On Context Semantics, Linear Logic and Computational Complexity

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Outline

The Need for a Unifying Framework

- Many systems are sound and complete w.r.t. relevant complexity classes (polytime functions, elementary functions, etc.).
- Proof techniques for soundness results are ad-hoc and cannot be easily generalized.
- As a consequence:
  - There are many open problems. For example: the impact of linearity costraint on the expressive power of higher-order recursion [Hofmann97,BNS00] has not been analyzed precisely.
  - It is not known in general whether combining different systems characterizing the same class would break soundness.
- The above mentioned systems are all extensionally complete but definately not intensionally complete.

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Context Semantics

- *Intensional* ways of giving semantics to logics and programming languages:
  - Game semantics [AJM92];
  - Geometry of interaction [Girard89];
  - Context semantics [GAL92a,GAL92b].
- Game semantics and geometry of interaction have been already used to study quantitative properties of programs and proofs [BaillotPedicini01,Ghica05].
- We chose context semantics because of its simplicity.
- We applied it to linear logic and its subsystems.
- Strong results can be obtained.

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Context Semantics vs. Normalization Time

From the context semantics of any linear logic proof-net G, a *weight* $W_G \in \mathbb{N} \cup \{\omega\}$ can be defined in such a way that:

Theorem

There is a polynomial $p : \mathbb{N}^2 \to \mathbb{N}$ such that for every proof-net G, G normalizes in at most $p(W_G, |G|)$ steps and the size of any reduct H of G is at most $p(W_G, |G|)$.

Theorem

Let G be a proof-net. Then, there is H with $G \to^{W_G} H$.

Syntax

- Formulae:

$$A ::= \alpha \mid A \multimap A \mid A \otimes A \mid !A \mid \forall \alpha.A \mid \mu \alpha.A$$

- Rules:

$$\frac{}{A \vdash A}\ A \qquad \frac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}\ U \qquad \frac{\Gamma \vdash B}{\Gamma, !A \vdash B}\ W \qquad \frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\ X$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\ R_{\multimap} \qquad \frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C}\ L_{\multimap}$$

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\ R_{\otimes} \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\ L_{\otimes}$$

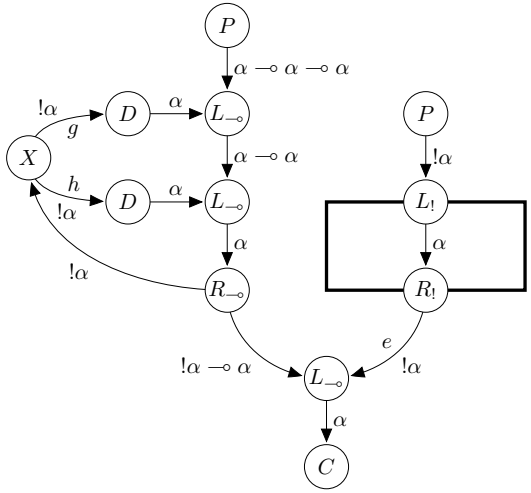
$$\frac{A_1, \ldots, A_n \vdash B}{!A_1, \ldots, !A_n \vdash !B}\ P_! \qquad \frac{A, \Gamma \vdash B}{!A, \Gamma \vdash B}\ D_! \qquad \frac{!!A, \Gamma \vdash B}{!A, \Gamma \vdash B}\ N_!$$

$$\frac{\Gamma \vdash A \quad \alpha \notin FV(\Gamma)}{\Gamma \vdash \forall \alpha.A}\ R_{\forall} \qquad \frac{\Gamma, A\{B/\alpha\} \vdash C}{\Gamma, \forall \alpha.A \vdash C}\ L_{\forall}$$

Proof-nets

- We adopt a system of intuitionistic proof-nets.
- Nodes for axioms and cuts are not needed.
- For example:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\alpha \vdash \alpha}\,A \quad \dfrac{}{\alpha \vdash \alpha}\,A}{\alpha, \alpha \multimap \alpha \vdash \alpha}\,L_! \quad \dfrac{}{\alpha \vdash \alpha}\,A}{\alpha, \alpha, \alpha \multimap \alpha \multimap \alpha \vdash \alpha}\,L_!}{!\alpha, \alpha, \alpha \multimap \alpha \multimap \alpha \vdash \alpha}\,D_!}{!\alpha, !\alpha, \alpha \multimap \alpha \multimap \alpha \vdash \alpha}\,D_!}{!\alpha, \alpha \multimap \alpha \multimap \alpha \vdash \alpha}\,X}{\alpha \multimap \alpha \multimap \alpha \vdash !\alpha \multimap \alpha}\,R_! \qquad \dfrac{\dfrac{}{\alpha \vdash \alpha}\,A \quad \dfrac{\dfrac{}{\alpha \vdash \alpha}\,A}{!\alpha \vdash !\alpha}\,P_!}{!\alpha \multimap \alpha, !\alpha \vdash \alpha}\,L_!}{\alpha \multimap \alpha \multimap \alpha, !\alpha \vdash \alpha}\,U$$



Copying - The Root of Complexity?

- Linear logic without exponentials and additives cannot express anything beyond polynomial time.
- Suppose we know the total number of times boxes in a proof-net G can *possibly* be duplicated. Call it W_G.
- How W_G relates to the complexity of normalizing G?
- Context semantics helps in giving strong results on this question.

Context Semantics - I

- The language $\mathscr{E}$ of *exponential trees* is defined by induction from the following sets of productions:

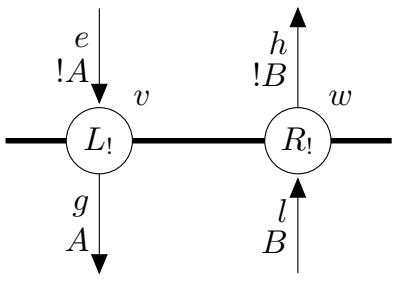
$$t ::= \mathsf{e} \mid \mathsf{r}(t) \mid \mathsf{l}(t) \mid \mathsf{p}(t) \mid \mathsf{n}(t, t).$$

- A *stack element* is either an exponential tree or one of the following characters: $\mathsf{a}, \mathsf{o}, \mathsf{s}, \mathsf{f}, \mathsf{x}$. $\mathscr{S}$ is the set of stack elements.
- Every proof-net G induces rewriting rules on

$$C_G = E_G \times \mathscr{E}^* \times \mathscr{S}^+ \times \{0, 1\}$$

where E_G is the set of G edges,

Context Semantics - II



$$
\begin{aligned}
(e, U, V \cdot t, +) &\rightarrow_G (g, U \cdot t, V, +) \\
(g, U \cdot t, V, -) &\rightarrow_G (e, U, V \cdot t, -) \\
(l, U \cdot t, V, +) &\rightarrow_G (h, U, V \cdot t, +) \\
(h, U, V \cdot t, -) &\rightarrow_G (l, U \cdot t, V, -) \\
(e, U, t, +) &\rightarrow_G (h, U, t, +) \\
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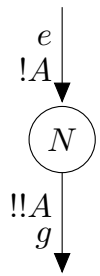
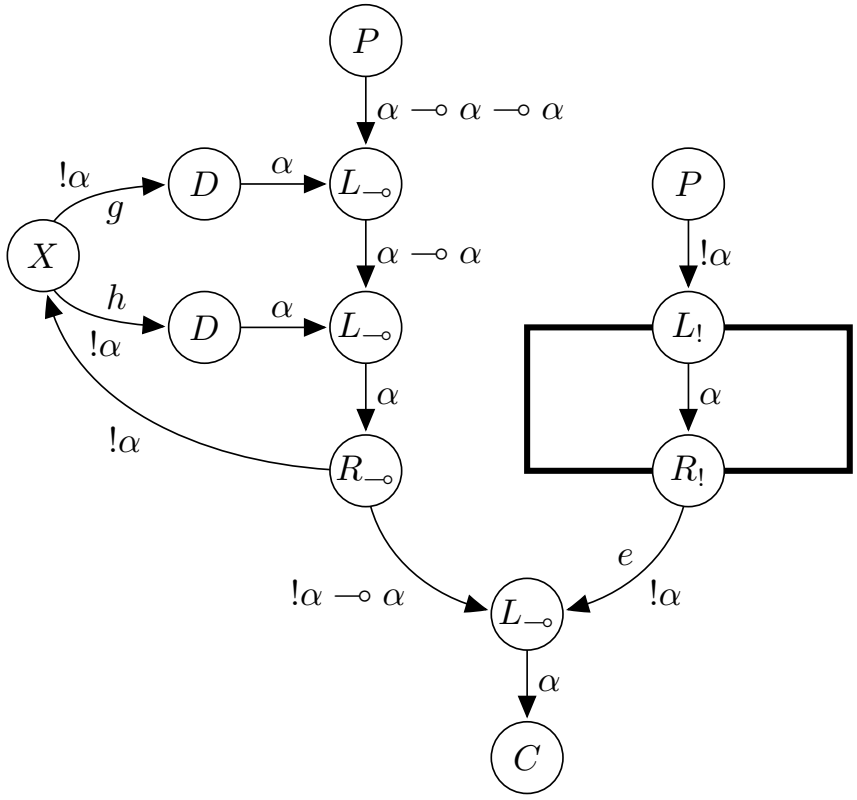
\end{aligned}
$$

Context Semantics - III



$$
\begin{aligned}
(e, U, V \cdot \mathsf{n}(t,u), +) &\rightarrow_G (g, U, V \cdot t \cdot u, +) \\
(g, U, V \cdot t \cdot u, -) &\rightarrow_G (e, U, V \cdot \mathsf{n}(t,u), -) \\
(e, U, \mathsf{p}(t), +) &\rightarrow_G (g, U, t, +) \\
(g, U, t, -) &\rightarrow_G (e, U, \mathsf{p}(t), -)
\end{aligned}
$$

An Example



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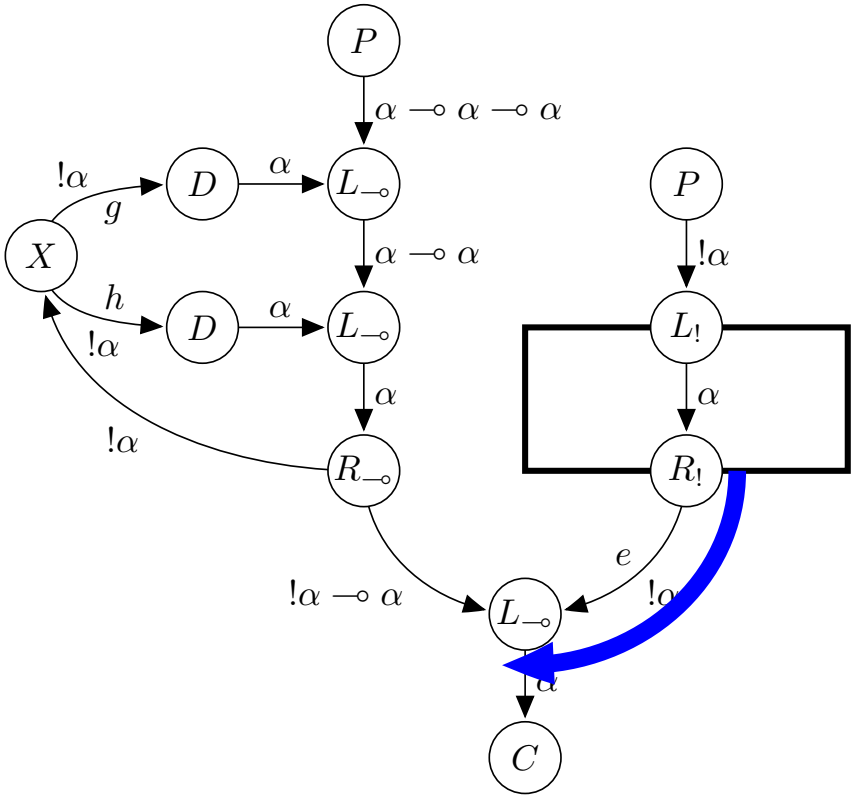

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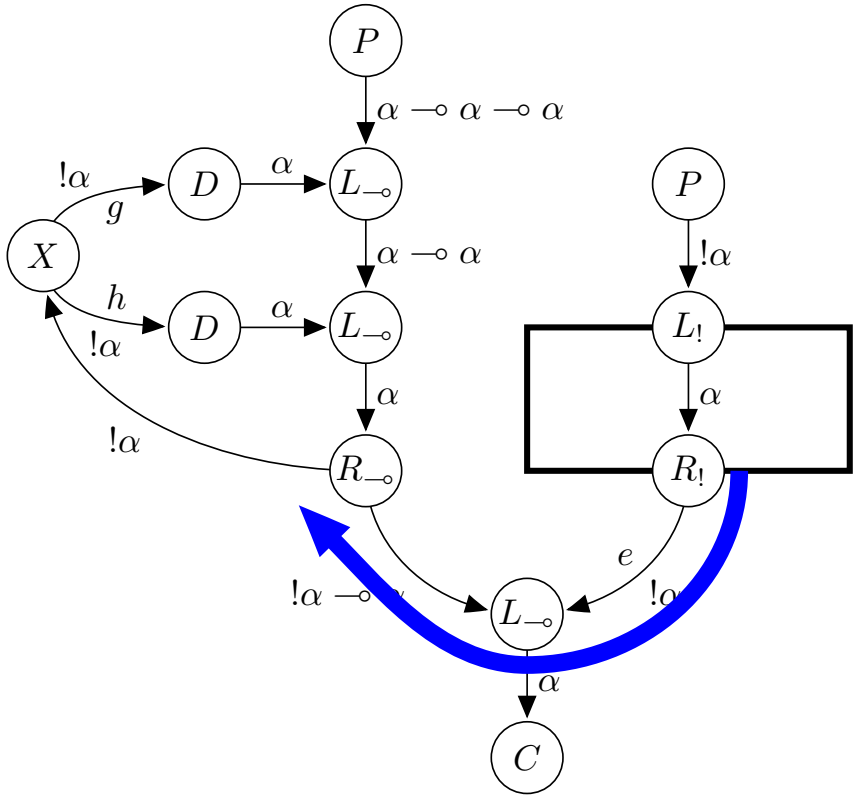

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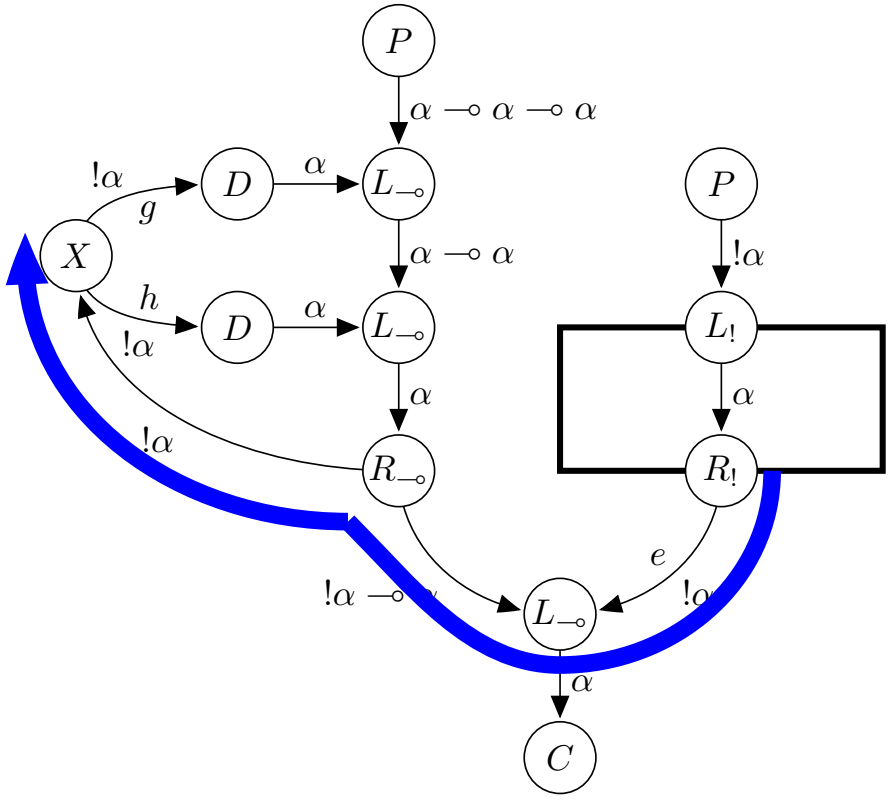

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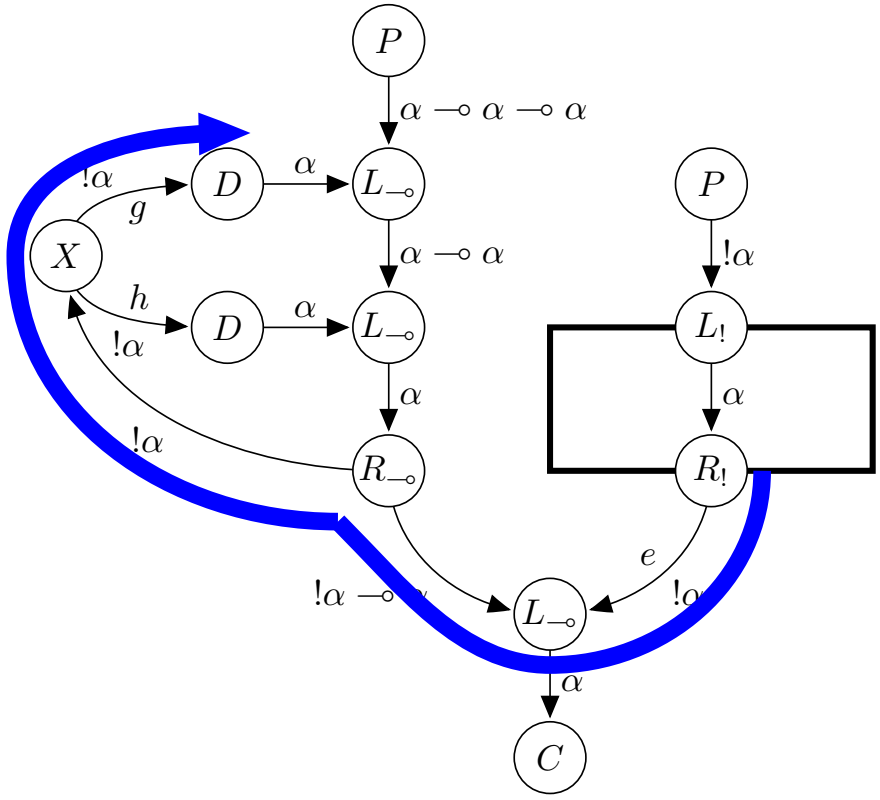

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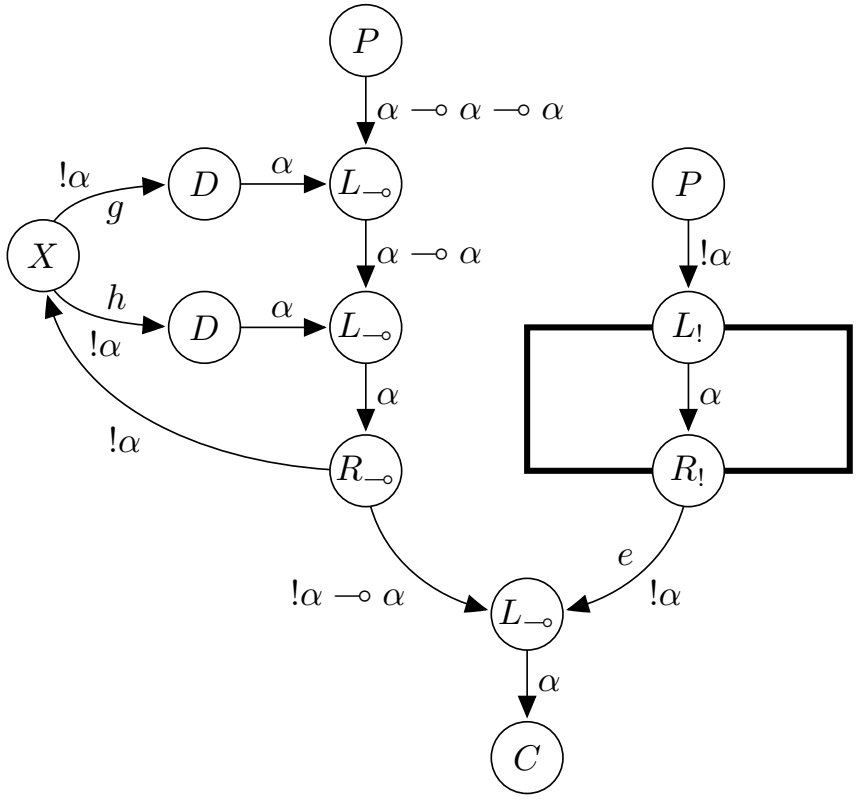

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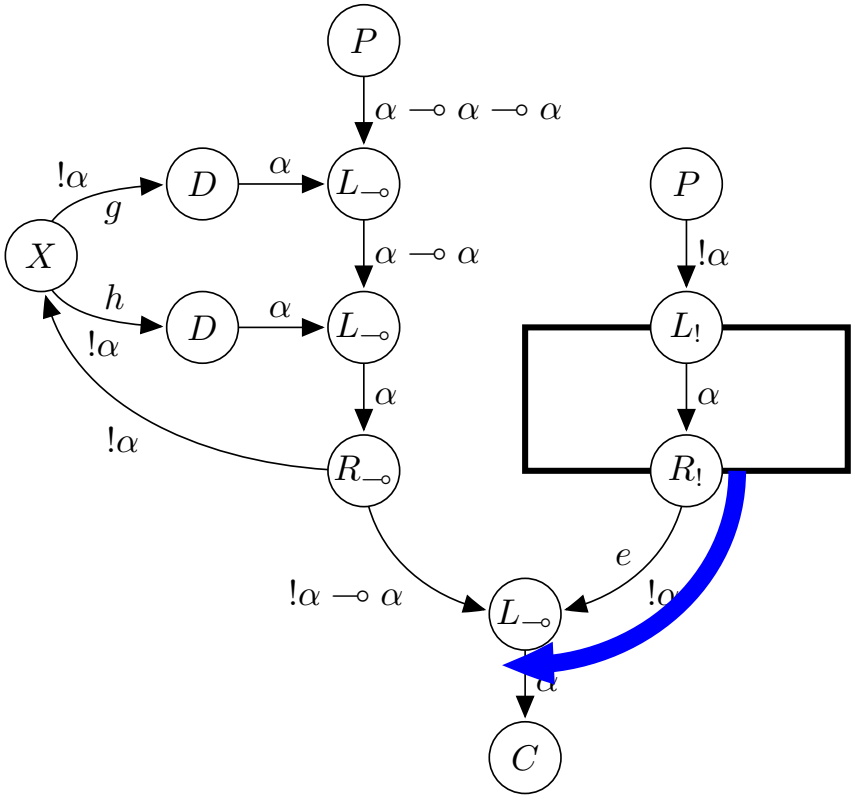

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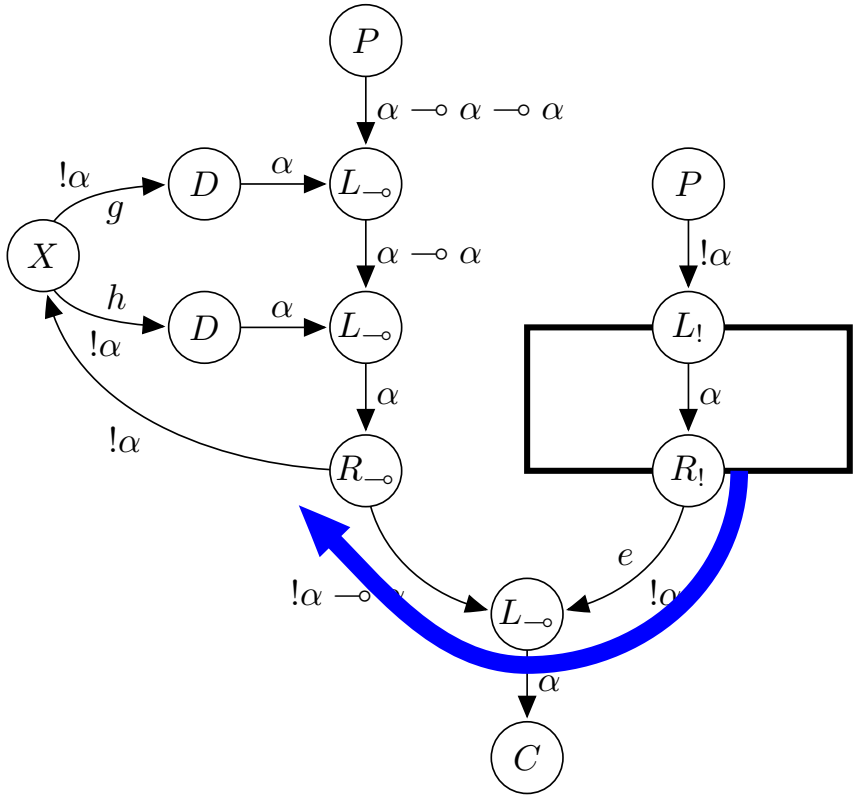

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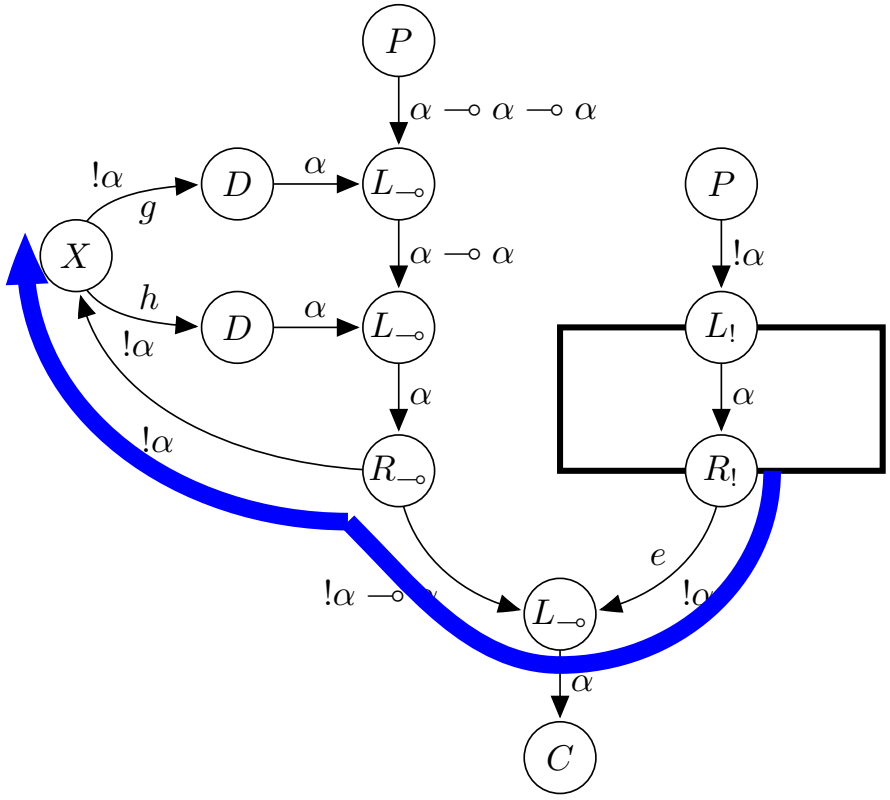

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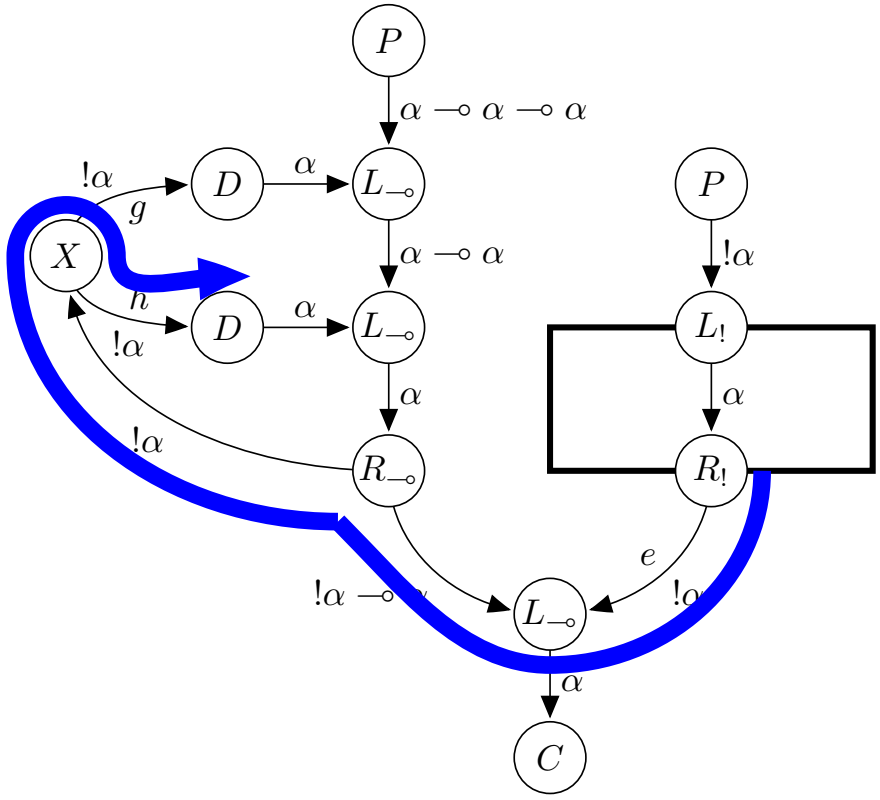

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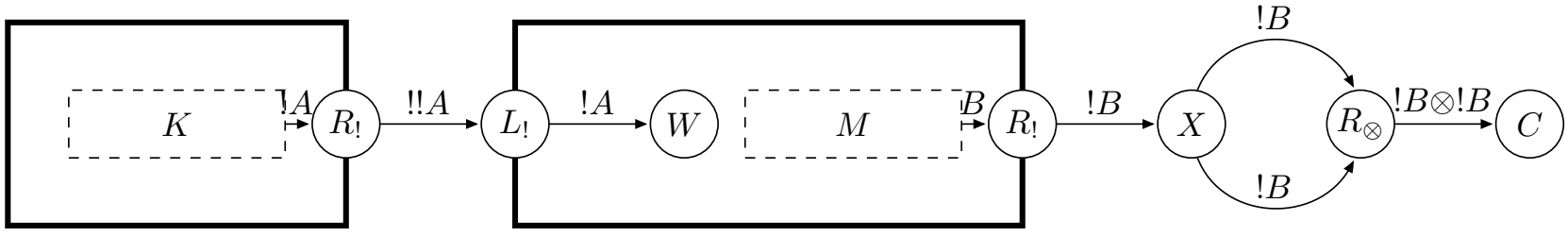

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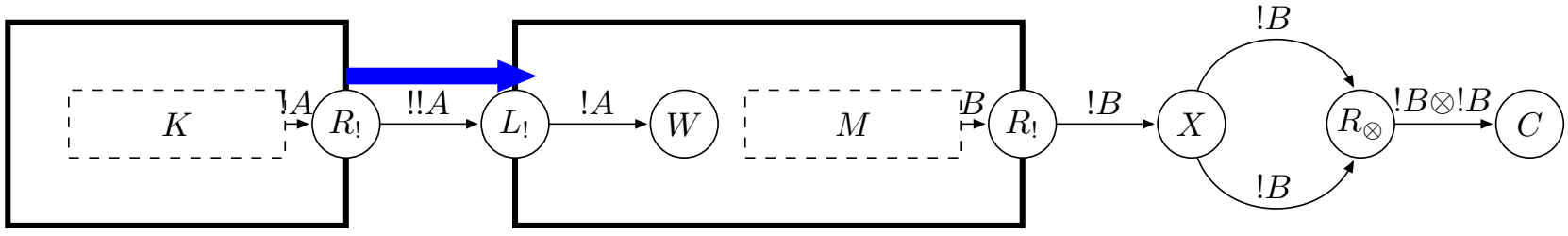

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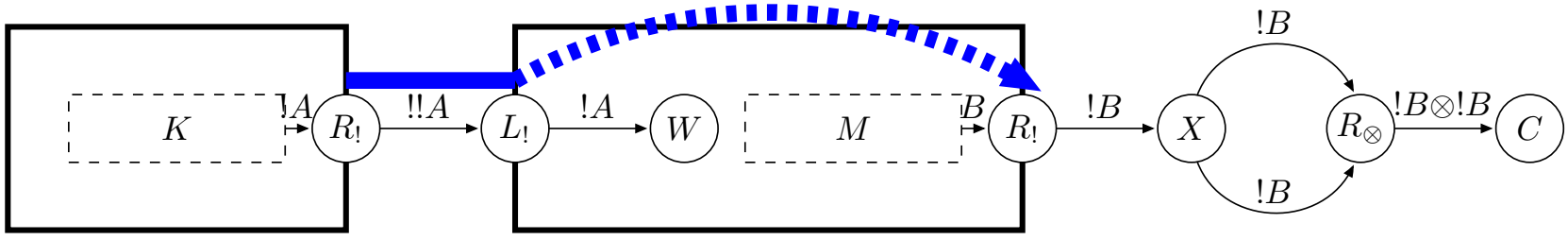

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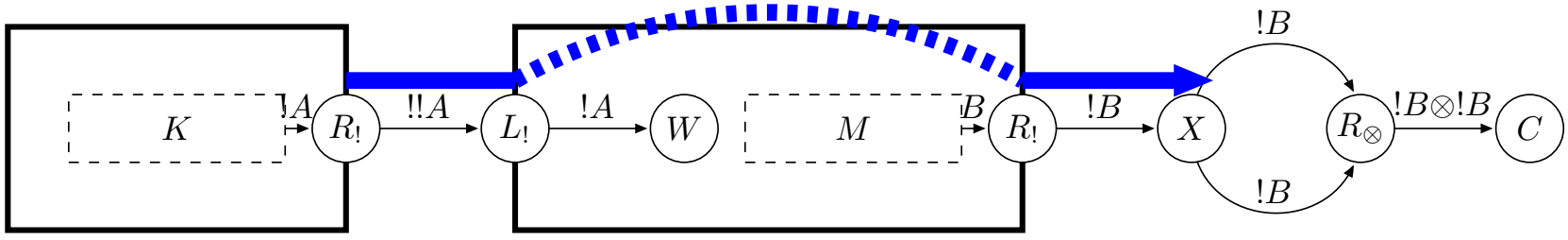

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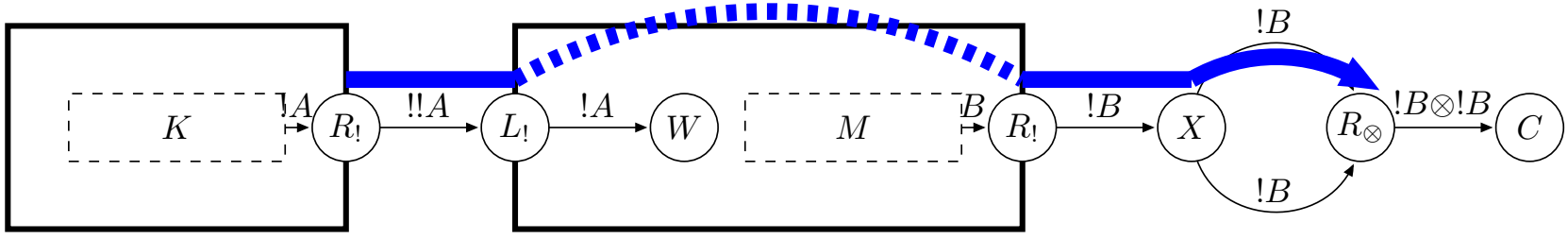

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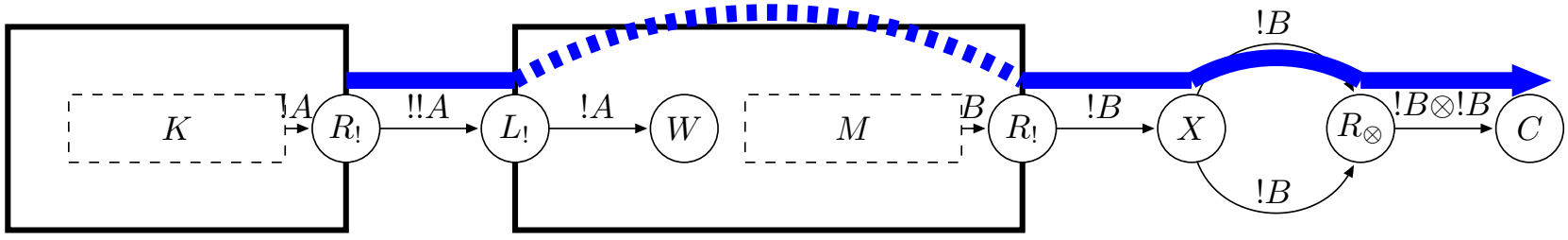

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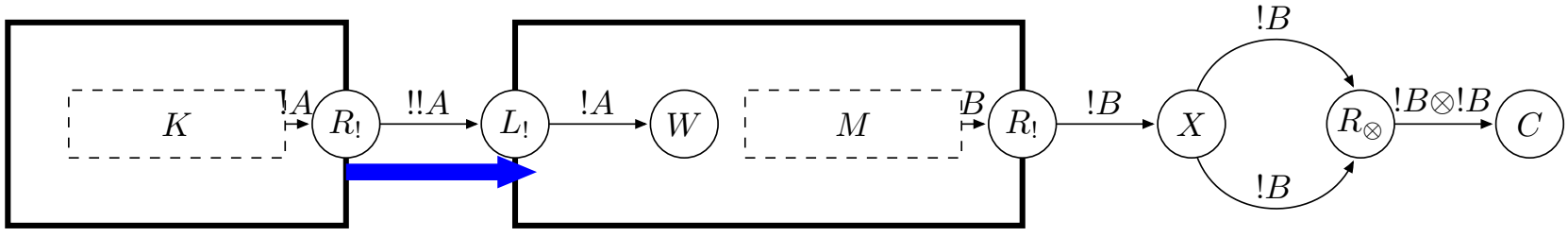

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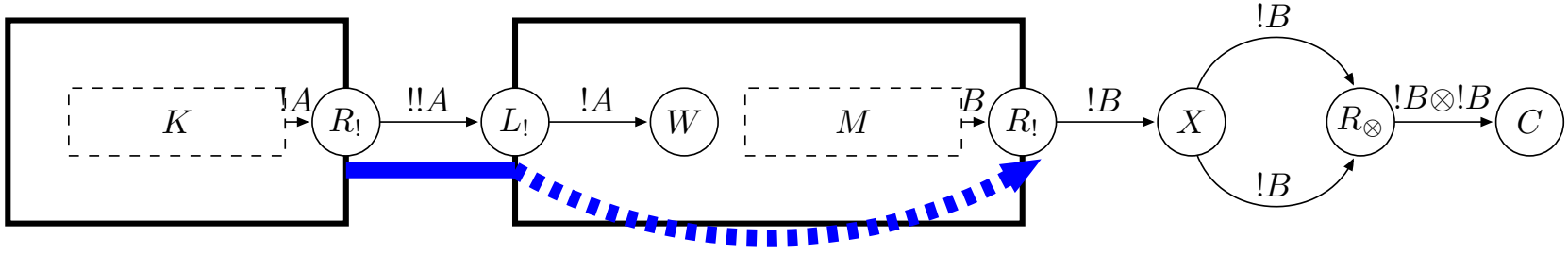

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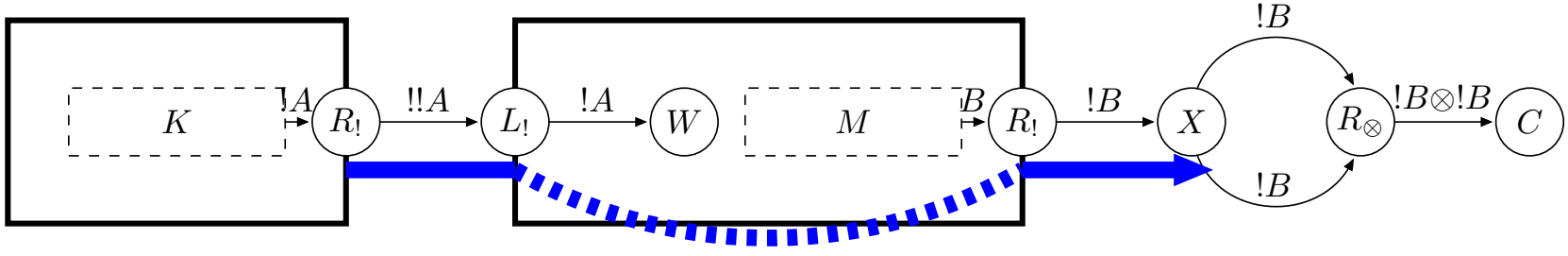

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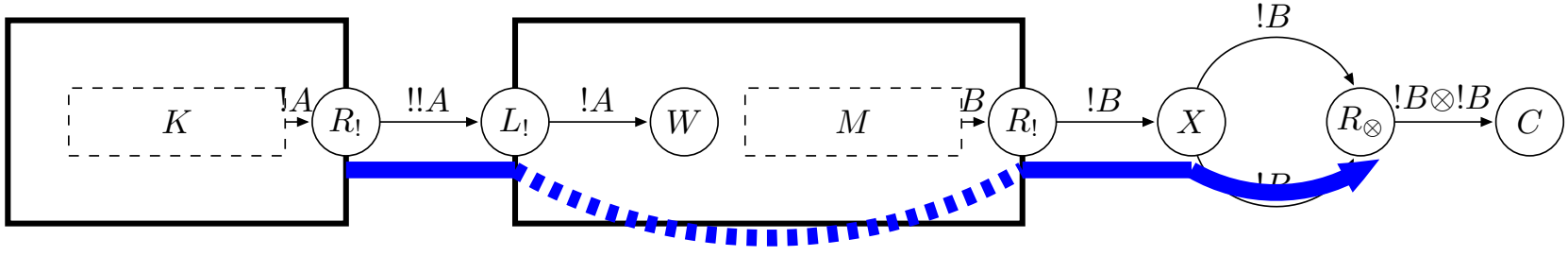

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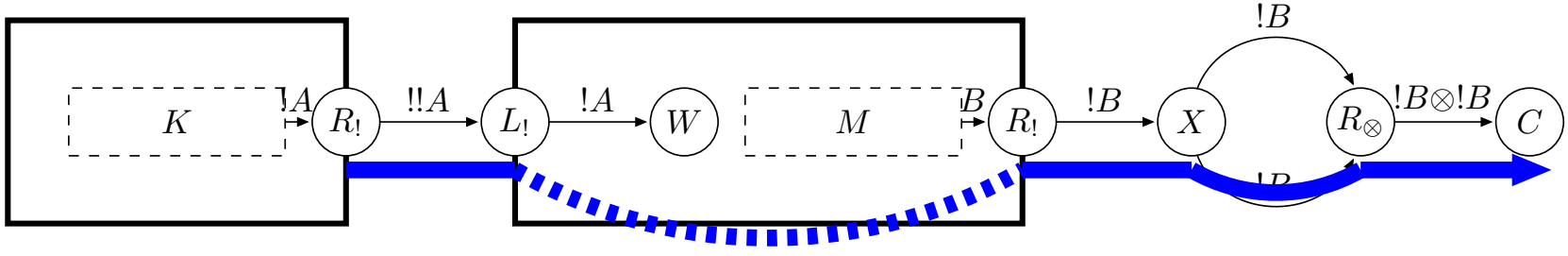

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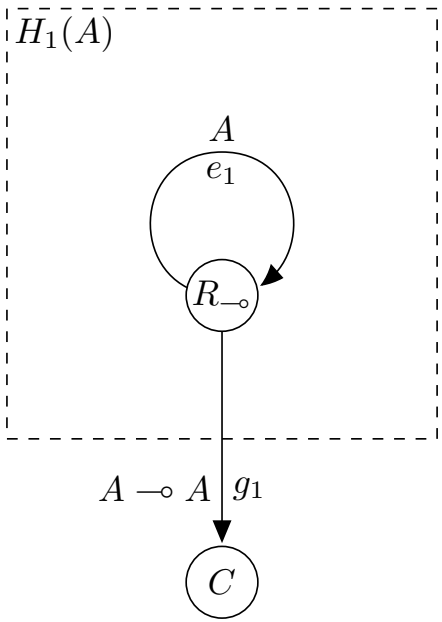

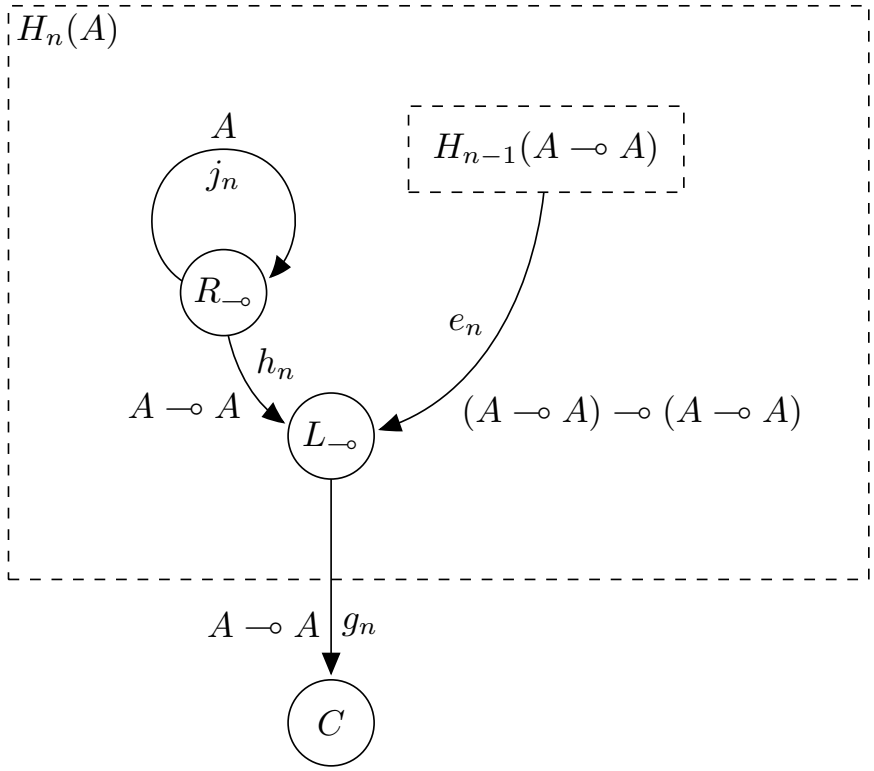

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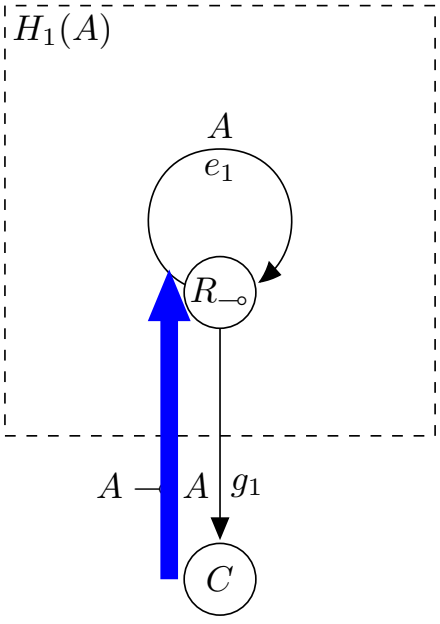

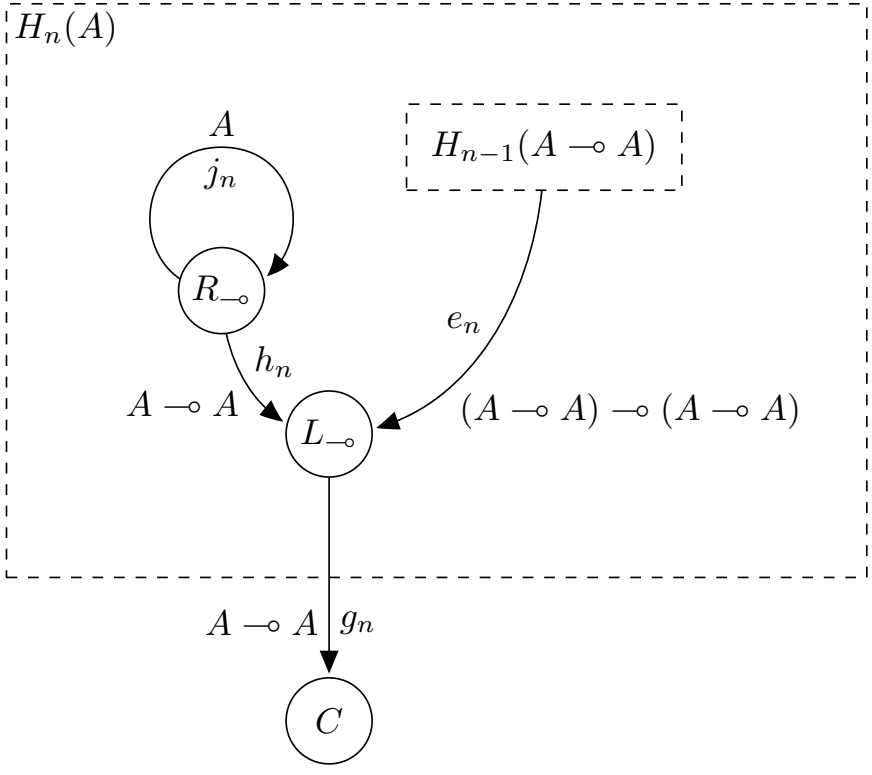

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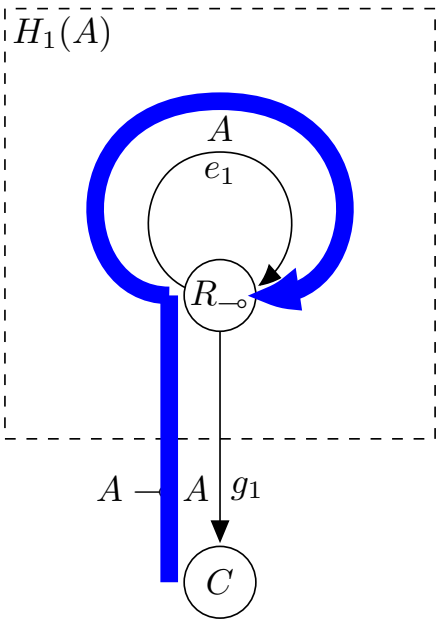

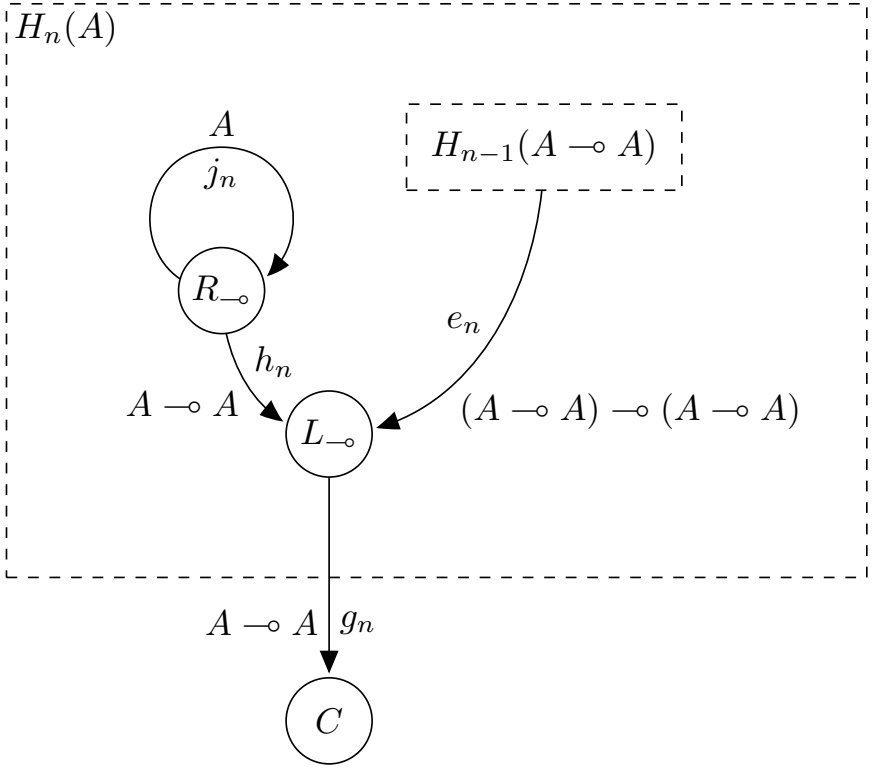

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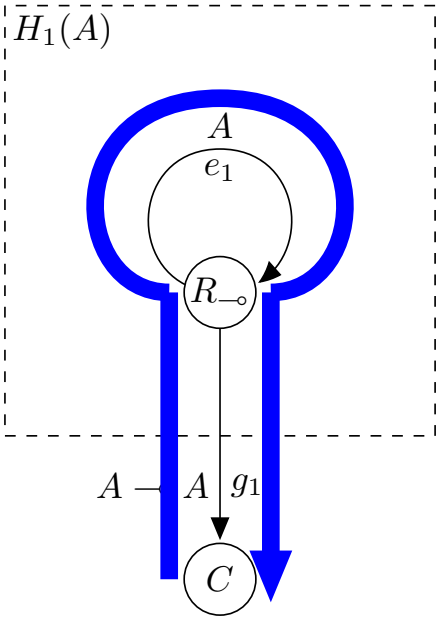

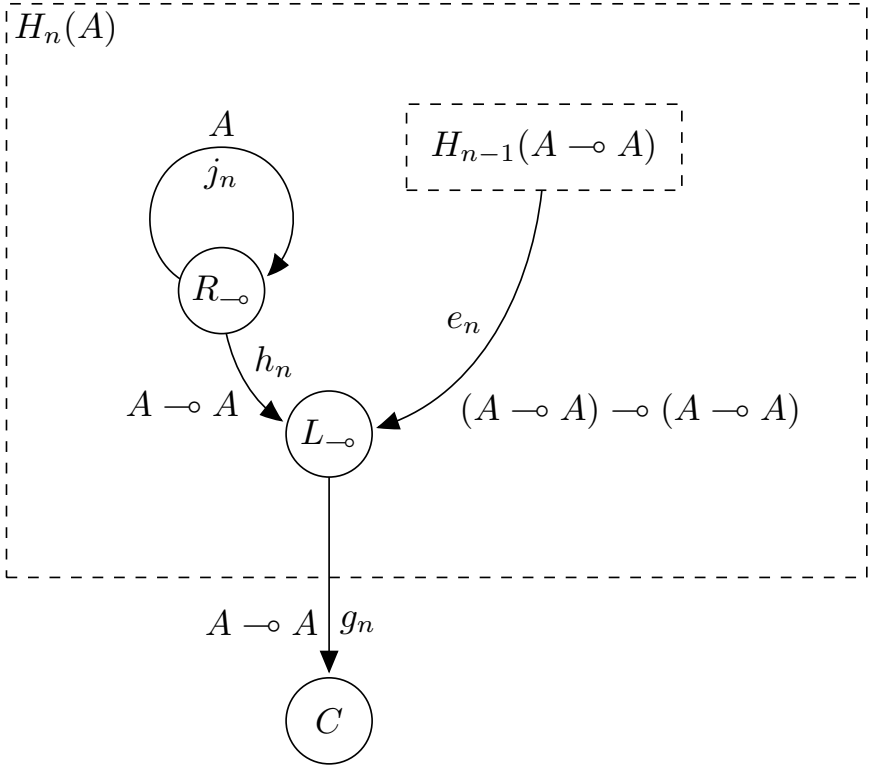

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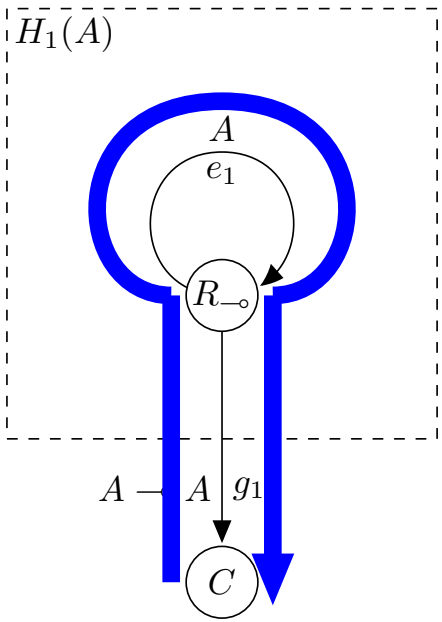

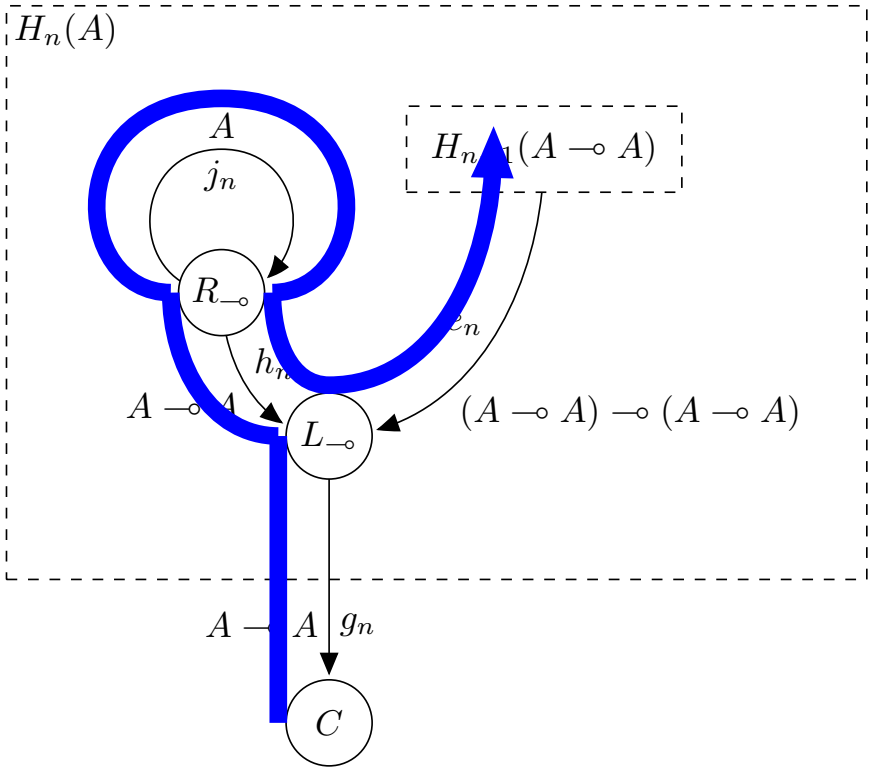

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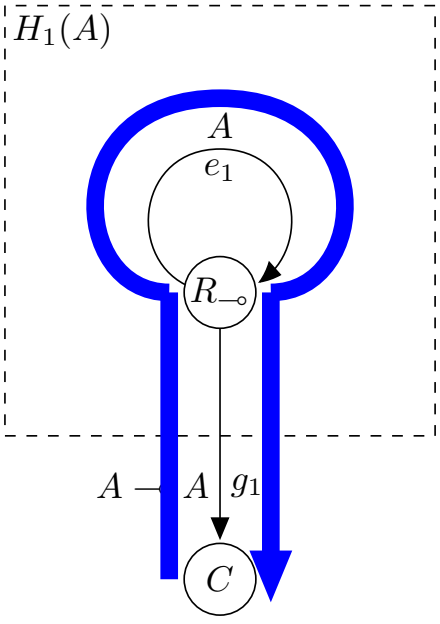

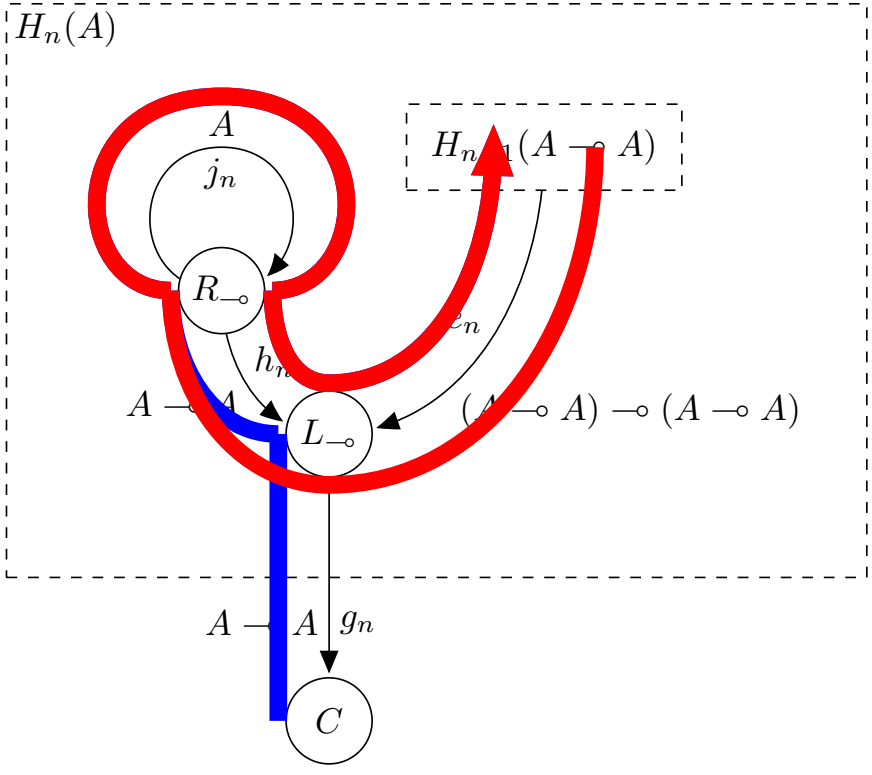

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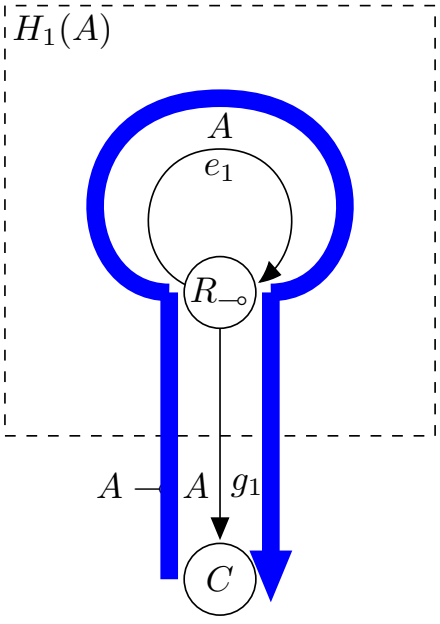

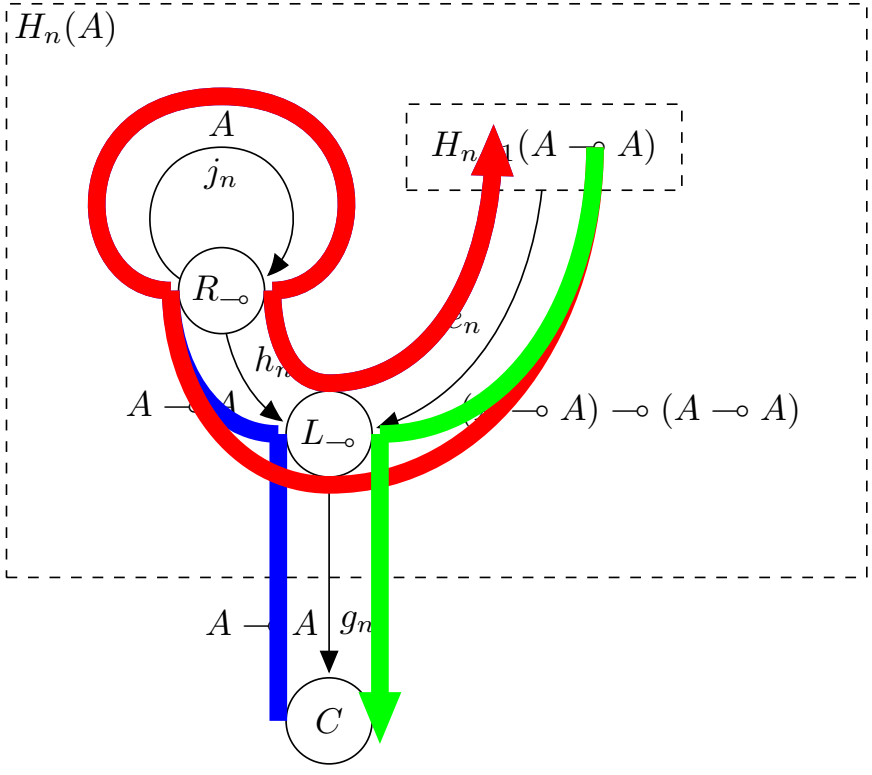

The Weight of a Proof-net

- If $e \in E_G$ is a box main premise, then every exponential tree t with $(e, U, t, +) \mapsto^* C$ (where C is final) corresponds to a different *copy* of the box *under* U.
- U is canonical for $e \in E_G$ if it is built up from copies of the boxes in which $e \in E_G$ is contained.
- The natural number $R_G(e, U)$ is the number of distinct copies of e under U.
- W_G is the sum of $R_G(e, U)$ over all box main premises and over all canonical sequences for them.

Key Property

- W_G is always positive, but is not guaranteed to strictly decrease at any normalization step.
- It is not clear whether the size of $|G|$ is related to W_G or not.
- However, we can define another quantity T_G which *both* decreases at every normalizatop step and majorizes $|G|$.
- As a consequence, we get:

Proposition (Monotonicity)

There are normalization strategies PW and LBL such that $W_G \geq W_H$ and $T_G > T_H$ whenever H is obtained from U using the PW strategy and, moreover, $W_G \leq W_H + 1$ whenever H is obtained from G using the LBL strategy.

One Direction

- T_G is polynomially related to W_G and $|G|$.
- The PW strategy is the worst one:
  - It takes as least as many steps as any other reduction strategy.
  - It maximizes the size of reducts.
- As a consequence, we get:

Theorem

There is a polynomial $p : \mathbb{N}^2 \to \mathbb{N}$ such that for every proof-net G, G normalizes in at most $p(W_G, |G|)$ steps and the size of any reduct H of G is at most $p(W_G, |G|)$.

The Other Direction

- Is W_G a gross overestimate on the time needed to normalize G?
- Consider the LBL strategy:
  - For every $n \in \mathbb{N}$, a cut at level $n+1$ is fired only if there are not cuts at levels from 1 to n.
  - For every $n \in \mathbb{N}$, a !-cut at level n is fired only if every cut at level n is either a W-cut or a !-cut.
  - A W-cut is fired only if every cut in the proof-net is a W-cut.
- W_G decreases by at most one at every step.
- As a consequence, we get:

Theorem

Let G be a proof-net. Then there is H with $G \rightarrow^{W_G} H$.

Consequences

- The two results can together be seen as a strengthening of the well-known correspondence between strongly normalizing nets and finiteness of regular paths.
- This has very interesting consequences: for example, a family $\mathscr{G}$ of proof-nets can be normalized in polynomial (respectively, elementary) time iff there is a polynomial (respectively, an elementary function) p such that $W_G \leq p(|G|)$ for every $G \in \mathscr{G}$.
- This will greatly help when giving new proofs of soundness for various subsystems of linear logic.
- As a side-effect, we have succeeded in defining an invariant cost-model for MELL proof-nets. If we consider W_G as *the cost* of normalizing G, proof-nets and Turing machines can simulate each other with a polynomial overhead in time.

Elementary Linear Logic

- Rules $N_!$ and $D_!$ are no more part of the sequence calculus.
- As a consequence:

Lemma (Stratification)

Let G be an elementary linear logic proof-net. If $(e, U, V, b) \to^* (g, W, Z, c)$, then $\|U\| + \|V\| = \|W\| + \|Z\|$.

- The usual soundness theorem can be reproved:

Proposition (ELL Soundness)

For every $n \in \mathbb{N}$ there is an elementary function $p_n : \mathbb{N} \to \mathbb{N}$ such that $W_G \leq p_{d(G)}(|G|)$ for every ELL proof-net G.

Soft Linear Logic

- It can be defined from ELL by replacing rule X with M as follows:

$$\frac{\Gamma, A, \ldots, A \vdash B}{\Gamma, !A \vdash B}\ M$$

- Exponential trees becomes simpler:

$$t ::= \mathrm{e} \mid \mathrm{m}(i)$$

where i ranges over natural numbers.

- We obtain:

Proposition (SLL Soundness)

For every $n \in \mathbb{N}$ there is a polynomial $p_n : \mathbb{N} \to \mathbb{N}$ such that $W_G \leq p_{\partial(G)}(|G|)$ for every SLL proof-net G.

Light Linear Logic - I

- (Multiplicative) Light linear logic can be obtained from ELL by enriching the language of formulae with a new modal operator § and splitting rule $P_!$ into two rules:

$$\frac{\Gamma \vdash B \quad |\Gamma| \leq 1}{!\Gamma \vdash !B}\ S_! \qquad \frac{\Gamma, \Delta \vdash A}{!\Gamma, \S\Delta \vdash \S A}\ S_\S$$

- At the proof-nets level, two box constructions, !-boxes and §-boxes, correspond to $S_!$ and $S_\S$.
- As for the underlying context semantics, !-boxes induce the usual rewriting rules on C_G, while the last two rules are not valid for §-boxes.

Light Linear Logic - II

- As an immediate consequence of the restrictions on rules for exponentials, we get

Lemma (Strong Determinacy)

For every context C there is at most one context D such that $C \to_G D$.

- So:

Proposition (LLL Soundness)

For every $n \in \mathbb{N}$ there is an polynomial $p_n : \mathbb{N} \to \mathbb{N}$ such that $W_G \leq p_{\partial(G)}(|G|)$ for every LLL proof-net G

Contributions

- We have proved the existence of deep connections between W_G and the cost of normalizing G.
- Copying has been proved to be the real source of complexity in the realm of linear logic.
- We have reproved strong soundness results for fragments of multiplicative and exponential linear logic.
- Most work is factored over the various fragments and done just once.

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Future Developments

- Is context semantics related to resource polynomials and weights in bounded linear logic?
- Extend the results to affine variants of the considered systems (easy).
- Extend the results to pure, untyped, nets.

Questions?