aligned}$$

- Test

$$\mathrm{Select}(;\overline{x},\overline{y},\overline{z}) = \begin{cases} \overline{y} & \text{if } \mathrm{hd}(\overline{x}) = \mathbf{1} \\ \overline{z} & \text{otherwise} \end{cases}$$

Safe Recursive functions over $\mathcal{K}$

and closed under both schemas

- safe composition

$$f(\overline{x};\overline{y}) = g(h_1(\overline{x};);h_2(\overline{x};\overline{y}))$$

- safe recursion

$$f(\epsilon,\overline{x};\overline{y}) = g(\overline{x};\overline{y})$$
$$f(a.\overline{z},\overline{x};\overline{y}) = g(\overline{z},\overline{x};f(\overline{z},\overline{x};\overline{y}),\overline{y})$$

Polynomial time functions PTIME$_{\mathcal{K}}$

Theorem (Bournez-Cucker-de Naurois-Marion (03))

Over any structure $\mathcal{K}$, the set of safe recursive functions over $\mathcal{K}$ is exactly PTIME$_{\mathcal{K}}$.

Proof.

This proof implies Bellantoni and Cook's one and is more direct. □

About space

A priori, there is no valid notion of space over arbitrary structures.

Theorem (Michaux)

Over $(\mathbb{R}^+, 0, 1, =, +, -, *, <)$, *any computable function can be computed in constant working space.*

But, Paulin de Naurois gives a logarihmic cost, see his talk at the ICC workshop next week !

The class FPAR$_{\mathcal{K}}$

FPAR$_{\mathcal{K}}$ is the set of functions computable in parallel poly-time.
That is, by a P-uniform family of circuits of polynomial depth.

Theorem (Bournez-Cucker-deNaurois-Marion (04))

A function: $\mathbb{K}^* \to \mathbb{K}^*$ is computed in FPAR$_{\mathcal{K}}$ *if and only if it is defined as a safe recursive function with substitutions over $\mathcal{K}$.*

What we've seen

- Data Ramification Principle
  1. Normal/Safe recursions
  2. Tiering
  3. Simply typed λ-calculus
- Characterizations of PTIME and PSPACE
- Capture Turing Machine over arbitrary structures

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Conclusion

- Intrinsic characterizations
  - "resource" is inside Data Ramification Principle
  - Syntactic complexity characterization
    - we may extract bound,
    - but, low algorithmic expressiveness
  - quite robust wrt model of computations
- Can we apply data ramification to other models of computation ?
- Studying intentional characterization of complexity classes.
- Developing automatic resource analysis by mean of static analysis.