MLL normalization and transitive closure: circuits, complexity, and Euler tours

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Problem: Given a proofnet in multiplicative linear logic (MLL), what is the computational complexity of determining its normal form?

[sensitive to size of output]

Decision problem: Given two proofnets in multiplicative linear logic, what is the computational complexity of determining if they have the same normal form?

[output insensitive - more interesting]

Why study the complexity of normalization?

Linearity is the key ingredient in understanding the complexity of type inference.

Approximation: every occurrence of a bound variable has the same type (simple types, ML, intersection types).
Thus linearity subverts approximation: it renders type inference synonymous with normalization.

Why study the complexity of normalization?

MLL is a baby programming language:

$\otimes$ is a linear pairing of
expressions (`cons`)
expression and continuation (@)

$\parr$ is a linear unpairing of
expressions (π, π')
expression and continuation (λ)

complexity of normalization = complexity of interpreter

Plan...

Preliminaries
Complexity of normalization: some results...
Some technical details...

Some preliminaries...

Sequent rules for multiplicative linear logic

$$\frac{}{\alpha^\perp,\alpha}\,\mathrm{Ax} \qquad \frac{\Gamma,\alpha \quad \alpha^\perp,\Delta}{\Gamma,\Delta}\,\mathrm{Cut} \qquad \frac{\Gamma,\alpha \quad \Delta,\beta}{\Gamma,\Delta,\alpha\otimes\beta}\,\otimes \qquad \frac{\Gamma,\alpha,\beta}{\Gamma,\alpha\parr\beta}\,\parr$$

An ML(L) metalanguage: MLL proofs, written in *linear* ML

Note: $\alpha\multimap\beta=\alpha^\perp\parr\beta$, $\alpha^{\perp\perp}=\alpha$, two sided sequents ($\alpha^\perp$ on left is like α on right), etc.

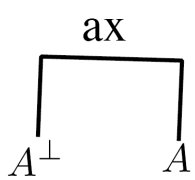
$$\frac{}{x{:}\alpha\vdash x{:}\alpha}$$

$$\frac{\Gamma\vdash E{:}\alpha \quad \Delta\vdash F{:}\beta}{\Gamma,\Delta\vdash (E,F){:}\ \alpha\otimes\beta} \qquad \frac{\Gamma\vdash F{:}\alpha\otimes\beta \quad \Delta,x{:}\alpha,y{:}\beta\vdash E{:}\sigma}{\Gamma,\Delta\vdash \mathrm{let}\ (x,y)=F\ \mathrm{in}\ E{:}\ \sigma}$$

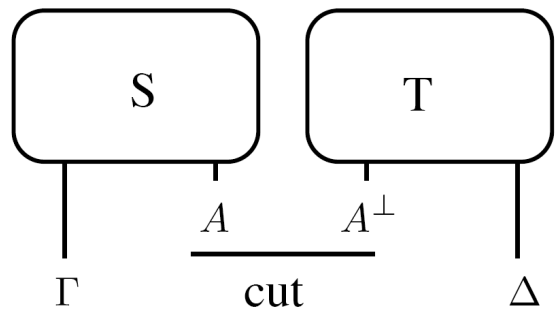
$$\frac{\Gamma,x{:}\alpha\vdash E{:}\beta}{\Gamma\vdash \lambda x.E{:}\ \alpha\multimap\beta} \qquad \frac{\Gamma\vdash E{:}\alpha\multimap\beta \quad \Delta\vdash F{:}\beta}{\Gamma,\Delta\vdash E\ F{:}\beta}$$

Proofnets for multiplicative linear logic

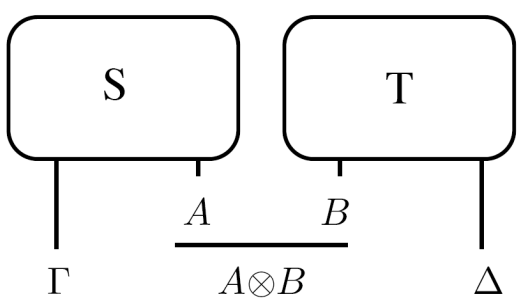
$$\frac{}{A^\perp, A}\ \mathrm{Ax}$$



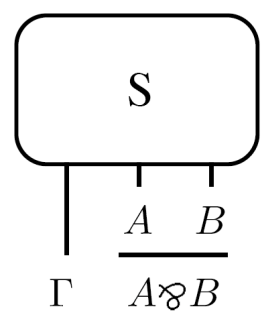
$$\frac{\Gamma, A \qquad A^\perp, \Delta}{\Gamma, \Delta}\ \mathrm{Cut}$$



$$\frac{\Gamma, A \qquad \Delta, B}{\Gamma, \Delta, A \otimes B}\ \otimes$$



$$\frac{\Gamma, A, B}{\Gamma, A \parr B}\ \parr$$



How complex is an MLL proof?

Are the axiom formulas *atomic* or *non-atomic?*
(A *is a propositional variable;* α is an arbitrary formula.)

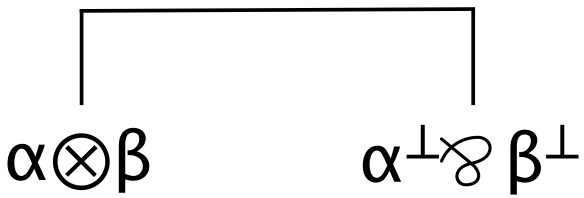
$$\frac{}{A^{\perp}, A}\,\mathrm{Ax} \qquad\qquad \frac{}{\alpha^{\perp}, \alpha}\,\mathrm{Ax}$$

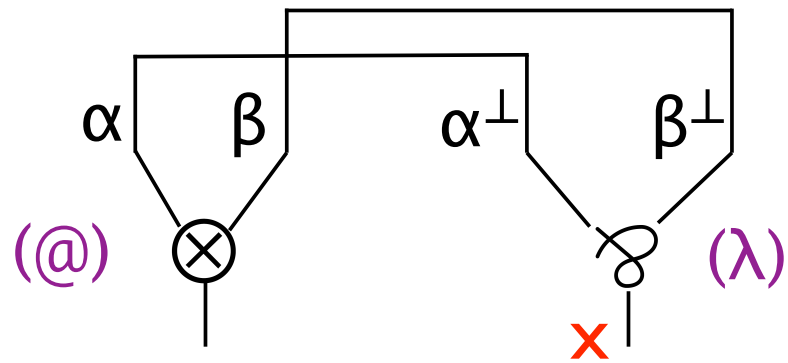
What is the logical depth of cut formulas?

$$\frac{\Gamma, \alpha \qquad \alpha^{\perp}, \Delta}{\Gamma, \Delta}\,\mathrm{Cut} \qquad\qquad \frac{\Gamma \vdash E : \alpha \multimap \beta \qquad \Delta \vdash F : \alpha}{\Gamma, \Delta \vdash E\ F : \beta}$$

The answers to these questions affect the *computational complexity* of normalization, and the *expressive power* of the logic.

η-expansion:
(*linear* in formula size, possibly *exponential* in formula depth)





$$\overline{\alpha\otimes\beta,\ \alpha^{\perp}\parr\beta^{\perp}}$$

$$\dfrac{\dfrac{\overline{\alpha,\alpha^{\perp}} \qquad \overline{\beta,\beta^{\perp}}}{\alpha\otimes\beta,\ \alpha,\ \beta}}{\alpha\otimes\beta,\ \alpha\parr\beta}$$

(programming language equivalents:)

x:A × B	(fst x,snd x)
f:A → B	λy:A. fy

Normalization (Computation)

$$\dfrac{\dfrac{\Gamma,\alpha^\perp \quad \Delta,\beta^\perp}{\Gamma,\Delta,\alpha^\perp\otimes\beta^\perp}\otimes \quad \dfrac{\Sigma,\alpha,\beta}{\Sigma,\alpha\parr\beta}\parr}{\Gamma,\Delta,\Sigma} \quad\Longrightarrow\quad \dfrac{\Gamma,\alpha^\perp \quad \dfrac{\Delta,\beta^\perp \quad \Sigma,\alpha,\beta}{\Delta,\Sigma,\alpha}}{\Gamma,\Delta,\Sigma}$$

$$\dfrac{\dfrac{}{\alpha,\alpha^\perp} \quad \dfrac{}{\alpha,\alpha^\perp}}{\alpha,\alpha^\perp} \quad\Longrightarrow\quad \dfrac{}{\alpha,\alpha^\perp}$$

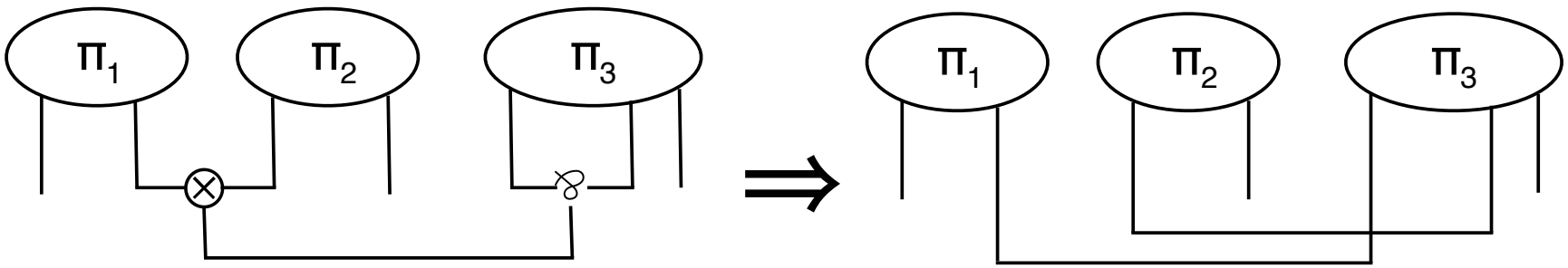
(Normalization always makes the proof get smaller.)

ML(L) programming language analogs (none for Ax-Cut ...)

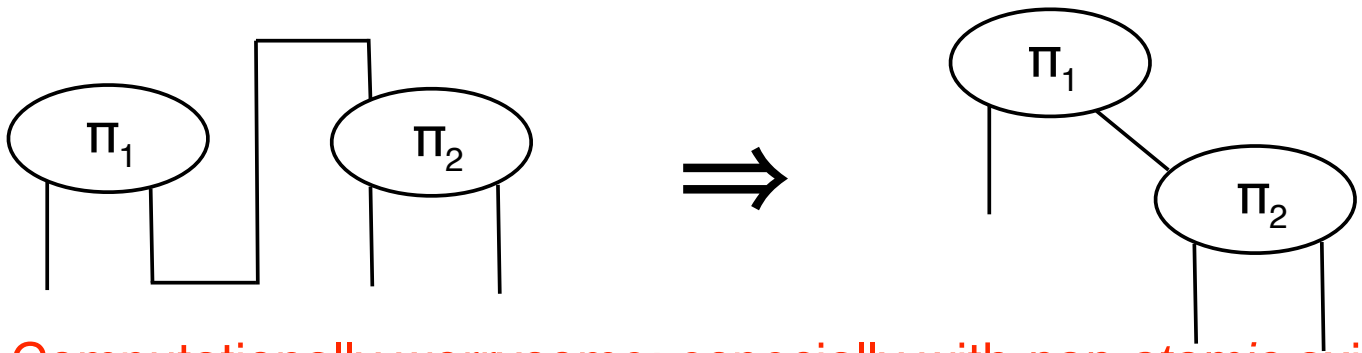
let (x,y)=(U,V) in E $\Rightarrow$ E[U/x,V/y]

$(\lambda x.E)F$ $\Rightarrow$ E[F/x]

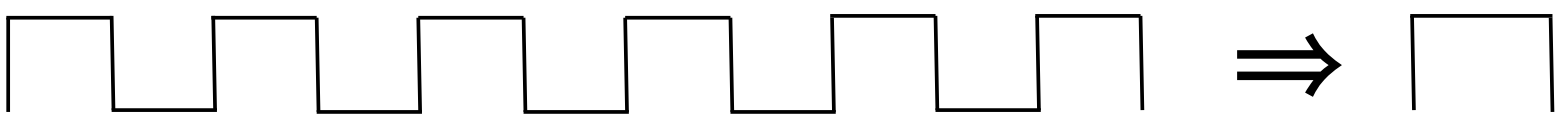
Normalization (Computation) -- proofnet version



Nice, easy, friendly, parallelizable, local...



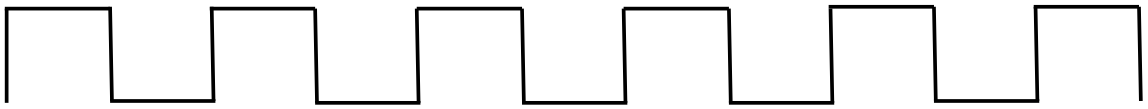
Computationally worrysome: especially with *non-atomic* axioms...



Transitive closure on edges: *not local!*

Computationally problematic if done repeatedly.

When does this happen?



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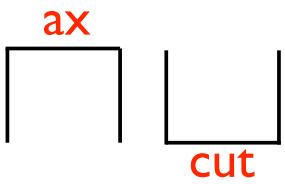
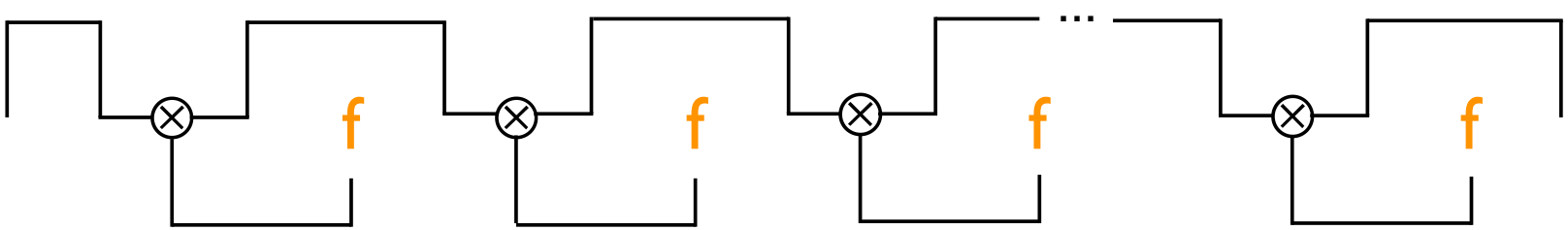
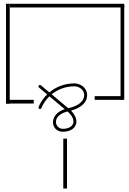


$(\lambda x.x)((\lambda x.x)((\lambda x.x)(\ldots((\lambda x.x)\ y)\ldots)))$ LOGSPACE

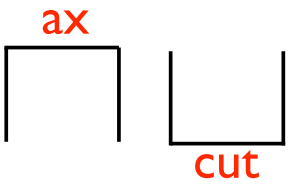
[dual to] $(\ldots(((\lambda x.x)\ (\lambda x.x))\ (\lambda x.x))\ldots)\ (\lambda x.x)\ y$ PTIME



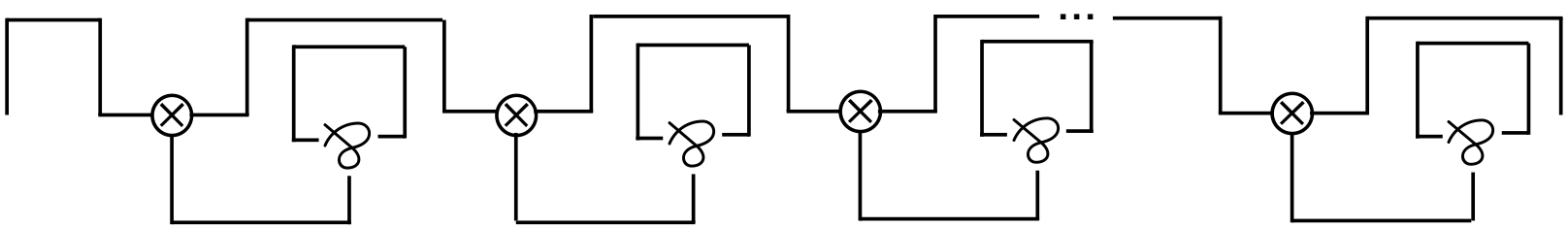
$f = (\lambda x.x) =$ 

also *(and more interesting!)* parity function, permutation, unbounded fan-in Boolean operation, transitive closure,...

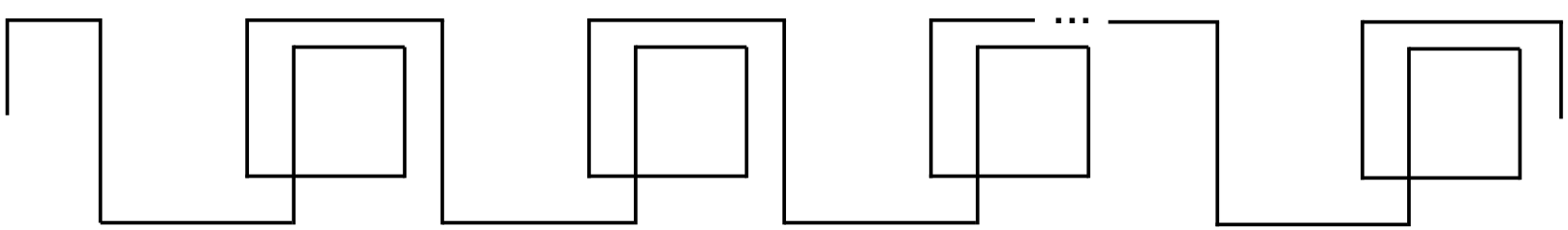
(Complexity/expressiveness: what can you compute with a "flat" proofnet?)



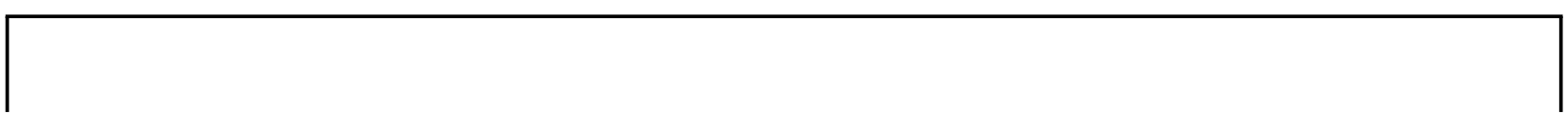
$(\lambda x.x)((\lambda x.x)((\lambda x.x)(\ldots((\lambda x.x)\ y)\ldots)))$



local, parallel



global, sequential (costly?)



MLLu: MLL with unbounded fanout *[Terui, 2004]*

$$\frac{\Gamma_1, A_1 \quad \Gamma_2, A_2 \quad \ldots \quad \Gamma_n, A_n}{\Gamma_1, \Gamma_2, \ldots, \Gamma_n, \otimes_n(A_1, A_2, \ldots, A_n)} \otimes_n$$

$$\frac{\Gamma, A_1, A_2, \ldots, A_n}{\Gamma, \parr_n(A_1, A_2, \ldots, A_n)} \parr_n$$

Why MLLu is needed: to simulate unbounded **fanout** in circuits (and *not* unbounded fanin!) in constant depth

[computational behavior (theorems, complexity results) virtually identical to MLL.]

Complexity of normalization: some results...

For MLL with *atomic* axioms, normalization of a proof is complete for LOGSPACE.

Containment: Given a proof of size n, its normal form can be computed in $O(\log n)$ space. *(Research: proved, and reproved?)*

Hardness: An arbitrary computation requiring $O(\log n)$ space (say, on a Turing machine with input of size n) can be compiled (reduced) to an MLL proof of size $O(n^k)$ -- whose normalization (simulating the Turing machine calculation) takes $O(\log n)$ space.

The reduction must use *less resources* than $O(\log n)$ space, for example NC_1 -- polynomial-sized circuits of depth [time] $O(\log n)$.

For MLL with *non-atomic* axioms, normalization of a proof is complete for PTIME.

Containment: Given a proof of size n, its normal form can be computed in $O(n^k)$ time. *(A trivial observation.)*

Hardness: An arbitrary computation requiring $O(n^k)$ time (say, on a Turing machine with input of size n) can be compiled (reduced) to an MLL proof of size $O(n^{ck})$ and depth $O(n^k)$ -- whose normalization (simulating the Turing machine calculation) takes $O(n^{ck})$ time.

The reduction must use *less resources* than PTIME, for example LOGSPACE.

Circuits + TC = MLLu: For MLLu with ***non-atomic*** axioms and ***formula depth*** d, normalization of a proof is ***equivalent*** to evaluating Boolean circuits with depth $\Theta(d)$, *modulo* a logic gate for constant-time *transitive closure* operation on acyclic graphs (using adjacency matrices).

Circuits + TC ⊆ MLLu: A Boolean circuit of size n and depth d, *including TC gates*, can be simulated *with uniform types* by an MLLu proof of size $O(n)$ and formula depth $O(d)$. *[TC on acyclic graphs is a constant-depth MLLu computation.]*

MLLu ⊆ Circuits + TC: An MLLu proof of size n and (formula) depth d can be simulated by a Boolean circuit with TC gates, of size $O(n^k)$ and depth $O(d)$. *(Terui, 2004)*

The reduction must use *less resources*, for example NC_1 -- polynomial-sized circuits of depth time $O(\log n)$.

What's Old?

Papers by Kazushige Terui (LICS 2004) and myself (ICTCS 2003), others...

What's New?

MLLu Boolean computations "without garbage": A size- and depth-preserving coding of Boolean circuits in MLLu which does not create garbage (output with extra terms and a computation-dependent type). This coding improves a garbage-dependent coding *without* uniform types of Boolean logic in MLLu, due to Terui (2004).

LOGSPACE-hardness: MLL normalization with *atomic* axioms is as hard as any problem requiring LOGSPACE. (Reduction from the *permutation problem.*)

Constant-depth transitive closure in MLLu: An MLLu proof whose normalization simulates transitive closure on an n-node acyclic graph. The proof has a constant formula depth (not dependent on n), and is a baroque elaboration of the LOGSPACE-hardness argument.

Some technical details...

Boolean logic *à la* Church: linear, but *affine*

Very, very, old...

```
- fun True x y= x;
val True = fn : 'a -> 'b -> 'a
- fun False x y= y;
val False = fn : 'a -> 'b -> 'b

- fun Not p= p False True;
val Not = fn : (('a -> 'b -> 'b) -> ('c -> 'd -> 'c) -> 'e) -> 'e
- fun And p q= p q False;
val And = fn : ('a -> ('b -> 'c -> 'c) -> 'd) -> 'a -> 'd
- fun Or p q= p True q;
val Or = fn : (('a -> 'b -> 'a) -> 'c -> 'd) -> 'c -> 'd

- Or False True;
val it = fn : 'a -> 'b -> 'a
- And True False;
val it = fn : 'a -> 'b -> 'b
- Not True;
val it = fn : 'a -> 'b -> 'b
```

Paradise lost: loss of linearity

```
- fun Same p= p True False;
val Same = fn : (('a -> 'b -> 'a) -> ('c -> 'd -> 'd) -> 'e) -> 'e
- Same True;
val it = fn : 'a -> 'b -> 'a
- Same False;
val it = fn : 'a -> 'b -> 'b

- fun K x y= x;
val K = fn : 'a -> 'b -> 'a

- fun Bizarre p= K (Same p) (Not p);
val Bizarre = fn :
   (('a -> 'a -> 'a) -> ('b -> 'b -> 'b) -> 'c) -> 'c
- Bizarre True;
val it = fn : 'a -> 'a -> 'a
- Bizarre False;
val it = fn : 'a -> 'a -> 'a
```

Paradise regained: copying and linearity

```
- fun Copy p= p (True,True) (False,False);
val Copy = fn
  : (('a -> 'b -> 'a) * ('c -> 'd -> 'c)
     -> ('e -> 'f -> 'f) * ('g -> 'h -> 'h) -> 'i)
    -> 'i
- Copy True;
val it = (fn,fn) : ('a -> 'b -> 'a) * ('c -> 'd -> 'c)
- Copy False;
val it = (fn,fn) : ('a -> 'b -> 'b) * ('c -> 'd -> 'd)

- fun nonBizarre p=
      let val (p',p'')= Copy p in K (Same p') (Not p'') end;
val nonBizarre = fn
  : (('a -> 'b -> 'a) * ('c -> 'd -> 'c)
     -> ('e -> 'f -> 'f) * ('g -> 'h -> 'h)
        -> (('i -> 'j -> 'i) -> ('k -> 'l -> 'l) -> 'm)
           * (('n -> 'o -> 'o) -> ('p -> 'q -> 'p) -> 'r))
    -> 'm
- nonBizarre True;
val it = fn : 'a -> 'b -> 'a
- nonBizarre False;
val it = fn : 'a -> 'b -> 'b
```

Continuation-passing style

```
- fun Notgate p k= k (Not p);
val Notgate = fn
  : (('a -> 'b -> 'b) -> ('c -> 'd -> 'c) -> 'e) -> ('e -> 'f) -> 'f
- fun Andgate p q k= k (And p q);
val Andgate = fn :
  ('a -> ('b -> 'c -> 'c) -> 'd) -> 'a -> ('d -> 'e) -> 'e
- fun Orgate p q k= k (Or p q);
val Orgate = fn :
  (('a -> 'b -> 'a) -> 'c -> 'd) -> 'c -> ('d -> 'e) -> 'e
- fun Copygate p k= k (Copy p);
val Copygate = fn
  : (('a -> 'b -> 'a) * ('c -> 'd -> 'c)
     -> ('e -> 'f -> 'f) * ('g -> 'h -> 'h) -> 'i)
    -> ('i -> 'j) -> 'j
```

Circuit evaluation: type inference is normalization

```
- fun Circuit e1 e2 e3 e4 e5 e6=
  (Andgate e2 e3 (fn e7=>
  (Andgate e4 e5 (fn e8=>
  (Andgate e7 e8 (fn f=>
  (Copygate f (fn (e9,e10)=>
  (Orgate e1 e9 (fn e11=>
  (Orgate e10 e6 (fn e12=>
  (Orgate e11 e12 (fn Output=> Output)))))))))))))));
val Circuit = fn
  : (('a -> 'b -> 'a) -> 'c -> ('d -> 'e -> 'd) -> 'f -> 'g)
    -> ('h
        -> ('i -> 'j -> 'j)
           -> 'k
              -> ('l -> 'm -> 'm)
                 -> ('n -> 'o -> 'n) * ('p -> 'q -> 'p)
                    -> ('r -> 's -> 's) * ('t -> 'u -> 'u)
                       -> 'c * (('v -> 'w -> 'v) -> 'x -> 'f))
       -> 'h -> ('y -> ('z -> 'ba -> 'ba) -> 'k) -> 'y -> 'x -> 'g

- Circuit False True True True True False;
val it = fn : 'a -> 'b -> 'a
```

NB: Normalization and type inference are here *synonymous*.

The problem of garbage

Now try and make the calculations non-affine, as in MLL...

```
- fun True (x,y)= (x,y);
val True = fn : 'a * 'b -> 'a * 'b
- fun False (x,y)= (y,x);
val False = fn : 'a * 'b -> 'b * 'a

- fun I x= x;
val I = fn : 'a -> 'a
- fun Compose (f,g) x= f (g x);
val Compose = fn : ('a -> 'b) * ('c -> 'a) -> 'c -> 'b

- fun Erase b= let val (u,v)= b(I,I) in Compose (u,v) end;
val Erase = fn
  : (('a -> 'a) * ('b -> 'b) -> ('c -> 'd) * ('e -> 'c)) -> 'e -> 'd
- Erase True;
val it = fn : 'a -> 'a

- fun Or p q= let val (u,v)= p (True,q) in (Erase v) u end;
val Or = fn
  : (('a * 'b -> 'a * 'b) * 'c
     -> 'd * (('e -> 'e) * ('f -> 'f) -> ('g -> 'h) * ('d -> 'g)))
    -> 'c -> 'h
- Or True False;
val it = fn : 'a * 'b -> 'a * 'b
```

`True` and `False` have *different* types

The problem of garbage (2)

Now try and let `True` and `False` have a uniform type, so that type inference is **not** synonymous with normalization. (Then the type of `Or` does not depend on the type/normal form of its arguments...)

```
- fun True (x:'a, y:'a)= (x,y);
val True = fn : 'a * 'a -> 'a * 'a
- fun False (x:'a, y:'a)= (y,x);
val False = fn : 'a * 'a -> 'a * 'a
```

`True` and `False` have *identical* types (so do u,v)

```
- fun Or p q= let val (u,v)= p (True,q) in (Erase v) u end;
val Or = fn
  : (('a * 'a -> 'a * 'a) * 'b -> 'c * (('d -> 'd) * ('e -> 'e) ->
    ('f -> 'g) * ('c -> 'f))) -> 'b -> 'g
- Or True False;
stdIn:15.1-15.14 Error: operator and operand don't agree [circularity]
```

Partial solution: maintain and create *garbage* during the calculation...

```
- fun Or (p,p') (q,q')= let val (u,v)=p (True,q) in (u,(p',q',v)) end;
val Or = fn
  : (('a * 'b -> 'a * 'b) * 'c -> 'd * 'e) * 'f
    -> 'c * 'g -> 'd * ('f * 'g * 'e)
```

...but can we *eliminate* this garbage? ...

Booleans, the non-affine variation, *without garbage*

```
- fun TT (x:'a,y:'a)= (x,y);
val TT = fn : 'a * 'a -> 'a * 'a
- fun FF (x:'a,y:'a)= (y,x);
val FF = fn : 'a * 'a -> 'a * 'a

- val True= (TT: ('a * 'a -> 'a * 'a), FF: ('a * 'a -> 'a * 'a));
val True = (fn,fn) : ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)
- val False= (FF: ('a * 'a -> 'a * 'a), TT: ('a * 'a -> 'a * 'a));
val False = (fn,fn) : ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)

- fun Show (u,v)= (let val (x,y)= u(true,false) in x end,
                   let val (x,y)= v(true,false) in x end);
val Show = fn :
  (bool * bool -> 'a * 'b) * (bool * bool -> 'c * 'd) -> 'a * 'c
- Show True;
val it = (true,false) : bool * bool
- Show False;
val it = (false,true) : bool * bool
```

Symmetric logic gates

And (p,p')(q,q') = (p∧q,p'∨q') = (p∧q,∼(p∧q))

```
- fun And (p,p') (q,q')=
      let val ((u,v),(u',v')) = (p (q,FF), p' (TT,q'))
      in (u,Compose (Compose (u',v),Compose (v',FF))) end;
val And = fn
  : ('a * ('b * 'b -> 'b * 'b) -> 'c * ('d -> 'e))
    * (('f * 'f -> 'f * 'f) * 'g -> ('e -> 'h) * ('i * 'i -> 'd))
    -> 'a * 'g -> 'c * ('i * 'i -> 'h)
```

An occurrence:

```
(('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a) -> ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
* (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a) -> ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
   -> ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a) -> ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)
```

If `B= ('a * 'a -> 'a * 'a)`, this is `(B * B -> B)[B/'a]`

```
- And True False;
val it = (fn,fn) : ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)
- Show (And True True);
val it = (true,false) : bool * bool
- Show (And True False);
val it = (false,true) : bool * bool
- Show (And False True);
val it = (false,true) : bool * bool
- Show (And False False);
val it = (false,true) : bool * bool
```

Why is there no garbage?

And (p,p')(q,q') = (p∧q,p'∨q') = (p∧q,∼(p∧q))

```
fun And (p,p') (q,q')=
    let val ((u,v),(u',v')) = (p (q,FF), p' (TT,q'))
    in (u,Compose (Compose (u',v),Compose (v',FF))) end;
```

```
(u,v)   = (p (q,FF))
(u',v') = (p'(TT,q'))
```

When p=TT,

```
(u,v)   = (q, FF)
(u',v') = (q',TT)
```

When p=FF,

```
(u,v)   = (FF,q)
(u',v') = (TT,q')
```

Thus {v,v'}={TT,FF}, and

Compose (v,Compose(v',FF))= TT (identity function)
Compose (Compose (u',v),Compose (v',FF))= u'

"Symmetric garbage is self-annihilating"

Symmetric logic gates (2)

Or (p,p′)(q,q′) = (p∨q,p′∧q′) = (p∨q,∼(p∨q))

```
- fun Or (p,p') (q,q')=
      let val ((u,v),(u',v')) = (p (TT,q), p' (q',FF))
      in (u,Compose (Compose (u',v),Compose (v',FF))) end;
val Or = fn
  : (('a * 'a -> 'a * 'a) * 'b -> 'c * ('d -> 'e))
    * ('f * ('g * 'g -> 'g * 'g) -> ('e -> 'h) * ('i * 'i -> 'd))
    -> 'b * 'f -> 'c * ('i * 'i -> 'h)
- Or True False;
val it = (fn,fn) :  (’a * ’a -> ’a * ’a) * (’a * ’a -> ’a * ’a)

- Show (Or True True);
val it = (true,false) : bool * bool
- Show (Or True False);
val it = (true,false) : bool * bool
- Show (Or False True);
val it = (true,false) : bool * bool
- Show (Or False False);
val it = (false,true) : bool * bool
```

Symmetric logic gates (3)

```
- fun Not (x,y)= (y,x);
val Not = fn : 'a * 'b -> 'b * 'a

- Not True;
val it = (fn,fn) : ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)
- Show (Not True);
val it = (false,true) : bool * bool
- Show (Not False);
val it = (true,false) : bool * bool
```

Symmetric logic gates (4)

```
- fun Copy (p,p')= (p (TT,FF), p' (FF,TT));
val Copy = fn
  : (('a * 'a -> 'a * 'a) * ('b * 'b -> 'b * 'b) -> 'c)
    * (('d * 'd -> 'd * 'd) * ('e * 'e -> 'e * 'e) -> 'f)
    -> 'c * 'f
```

Set `'a = 'b = 'd = 'e` and `'c = 'f = ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)`:

```
  (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a) ->
   ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
* (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a) ->
   ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
->
  (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
* (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
```

```
[p= TT]:  Copy (p,p')= ((TT,FF), (TT,FF))   [second component reversed]
[p= FF]:  Copy (p,p')= ((FF,TT), (FF,TT))   [first component reversed]
```

```
- let val (p,q)= Copy True in (Show p, Show q) end;
val it = ((true,false),(true,false)) : (bool * bool) * (bool * bool)
- let val (p,q)= Copy False in (Show p, Show q) end;
val it = ((false,true),(false,true)) : (bool * bool) * (bool * bool)
```

Symmetric logic gates (5)

```
- fun Copy4 (p,p')=
  let val (((s,s'),(p,p')),((q,q'),(r,r')))=
          (p ((TT,TT),(FF,FF)), p' ((FF,FF),(TT,TT)))
      in ((s,p),(s',p'),(q,r),(q',r')) end;
val Copy4 = fn
  : ((('a * 'a -> 'a * 'a) * ('b * 'b -> 'b * 'b))
     * (('c * 'c -> 'c * 'c) * ('d * 'd -> 'd * 'd))
     -> ('e * 'f) * ('g * 'h))
    * ((('i * 'i -> 'i * 'i) * ('j * 'j -> 'j * 'j))
       * (('k * 'k -> 'k * 'k) * ('l * 'l -> 'l * 'l))
       -> ('m * 'n) * ('o * 'p))
    -> ('e * 'g) * ('f * 'h) * ('m * 'o) * ('n * 'p)

- let val (s,p,q,r)=Copy4 True in (Show s,Show p,Show q,Show r) end;
val it = ((true,false),(true,false),(true,false),(true,false)) :
(bool * bool) * (bool * bool) * (bool * bool) * (bool * bool)
```

MLL**u** *unbounded fanout is* essential
(why we cannot do the simulation in MLL)

Symmetric logic gates (6)

```
- fun Circuit e1 e2 e3 e4 e5 e6=
  (Andgate e2 e3 (fn e7=>
  (Andgate e4 e5 (fn e8=>
  (Andgate e7 e8 (fn f=>
  (Copygate f (fn (e9,e10)=>
  (Orgate e1 e9 (fn e11=>
  (Orgate e10 e6 (fn e12=>
  (Orgate e11 e12 (fn Output=> Output)))))))))))));
val Circuit = fn
  : < big type... >

- Circuit True False False False False True;
val it = (fn,fn) : ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)
- Show (Circuit True False False False False True);
val it = (true,false) : bool * bool
```

Unbounded fan-in

The **And** of an arbitrary number of Boolean values can be computed in a constant depth MLL proof:

```
- fun Showtype x F= K x (F x);
val Showtype = fn : 'a -> ('a -> 'b) -> 'a

- Showtype And (fn A=> (A True (And False (And True True))));
val it = fn
  : (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)
     -> ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
    * (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)
       -> ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
    -> ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)
       -> ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)
- Showtype And (fn A=> (A True (A False (A True True))));
val it = fn
  : (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)
     -> ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
    * (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)
       -> ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
    -> ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)
       -> ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)
```

Unbounded fan-in Associate the binary operator to the *left*: *depth* grows *linearly* (*size* grows *exponentially*) under η-expansion.

```
- Showtype And
   (fn A=> (And (And (A True False) True) True));
val it = fn
: ((((('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)
     -> ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
    * (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)
       -> ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
    -> (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)
        -> ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
        * (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)
           -> ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)))
   * ((('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)
       -> ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
      * (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)
         -> ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
      -> (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)
          -> ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
          * (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)
             -> ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)))
   -> ((('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)
        -> ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
       * (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)
          -> ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
       -> (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)
           -> ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
           * (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)
              -> ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)))
      * ((('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)
          -> ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
         * (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)
            -> ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
         -> (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)
             -> ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
             * (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)
                -> ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)))
  * ((((('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)
       -> ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
      * (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)
         -> ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
      -> (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)
          -> ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
          * (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)
             -> ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)))
     * (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)
         -> ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
        * (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)
           -> ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)
           -> (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)
               -> ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
               * (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)
                  -> ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)))
      -> ((('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)
           -> ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
          * (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)
             -> ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
          -> (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)
              -> ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
              * (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)
                 -> ('a * 'a -> 'a * 'a)
                    * ('a * 'a -> 'a * 'a))))
```

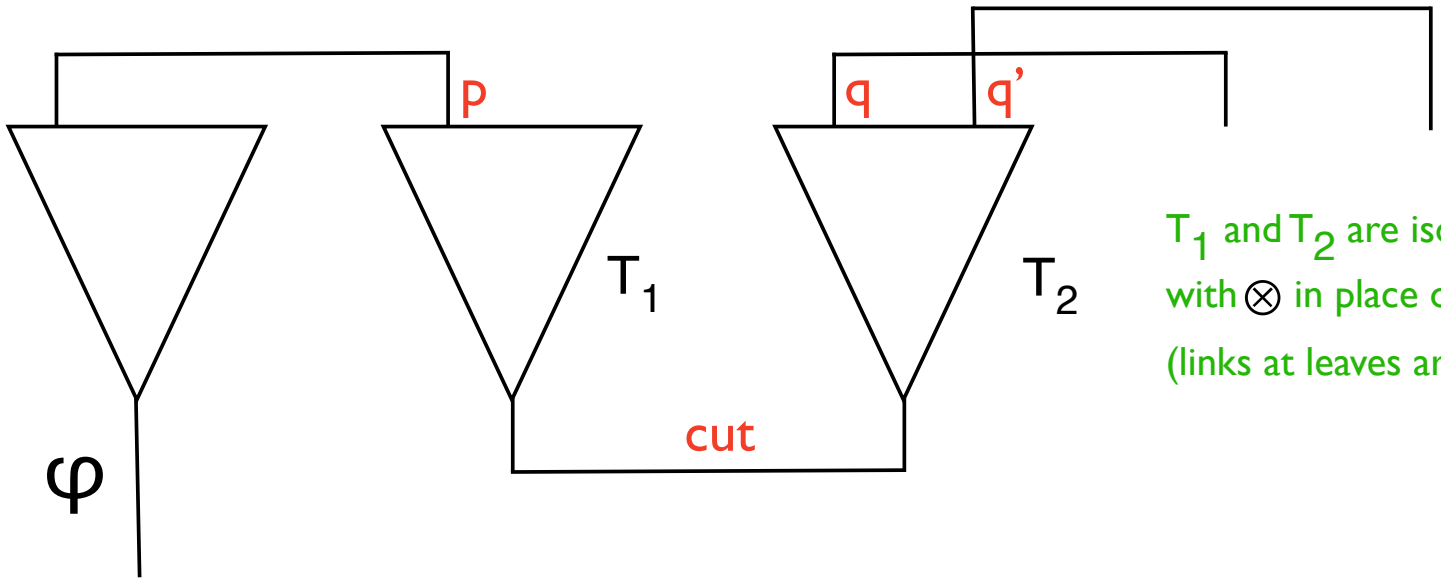
Following paths in a proof of φ: atomic MLL normalization in LOGSPACE

Idea: Use the *Geometry of Interaction* (GoI), following paths in a *proofnet* representing an MLL proof.

When axioms are atomic, paths can only be *polynomial* in length (to be explained).

When paths become *superpolynomial* in length [example: non-atomic axioms, formula depth $O(\log^2 n)$, paths can have length $2^{\log^2 n}$], attempts to parallelize *following paths* (in NC^k) will fail.

Following paths in a proof of φ: atomic MLL normalization in LOGSPACE

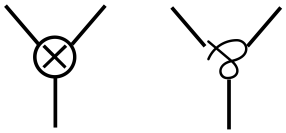


T_1 and T_2 are isomorphic, with ⊗ in place of ⅋

(links at leaves are different)

A LOGSPACE operation: at axiom link p, **guess** the link q such that the path (p,cut) and (q,cut) are isomorphic [modulo the gates]. Storing pointers p,q, etc. requires O(log n) bits -- **four** pointers suffice.

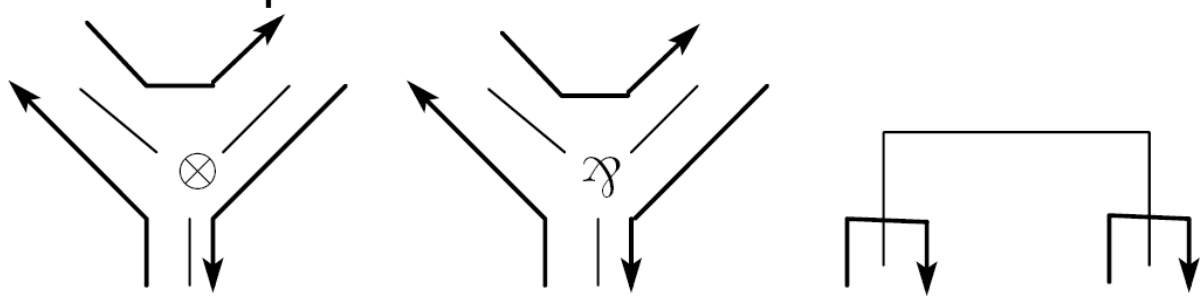
Isomorphic paths must enter nodes with same left/right orientation:



The usual path-following of the *geometry of interaction* is only modified by eliminating the *stack* (with height bounded by formula depth.

Terui's LOGSPACE algorithm (2002): the Euler tour

- 1st Stage
  - Copy axioms and links above conclusions.
  - Find a cut and place two pointers on it.
  - Make a trip as follows:

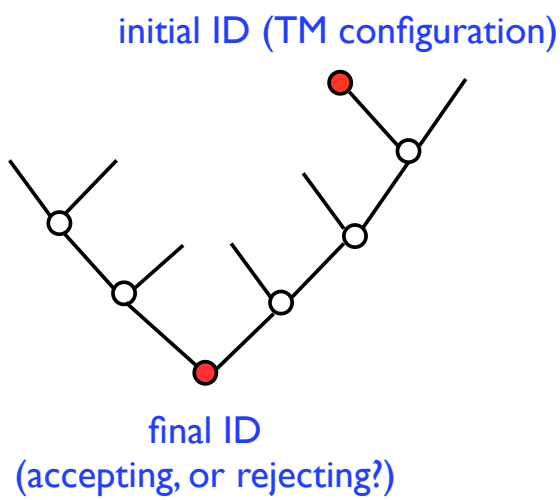


  - When two pointers arrive at axioms, connect the two axioms visited and continue the trip.
  - When they come back to the original cut, finish.
- 2nd Stage. Apply axiom reductions.
- Since both are logspace and logspace functions compose, the overall algorithm runs in logspace.

MLL normalization is as hard as LOGSPACE: the permutation problem

Problem: Given a permutation π on $\{1,2,\ldots,n\}$ and $1 \le i,j \le n$, are i and j on the same cycle of π?

Why this problem is LOGSPACE-hard: A LOGSPACE computation can be compiled into permutations on Turing machine IDs, with two cycles: the accept cycle, and the reject cycle. *Which cycle is the initial ID on?*



Why a binary tree? Without loss of generality, every ID is reached by a either the tape head *moving left* or *moving right* from a unique previous ID.

LOGSPACE = polynomial number of IDs/nodes (compiler from TM problem to permutation problem has limited resources).

To get a permutation, take an *Euler tour* of the tree.

Compiling this problem into MLL: check if i,j are on the same cycle, using a proof/program of constant depth (formula complexity). A programming problem...

Transitive closure on permutations, in constant-depth MLL *(without u)*

```
- val P=(False: ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a),
         False: ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a),
         False: ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a),
         False: ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a));
val P = ((fn,fn),(fn,fn),(fn,fn),(fn,fn))
  : <type omitted>

- fun Perm (S: ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a),
            P: ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a),
            Q: ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a),
            R: ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))=
      (Q,P,S,R);
val Perm = fn
  : (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
    * (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
    * (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
    * (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
    -> (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
       * (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
       * (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
       * (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
```

Transitive closure on permutations, in constant-depth MLL (2)

```
- fun Insert ((s,s'):
                  ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a),
               P: ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a),
               Q: ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a),
               R: ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))=
  ((TT: 'a * 'a -> 'a * 'a, Compose (s,s')),P,Q,R);
val Insert = fn
  : (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
    * (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
    * (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
    * (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
    -> (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
       * (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
       * (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a))
       * (('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)
```

Notice that `(s,s')` is either `(TT,FF)` or `(FF,TT)`, but in either case, `Compose (s,s')=FF`. Thus linearity is preserved because the *garbage is symmetric*.

Transitive closure on permutations, in constant-depth MLL (3)

```
- Show True;
val it = (true,false) : bool * bool
- fun Showvector (S,P,Q,R)= (Show S,Show P,Show Q,Show R);
val Showvector = fn
  : <type omitted>

- Showvector P;
val it = ((false,true),(false,true),(false,true),(false,true))
  : (bool * bool) * (bool * bool) * (bool * bool) * (bool * bool)
- Showvector (Insert P);
val it = ((true,false),(false,true),(false,true),(false,true))
  : (bool * bool) * (bool * bool) * (bool * bool) * (bool * bool)
- Showvector (Insert (Perm (Insert P)));
val it = ((true,false),(false,true),(true,false),(false,true))
  : (bool * bool) * (bool * bool) * (bool * bool) * (bool * bool)
```

For a permutation on $\{1,2,\ldots,n\}$, iterate `Compose(Perm,Insert)` n times.
Then index j is True when 1 and j are on the same cycle.

For a permutation on $\{1,2,\ldots,n\}$, iterate `Compose(Perm,Insert)` n times. Then index j is True when 1 and j are on the same cycle.

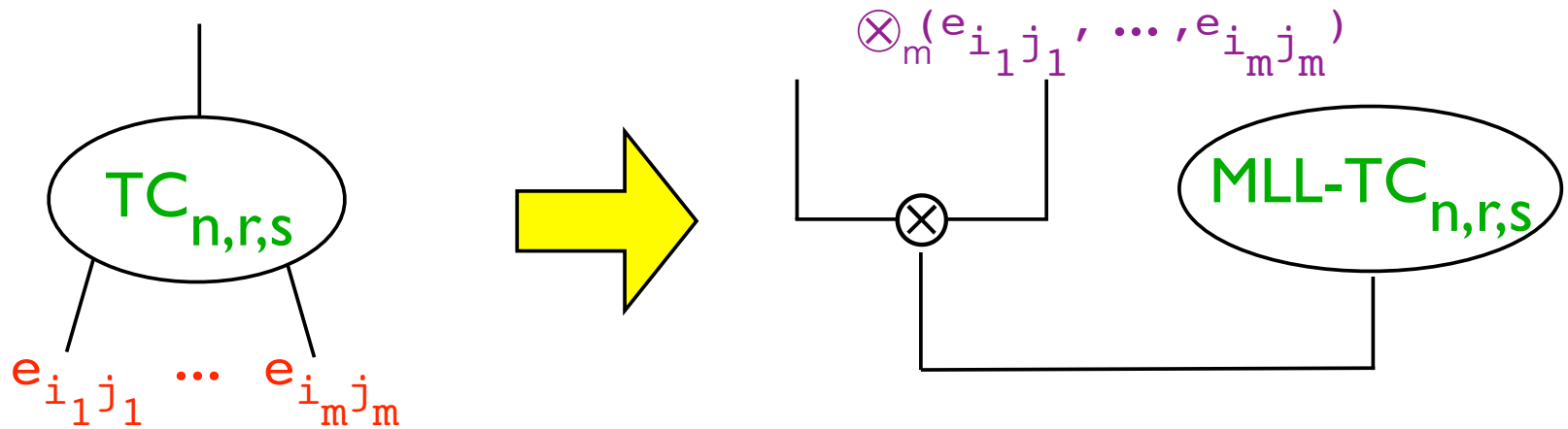
```
- Showvector (Insert (Perm (Insert P)));
val it = ((true,false),(false,true),(true,false),(false,true))
  : (bool * bool) * (bool * bool) * (bool * bool) * (bool * bool)

- fun ExtractR (s,p,q,(r',r''))=
     (r', Compose (r'', Compose (Compose (FF, Compose s),
                                 Compose (Compose p,Compose q))));
val ExtractR = fn
  : (('a -> 'b) * ('c -> 'a)) * (('d -> 'c) * ('e -> 'd)) * (('f
     -> 'e) * ('g -> 'f)) * (('b -> 'h) * 'i) -> ('g -> 'h) * 'i
```

[Recall: `Compose True = Compose (TT,FF) = FF = Compose False`]

```
- ExtractR (Insert (Perm (Insert P)));
val it = (fn,fn) : ('a * 'a -> 'a * 'a) * ('a * 'a -> 'a * 'a)
- Show (ExtractR (Insert (Perm (Insert P))));
val it = (false,true) : bool * bool
```

Transitive closure on acyclic graphs in constant depth MLLu



Boolean circuit gate

Constant-depth MLL proof

n	number of nodes in graph
m	size of adjacency matrix= n(n-1)/2
(i_t,j_t)	enumeration of adjacency matrix
r,s	vertices

Is there a path from vertex r to vertex s?

Lemma. Let $G=(V,E)$ be an undirected, acyclic graph, and let σ be an arbitrary enumeration of its edges. Each edge defines a transposition on V. Then the composition of the enumerated transpositions forms a cyclic permutation **(Euler tour!)** on the vertices of each connected component.

Example:

$\sigma = (P,Q)\ (S,P)\ (Q,R)$

	s	p	q	r
(PQ)	s	*q*	*p*	r
(SP)	*q*	*s*	p	r
(QR)	q	s	*r*	*p*

$(PQ)\cdot(SP)\cdot(SR) = (SQRP)$

Proof by induction on $|V|$. (It fails, easily, for cyclic graphs.)

Application: to check if two vertices are connected, use a fixed permutation (on edges) to broadcast message from one to the other.

To check if R is connected to S, use a fixed permutation (on edges) to broadcast message from R to S

Graph adjacency matrix $e_{i_1 j_1}$ $e_{i_2 j_2}$ $e_{i_m j_m}$ ($m=n^2$, Assume n even)

Swap data in position of p- and q-vector given by edge (u,v)

[equal to `True=(TT,FF)` or `False=(FF,TT)`]

```
fun Swap_ij (u,v) (p_1,...,p_n,q_1,...,q_n)=
  let val ((p'_i,p'_j),(q'_i,q'_j))= (u(p_i,p_j), v(q_i,q_j))
  in (p_1,...,p'_i,...,p'_j,p_n,...,q_1,...,q'_i,...,q'_j,...,q_n)
```

Insert `True` in position p_r, `False` in position q_r:

```
fun Insert_r (p_1,...,p_r,...,p_n,q_1,...,q_r,...q_n)=
  (p_1,...,(Compose p_r, TT),...,p_n,q_1,...,(TT, Compose q_r),...,q_n)
```

To check if R is connected to S, use a fixed permutation (on edges) to broadcast message from R to S (2)

Graph adjacency matrix $e_{i_1 j_1}$ $e_{i_2 j_2}$ $e_{i_m j_m}$ ($m=n^2$, n even)

Make *n* copies of each edge (**big** fanout):

```
let val ((e_{i1j1}1,e_{i1j1}2,...,e_{i1j1}n),
         (e_{i2j2}1,e_{i2j2}2,...,e_{i2j2}n),
         ...,
         (e_{imjm}1,e_{imjm}2,...,e_{imjm}n))=
      (Copy_n e_{i1j1}, Copy_n e_{i2j2},..., Copy_n e_{imjm}))
in ...
```

To check if R is connected to S, use a fixed permutation (on edges) to broadcast message from R to S (2)

Graph adjacency matrix $e_{i_1 j_1}$ $e_{i_2 j_2}$ $e_{i_m j_m}$ ($m=n^2$, Assume n even)

```
let val (F_1,...,F_k,...,F_n)= (...,

 (fn V=> (Swap_{i_1 j_1} e_{i_1 j_1 k})((Swap_{i_2 j_2} e_{i_2 j_2 k})(...((Swap_{i_m j_m} e_{i_m j_m k})
            Insert_r V), ...),
   ...)

 in let val (p_1,...,p_n,q_1,...,(q'_s,q''_s),...,q_n)=

         F_1(F_2(...(F_n (True,..., True, False,...,False))...))

 in (Compose (q'_s, (Compose p_1)∘...∘(Compose p_n)),

     Compose (q''_s, (Compose q_1)∘...∘(Compose q_{s-1})

            ∘(Compose True)∘(Compose q_{s+1})...∘(Compose q_n)))
```

Summary

Parity is everywhere -- it is the canonical programming technique in MLL. A close cousin is the *symmetry of garbage*.

de Morgan symmetry: the key to computing Boolean functions in MLL with a uniform type, and without garbage.

Circuits + ***TC*** = *MLL**u***: ????

u: Unlike Boolean circuits, MLL *does not support* constant-depth unbounded Boolean *fan-out* (it has constant-depth unbounded Boolean *fan-in*).

TC: Unlike MLL, Boolean circuits do not have constant-depth *transitive closure* on acyclic graphs.

Complexity and geometry of interaction: *following paths* is the key to the semantics. It is also the key to the LOGSPACE-completeness of MLL with atomic axioms.

Transitive closure on acyclic graphs: the natural consequence of programming in bounded depth *(parallelism)*. Transitive closure, coded in MLL, *creates its own transitive closure problem*. (In fact, the same one -- it is an ***analog computation***.)

Fin

Proof of lemma by induction on $|V|$. Assume G is a single connected component with $n+1$ vertices, including edge $(j,n+1)$:

$\sigma = \ldots (j,n+1) \ldots$

Without edge $(j,n+1)$ *(by induction):*

$$\begin{array}{ccccccc} 1 & 2 & \ldots & \pi^{-1}(k) & \ldots & n \\ \downarrow & \downarrow & & \downarrow & & \downarrow \\ \pi(1) & \pi(2) & \ldots & k & \ldots & \pi(n) \end{array}$$

Cycle:

$(\ldots \pi^{-1}(k)\ k\ \pi(k) \ldots)$

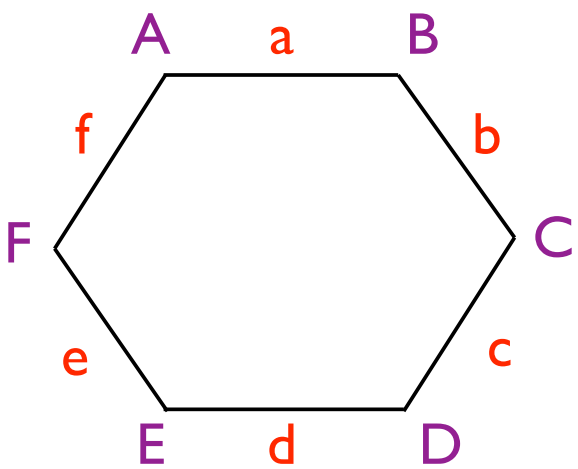
With edge $(j,n+1)$:

$$\begin{array}{cccccc} 1 & j & k & \pi^{-1}(k) & n & n+1 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & k & & & & n+1 \\ & n+1 & & & & k \\ & \downarrow & & & & \downarrow \\ \pi(1) & \pi(j) & \pi(k) & n+1 & \pi(n) & k \end{array}$$

Cycle:

$(\ldots \pi^{-1}(k)\ n+1\ k\ \pi(k) \ldots)$

The lemma fails for a cyclic graph



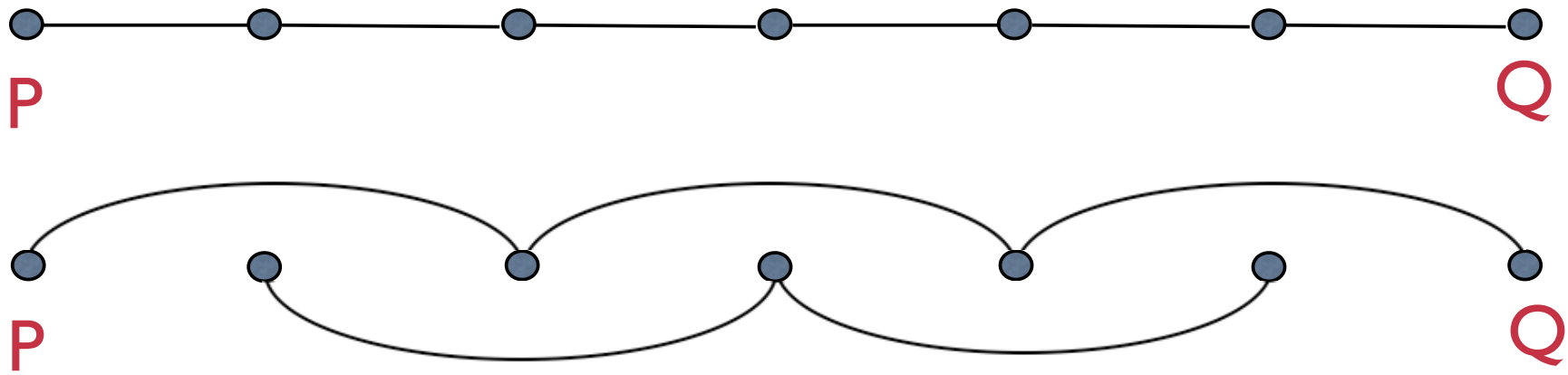
σ = acebdf

ABCDEF
abcdef
bacdef
ba***dc***ef
badc***fe***
b***da***cfe
bda***fc***e
edafc***b***

π = (aec)(bdf)

Transitive closure on acyclic graphs in constant depth MLLu

(this you cannot do with a Boolean circuit in constant depth, because you cannot compute parity in constant depth)



There is a path from P to Q when the parity is odd...