Resources Control Graphs

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Motivations

Programs analysis deal about uniform properties:
- Do all the executions terminate?
- Are all the executions performed within a given time/space bound?

Importance of the use of resources.
- Time and space are resources usually considered.
- Termination is usually reduced to finding a decreasing well-founded ordering, i.e. total usage of a finite resource.
- Only specific resources may be considered (non-overflow of a specific buffer or stack).
- Design a single tool for this kind of analysis.
The core idea
Control Flow Graphs

We’re working here over counters machines.

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2 : $y ++$
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An execution of the program \( \sim \) A path in the CFG.
Petri nets and VASS

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- \(S = \{s_1, \ldots, s_n\}\) are the vertices.
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- For each vertex \(s_i\), there is:
  - A set of valuations \(V_i\)
  - A set of admissible valuations \(V_i^+\).
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  - A set of *valuations* \(V_i\)
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- \(V = \bigcup V_i, V^+ = \bigcup V_i^+\).
- \(W_{i,j} = \mathcal{F}(V_i, V_j), W = \bigcup W_{i,j}\) is the set of weights.
- \(\omega : A \rightarrow W\) such that if \(a = (s_i, s_j)\), then \(\omega(a) \in W_{i,j}\).
Configurations and walks

Let $G = (S, A)$ be a graph.

- A vertex is an element of $S$.
- A path is a sequence of vertices $s^1, \ldots, s^n$ such that there is an edge between $s^i$ and $s^{i+1}$. 
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Let $R = (G, V, V^+, W, \omega)$ be a RSS.

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- A configuration $(s_i, x)$ is *admissible* if $x \in V_i^+$. 
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- A *walk* is a sequence of configurations $(s_1, x_1), \ldots, (s_n, x_n)$ such that:
  - $s_1, \ldots, s_n$ is a path.
  - If $a^i$ is the edge between $s_i$ and $s_{i+1}$, then $x_{i+1} = \omega(a^i)(x^i)$.
Turing Machines

A TM can be represented by a RSS in the following way:

- Underlying graph is the automaton of the TM.
- Valuations (and admissible valuations) are bi-infinite strings over \( \{0, 1\} \).
- Weights perform the corresponding operations.

Each execution of the TM corresponds to an (admissible) walk in the RSS and each (admissible) walk in the RSS corresponds to an execution of the TM.

By Rice’s theorem, extensionnal properties of RSS are not decidable.
Facts and notations

Let \( R = (G, V, V^+, W, \omega) \) be a RSS. Let \( (s^0, x^0), \ldots, (s^n, x^n) \) be a walk following edges \( a^1, \ldots, a^n \).

- \( x^n = \omega(a^n) \circ \cdots \circ \omega(a^1)(x^0) \).
- Functions composition is done in reverse order.
- Weight functions are usually somewhat uniform.
- We write \( f \circ g \) instead of \( g \circ f \) and \( x \otimes f \) instead of \( f(x) \).
- \( x^n = x^0 \otimes \omega(a^1) \circ \cdots \circ \omega(a^n) \).
- \( \circ \) is associative.
- \( \overline{W} \) is the closure of \( \bigcup \omega(a_i) \) by \( \circ \).
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- $R$ uniformly terminates if there is no infinite admissible walk.
- $R$ is $f$-resource aware if for each admissible walk $(s^0, x^0), \ldots, (s^i, x^i)$, $x^i \prec f(x^0)$ ($f$ increasing).
A first result

$R$ does not uniformly terminate $\Rightarrow \exists$ an admissible cycle $(s, x) \xrightarrow{*} (s, x')$ such that $x \preceq x'$. If $R$ is monotonic and positive, the converse is true.
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If an infinite admissible walk exists, extract from it an infinite sequence of configurations with the same vertex. Since \( \prec \) is a well ordering, \( x \preceq x' \) can be extracted from this sequence.
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If the cycle exists, follow it infinitely many time. Monotonicity ensure that valuations reached increase and positivity that this keeps everything admissible.
Resource Control Graph

Let $p$ be a program. It’s Resource Control Graph (RCG) is a RSS where:

- The underlying graph is the Control Flow Graph.
- Admissible valuations approximate the state of memory.

Approximations allow to restrict weighting functions to some uniform family of functions (e.g. $\lambda x.x + \alpha$).
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Unif. term. of the RCG $\Rightarrow$ u.t. of the program.
Weighted graphs and Non Size Increasingness
Weighted graphs as RSS

- A weighted graph is a graph $G = (S, A)$ together with a weighting function $\omega : A \rightarrow \mathbb{Z}$.

- The weight of a path is the sum of the weights of each edges.
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A weighted graph uniformly terminates if and only if it contains no cycle of weight \( \geq 0 \).

It is \( \lambda x.x + \alpha \)-resource aware if it contains no cycle of weight \( > 0 \).

Both criterions can be decided in polynomial time.
Non Size Increasingness

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This leads to the following RCG:

- \( V^+ = \mathbb{N} \) (total space usage is \( \geq 0 \)).
- \( \omega(\text{cons}) = \lambda x.x + 1, \omega(\text{tail}) = \lambda x.x - 1. \)
- \( \overline{W} = \bigcup \lambda x.x + \alpha, V = \mathbb{Z}. \)
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Admissible valuations play exactly the same role as M. Hofmann’s diamonds: the valuation is equal to the number of diamonds needed in the current state.
Going further

- We can use more types if we have an appropriate size function.
- We can choose not to consider all the lists, *i.e.* only control some buffers.
- We can similarly control depth of data stacks (*push, pop*).
- We can similarly control depth of recursion stack (*call, return*).
Size Change Principle
Original SCP

(Lee, Jones, Ben Amram)

\[ f(a + 1, b) \rightarrow g(a, b, b) \]
\[ g(a, b, c + 1) \rightarrow g(a, b, c) \]
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Multipaths, threads.
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Multipaths, threads.
\(\text{FLOW}^\omega, \text{DESC}^\omega\)

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\begin{align*}
  \text{FLOW}^\omega = \text{DESC}^\omega & \Rightarrow \text{termination} \\
  \text{(PSPACE-complet)}.\n\end{align*}
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SCP \equiv \text{uniform termination for all } n.
Not u.t. \iff \exists \text{ cycle of dec. idempotent weight.}
Using matrices

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(Abel and Altenkirch)
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\end{array}
\]

\(W\): matrices over the three valued set.
\(\otimes\): matrices multiplication.
Matrices Multiplication Systems with States
A MMSS is a RSS where:

- \( V_i = bZ^{k_i}, \quad V_i^+ = \mathbb{N}^{k_i} \)
- \( W_{i,j} = \mathcal{M}_{k_i,k_j} \)
- \( \odot = \odot = \times \)

If we use column-vectors instead of row-vectors, we need to transpose matrices and perform multiplication in reverse order.
Uniform termination of MMSS is not decidable
Uniform termination

Uniform termination of MMSS is not decidable

Reduce to uniform termination of counter machines.

- $x++$, $x--$: keep $1$ as first component of vector.
- $x \neq 0$: $x--; x++$.
- $x = 0$: multiplication by $-1$.
Polynomial bound

(Niggl and Wunderlich)

- Use a matrix over \( \{0, 1, \infty\} \) as a certificate for polynomiality of instructions.
- Certificate for \( P_1; P_2: M_1 \times M_2 \).
- Certificate for tests: max.

Shape of the certificate of a program can lead to polytime bound.
Difficulty of instructions

- $x++, x--, x \neq 0$ can be modelised with VASS (*i.e.* with vectors).
- $x = 0$ needs MMSS (*i.e.* a matrix) to be modelised. (it is a well-known fact that Petri nets cannot test if a place is empty)
- First order program can be represented by several matrices, *i.e.* a tensor (given an enumeration of the edges).
- Tensor Multiplication Systems should be able to modelise high order programs.
- The higher the order, the more dimensions are required.
Conclusion

- New tool, generalisation of existing one (Petri nets/VASS).
- Several analysis can be rewritten using this tool.
- Is it possible to combine these analysis?
- Can we also rewrite other analysis?
- Can we discover new analysis?