Recursive Analysis And Real Recursive Functions

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Computing on the reals

- There are several machines, models for computing over the reals
  - Differential Analyzer
  - General Purpose Analog Computer (GPAC)
  - Computable Analysis (or *Recursive Analysis*)
  - $\mathbb{R}$-recursive functions (or *real recursive functions*)
  - Polynomial Differential Equations (PolyDE)

Discrete Case

- There are several models for computation over integers
  - Computable functions
  - Turing machines
  - λ-calculus
  - ...
- But those models are "equivalent".

Church-Turing thesis

All reasonable discrete models of computation compute exactly the same functions.

Linking models of “real” computation

The models of computable analysis and $\mathbb{R}$-recursive functions deal with similar functions but lack real relations between their classes. Investigating such links can help giving an analog characterization of what may be considered reasonable in computation over the reals.

A step towards a Church Thesis for computation over the reals?

Purpose of this talk

We are going to present

- some background results: characterizing elementary functions and all levels of the Grzegorczyk hierarchy as restrictions of real recursive functions;
- a zero-finding operator to extend this result and obtain a characterization of recursive functions;
- a limit operator that bridges the step from discrete functions to real recursive functions.

Recursive and Sub-recursive functions

$$\mathcal{R}ec = [0, S, U; COMP, REC, MU]$$

$$\mathcal{PR} = [0, U, S; COMP, REC]$$

$$\mathcal{E} = [0, S, U, +, \ominus; COMP, BSUM, BPROD]$$

$$\mathcal{E}_n = [0, S, U, +, \ominus, E_{n-1}; COMP, BSUM, BPROD]$$

With

$E_2(x) = 2^x$

$E_{n+1}(x) = E_n^{[x]}(1)$ for $n \geq 2$ with $f^{[0]}(x) = x$, $f^{[d+1]}(x) = f(f^{[d]}(x))$

Computable analysis: representing a real number

Definition [Representation of a real]

A real number is represented by a sequence of integers:
Let $x \in \mathbb{R}$.
There exists $(x_n) \in \mathbb{Q}^{\mathbb{N}}$ such that $\forall i, |x - x_i| < \frac{1}{2^i}$.
Let $\nu_{\mathbb{Q}}$ be a representation of the rational numbers.
$(m_n) \in \mathbb{N}^{\mathbb{N}}$ represents x iff $\forall i, |x - \nu_{\mathbb{Q}}(m_i)| < \frac{1}{2^i}$.

Definition [Notation]

We will write $(m_n) \rightsquigarrow x$ if the sequence (m_n) represents x.

Computable functions [Weihrauch00,Ko91]

Definition [Computable functions]

A function $f : [a, b] \to \mathbb{R}$ with $a, b \in \mathbb{Q}$ is computable (resp: elementarily computable) iff there exists $\phi : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ recursive (resp: elementary) such that

$$\forall X \rightsquigarrow x, (\phi(X)) \rightsquigarrow f(x).$$

Complexity for functions from recursive analysis

Definition [Complexity]

Let $f : G \to \mathbb{R}$ a computable function. $t : G \times \mathbb{N} \to \mathbb{N}$ is a complexity bound for f if there exists ϕ computing f such that

$$\forall x \in G, \forall n > 0, t(x,n) \geq \sup_{X \rightsquigarrow x} \{ T((\phi(X))_n) \}$$

where $T(((\phi(X))_n))$ is the time taken to compute $(\phi(X))_n$.

Definition [Uniform Complexity]

$t' : \mathbb{N} \to \mathbb{N}$ is a uniform complexity bound if
$\forall x \in G \cap [-2^n, 2^n], t(x,n) \leq t'(n)$

In other words, the complexity of a function is the number $t(n)$ of bits of the inputs that need to be read to get n bits of the output.

Real recursive functions [Moore96]

Definition [$\mathcal{G}$]

$$\mathcal{G} = [0, 1, U; \mathrm{COMP}, \mathrm{INT}, \mathrm{MU}]$$

- INT: given g, h, $\mathrm{INT}(g, h)$ is the solution of the differential equation $\begin{cases} f(\overrightarrow{x}, 0) & = & g(\overrightarrow{x}) \\ \frac{\partial f}{\partial y}(\overrightarrow{x}, y) & = & h(\overrightarrow{x}, y, f(\overrightarrow{x}, y)) \end{cases}$
- MU: given $f : \mathcal{D} \subset \mathbb{R}^{n+1} \to \mathbb{R}$, $\mathrm{MU}(f) : \overrightarrow{x} \mapsto \begin{cases} y_- = \sup_{y \leq 0}\{f(\overrightarrow{x}, y) = 0\} \text{ if } |y_-| \leq |y_+| \\ y_+ = \inf_{y > 0}\{f(\overrightarrow{x}, y) = 0\} \text{ if } |y_+| < |y_-| \end{cases}$

Real recursive functions [Campagnolo01]

Definition [$\mathcal{L}$]

$$\mathcal{L} = [0, 1, -1, \pi, U, \theta_3; \mathrm{COMP}, \mathrm{CLI}]$$

With

- U : projections
- $\theta_3 : \begin{cases} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & \max(0, x^3) \end{cases}$
- COMP: composition
- CLI: given g, h, c such that h' bounded by c.
  $f = \mathrm{CLI}(g, h, c)$ is the maximal solution of
  $$\begin{array}{rcl} f(\overrightarrow{x}, 0) & = & g(\overrightarrow{x}) \\ \frac{\partial f}{\partial y}(\overrightarrow{x}, y) & = & h(\overrightarrow{x}, y) f(\overrightarrow{x}, y) \end{array}$$

Properties of $\mathcal{L}$

Proposition [Campagnolo]

All functions from $\mathcal{L}$ are continuous, defined everywhere and of class $\mathcal{C}^2$.

For a class $\mathcal{F}$ of functions $\mathbb{R} \to \mathbb{R}$, $DP(\mathcal{F})$ is the set of functions $\mathbb{N} \to \mathbb{N}$ that have an extension in $\mathcal{F}$.

Proposition [Campagnolo]

$$DP(\mathcal{L}) = \mathcal{E}$$

Where $\mathcal{E} = [0, S, U, \ominus; \mathrm{COMP}, \mathrm{BSUM}, \mathrm{BPROD}]$ is the class of discrete elementary functions.

What about recursive functions?

This result gives a characterization of $\mathcal{E}$ (and has been extended to all levels of the Grzegorczyk hierarchy).

We will now present an operator that will extend the discrete μ.

A real μ operator

Remark: A naive "return the smallest root" operator yields unwanted functions (see [Moore96]).

Definition [UMU]

Given $f : \mathcal{D} \times \mathcal{I} \subset \mathbb{R}^{k+1} \to \mathbb{R}$ differentiable such that:

- $\forall \overrightarrow{x} \in \mathcal{D}$, the function $g_{\overrightarrow{x}} : y \mapsto f(\overrightarrow{x}, y)$ is non decreasing,
- $g_{\overrightarrow{x}}$ has a unique root $y_{\overrightarrow{x}} \in \mathcal{I}^\circ$,
- $\frac{\partial f}{\partial y}(\overrightarrow{x}, y_{\overrightarrow{x}}) > 0$.

$$\mathrm{UMU}(f) = \begin{cases} \mathbb{R}^k & \longrightarrow & \mathbb{R} \\ \overrightarrow{x} & \mapsto & y \text{ such that } f(\overrightarrow{x}, y) = 0 \end{cases}$$

Proposition

UMU preserves $\mathcal{C}^2$.

$\mathcal{H} = \mathcal{L} + \mathrm{UMU}$

Definition [$\mathcal{H}$]

$$\mathcal{H} = [0, 1, U, \theta_3; \mathrm{COMP}, \mathrm{CLI}, \mathrm{UMU}]$$

Proposition

$$\mathcal{L} \subset \mathcal{H}$$

Proof:

- $-1 = \mathrm{UMU}(x \mapsto x + 1)$
- $x \mapsto \frac{1}{1+x^2} = \mathrm{UMU}\left(x, y \mapsto (1 + x^2)y - 1\right)$;
  $\arctan(0) = 0$ and $\arctan'(x) = \frac{1}{1+x^2}$;
  $\pi = 4 \arctan(1)$

Result: $DP(\mathcal{H}) = \mathcal{R}ec(\mathbb{N})$

Theorem

$$DP(\mathcal{H}) = \mathcal{R}ec(\mathbb{N})$$

Where $\mathcal{R}ec(\mathbb{N})$ denotes the set of discrete partial recursive functions.

Proof: we have to demonstrate both directions.

- $DP(\mathcal{H}) \subset \mathcal{R}ec(\mathbb{N})$ comes from the fact that UMU preserves computability (in the sense of recursive analysis).
- $\mathcal{R}ec(\mathbb{N}) \subset DP(\mathcal{H})$ can be proven using a normal form theorem in $\mathcal{R}ec(\mathbb{N})$ and translating the discrete μ into our UMU.

Proof of $\mathcal{R}ec(\mathbb{N}) \subset DP(\mathcal{H})$

Let $\phi \in \mathcal{R}ec(\mathbb{N})$. There exists χ and ψ elementary such that $\phi = \chi \circ \mu(\psi)$.

$$\sigma(m, n) = \prod_{z<n} \psi(m, z)$$

$$\kappa(m, n) = 1 \ominus (1 \ominus (1 \ominus \sigma(m, n) + \sigma(m, n + 1)))$$

$$\iota(m, n) = 1 \ominus \kappa + 2 \times (1 \ominus \sigma)$$

$\iota \in \mathcal{E}$ has a single one and it coincides with ψ's first zero.
We then extend ι into a function i from $\mathcal{L}$ and use some tricks so that UMU can be applied to $i - 1$.
Finally, with h an extension to $\mathcal{L}$ of χ, $\mathrm{COMP}(UMU(i - 1), h)$ is an extension to $\mathcal{H}$ of ϕ. ∎

Consequences

Corollary

$$\mathcal{L} \subsetneq \mathcal{H}$$

Corollary ["Normal Form"]

A function from $\mathcal{H}$ can be written with at most 3 nested UMU.

We may need 2 UMU to obtain π and -1. The other UMU comes from the simulation of the discrete μ.

Characterizing computable analysis classes

Those results give analog characterizations of $\mathcal{E}$ and $\mathcal{R}ec(\mathbb{N})$.
With a limit operator, we can extend those characterizations to obtain characterizations of $\mathcal{E}(\mathbb{R})$ and $\mathcal{R}ec(\mathbb{R})$.

Operator LIM

Definition [LIM schema]

Given $f : \mathbb{R} \times \mathcal{D} \subset \mathbb{R}^{k+1} \to \mathbb{R}^l$,
If there are $K : \mathcal{D} \to \mathbb{R}$ and $\beta : \mathcal{D} \to \mathbb{R}$ a *polynomial* such that

$$\forall \overrightarrow{x}, \forall t \geq \|\overrightarrow{x}\|, \|\frac{\partial f}{\partial t}(t, \overrightarrow{x})\| \leq K(\overrightarrow{x}) \exp(-t\beta(\overrightarrow{x})).$$

Then, $F = \mathrm{LIM}(f, K, \beta)$ is defined by $F(\overrightarrow{x}) = \lim_{t\to\infty} f(t, \overrightarrow{x})$ provided it is $\mathcal{C}^2$.

Theorems

We will write $\mathcal{C}^*$ where $\mathcal{C} = [\mathcal{F}; \mathcal{O}]$ to denote the class $[\mathcal{F}; \mathcal{O}, \mathrm{LIM}]$.

Theorem

For functions of class $\mathcal{C}^2$ defined on a compact domain,

$$\mathcal{L}^* = \mathcal{E}(\mathbb{R}).$$

Theorem

For functions of class $\mathcal{C}^2$ defined on a compact domain,

$$\mathcal{H}^* = \mathcal{R}ec(\mathbb{R}).$$

first directions of the proofs

We need to prove that $\mathcal{L}^* \subseteq \mathcal{E}(\mathbb{R})$ and $\mathcal{H}^* \subseteq \mathcal{R}ec(\mathbb{R})$.
To prove those results, we first recall the properties of the operators to preserve $\mathcal{R}ec(\mathbb{R})$ and $\mathcal{E}(\mathbb{R})$[1] then show that LIM also preserves those classes.

[1]except UMU of course.

Second directions of the proofs

Then show that $\mathcal{E}(\mathbb{R}) \subseteq \mathcal{L}^*$.
To do that, we will prove a more general property.

Proof (suite)

Proposition

Given C with $\mathcal{E} \subset C$ and $\mathrm{COMP}(C) \subset C$
and $\mathcal{C}$ with $\mathcal{L} \subset \mathcal{C}$, and $\mathrm{COMP}(\mathcal{C}) \subset \mathcal{C}$ and $\int(\mathcal{C}) \subset \mathcal{C}$
If

$$C \subseteq DP(\mathcal{C}),$$

then for functions of class $\mathcal{C}^2$ defined on a compact domain, whose derivatives have a modulus of continuity in C,

$$C(\mathbb{R}) \subseteq \mathcal{C}^*.$$

Consequences

Corollary ["Normal Form"]

A function from $\mathcal{L}^*$ can be written with at most 2 nested LIM

One limit to obtain $1/x$ and another from the limit mechanism.

Other result

From the proposition used to prove the second direction of the proof, we can get some new inclusions, for example:

Proposition

Let $\bar{D} = [0, 1, -1, U, \theta_3; \mathrm{COMP}, \bar{I}]$. We know that $\bar{D} \supset \mathcal{PR}$. Hence $\bar{D}^* \supset \mathcal{PR}(\mathbb{R})$.

Results

From $DP(\mathcal{L}) = \mathcal{E}(\mathbb{N})$ and $DP(\mathcal{L}_i) = \mathcal{E}_i(\mathbb{N})$, we obtained:

- $DP(\mathcal{H}) = \mathcal{R}ec(\mathbb{N})$
- For $\mathcal{C}^2$ functions defined on a compact,
  - $\mathcal{L}^* = \mathcal{E}(\mathbb{R})$
  - $\mathcal{L}_i^* = \mathcal{E}_i(\mathbb{R})$
  - $\mathcal{H}^* = \mathcal{R}ec(\mathbb{R})$.

Perspectives

- Improving normal forms theorems?
- Can we omit θ_3?
- Studying the link between UMU and LIM
- Is it possible to have $\mathcal{C}^1$ or $\mathcal{C}^0$ instead of $\mathcal{C}^2$?
- What about complexity classes?