Multiplexor Categories and Models of Soft Linear Logic

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GEOCAL'06

Workshop on ICC

Marseille - Luminy

Feb 13-17, 2006

Categorical logic: an introduction

Given any (reasonable) logical system, one builds a category where

- objects are formulas
- morphisms are (equivalence classes) of proofs

Under this interpretation, the logical connectives become functors and semantics becomes a structure-preserving functor from this syntactic category.

Some examples:

Logic	Categorical notion
Intuitionsitic logic	ccc
IMLL	SMCC
IMALL	SMCC + fin. (co)products
IMELL	SMCC + comonad
Linear logic	Seely model

Question: What is the categorical notion of a polynomial time linear logic?

Polynomial time linear logics: (A short history)

Bounded Linear Logic (BLL): J.-Y. Girard, A. Scedrov, P. Scott (1992)

First linear logic characterization of polynomial time computation. Based on second order intuitionistic multiplicative exponential linear logic, but uses polynomials explicitly in types to control the use of contraction. It was realized that controlling contraction (more generally, the exponential rules) is an indirect way of controlling time complexity.

Light Linear Logic (LLL): J.-Y. Girard (1998)

Here polynomials no longer appear explicitly in the types, but a new modality is introduced as well as a notion of hybrid sequent. The latter is avoided in

Light Affine Logic (LAL): (A. Asperti (1998)) by adding the unrestricted rule of weakening.

Soft Linear Logic (SLL): Y. Lafont (2002)

More recently Lafont introduced a logical system called *Soft Linear Logic.* It can be seen as a subsystem of BLL which is expressive enough to encode polynomial time. Briefly, SLL is second order intuitionistic multiplicative linear logic, together with a weaker (softer) notion of the exponential operation ! of linear logic. In SLL the usual rules for the exponential ! are replaced with weaker derived rules called *soft promotion* and *multiplexing.*

Project goal: Find a categorical framework for polynomial time linear logics.

Logic	Categorical notion
polytime LL	???

Today's talk: The categorical notions behind soft linear logic.

In fact, soft linear logic has a very natural categorical interpretation, as we shall see.

Outline for remainder of talk

We begin with an introduction to soft linear logic and the corresponding notion of *multiplexor category*. We shall see that a multiplexor category provides a denotational semantics for SLL.

We consider various ways of constructing a multiplexor category from a symmetric monoidal closed category (SMCC) having countable limits, which leads to a large class of models.

More generally, we give a categorical construction (completion) for building a multiplexor category from a SMCC having only a *bounded* form of multiplexing.

As an application to illustrate these ideas, we construct a realizability model for SLL based on the Hofmann-Scott model for BLL.

Finally, we discuss connections with Baillot's Stratified Coherence Spaces for LLL and other future work.

Soft Linear Logic (Y. Lafont)

We assume familiarity with the intuitionistic sequent calculus for propositional ILL, where sequents are of the form $\Gamma \vdash A$, where Γ is a finite sequence of formulas called the hypotheses, and A is a formula called the conclusion. The rules for SLL are the following organized in three groups:

-structural rules: *exchange*, *identity*, and *cut*

$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C} \qquad \frac{}{A \vdash A} \qquad \frac{\Gamma \vdash A \quad \Delta, A \vdash C}{\Gamma, \Delta \vdash C}$$

-multiplicative logical rules:

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B} \qquad \frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C}$$

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C} \qquad \frac{}{\vdash \mathbf{1}} \qquad \frac{\Gamma \vdash C}{\Gamma, \mathbf{1} \vdash C}$$

-exponential logical rules: *soft promotion* and *multiplexing*

$$\frac{\Gamma \vdash A}{!\Gamma \vdash\ !A} \qquad \frac{\Gamma, A^{(n)} \vdash C}{\Gamma, !A \vdash C}$$

In multiplexing, the *rank* n can be any natural number. In particular, we get *weakening* for $n = 0$, and *dereliction* for $n = 1$.

But we do not have *contraction*.

We can recover Girard's exponentials by adding *digging*:

$$\frac{\Gamma, !!A \vdash C}{\Gamma, !A \vdash C}$$

We shall need the following theorem due to Lafont.

Theorem. SLL satisfies cut elimination.

We prove this syntactically by showing that the following generalized version of cut can be eliminated:

$$\frac{\Gamma \vdash A \qquad \Delta, A, \Theta \vdash C}{\Gamma, \Delta, \Theta \vdash C}$$

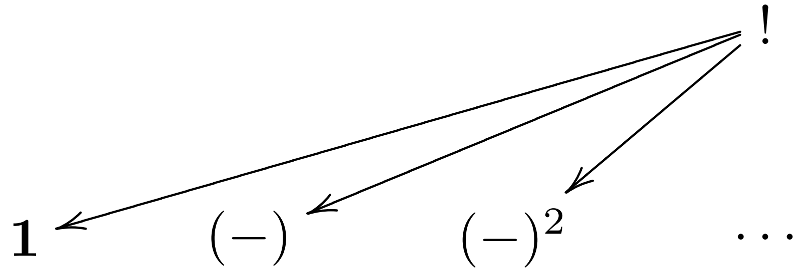
Multiplexor Categories

Definition. A *multiplexor category* consists of a triple $(\mathcal{C}, !, \{m_n\}_{n\geq 0})$, where $\mathcal{C}$ is a symmetric monoidal closed category (SMCC) with tensor unit $\mathbf{1}$, ! is a symmetric monoidal endofunctor on $\mathcal{C}$, and for each natural number n,

$$m_n : ! \stackrel{\cdot}{\to} (-)^n$$

is a monoidal natural transformation (to the n-fold tensor product functor), called a multiplexor of rank n.

In other words, a multiplexor category consists of a SMCC $\mathcal{C}$ together with the following diagram of symmetric monoidal endofunctors on $\mathcal{C}$ and monoidal natural transformations:



Instantiating this picture with an object A in $\mathcal{C}$, we get an arrow $!A \to A^n$, for each natural number n. This models multiplexing.

The requirement that the modality ! be a symmetric monoidal endofunctor on $\mathcal{C}$ means the existence of $\mathcal{C}$-morphisms

$$\begin{array}{rcl} F_2 & : & !A \otimes !B \to !(A \otimes B) \\ F_0 & : & \mathbf{1} \to !\mathbf{1} \end{array}$$

natural in A and B. These must make a number of diagrams (not given here) involving the structural maps of $\mathcal{C}$ commute.

The requirement that the multiplexors m_n are moniodal natural transformations means that for each natural number n the following two diagrams must commute in $\mathcal{C}$

$$\begin{array}{ccc} !A \otimes !B & \xrightarrow{F_2} & !(A \otimes B) \\ {\scriptstyle m_n \otimes m_n}\downarrow & & \downarrow{\scriptstyle m_n} \\ A^n \otimes B^n & \xrightarrow{\sim} & (A \otimes B)^n \end{array} \qquad \begin{array}{ccc} \mathbf{1} & \xrightarrow{F_0} & !\mathbf{1} \\ \| & & \downarrow{\scriptstyle m_n} \\ \mathbf{1} & \xrightarrow{\sim} & \mathbf{1}^n \end{array}$$

We have the following theorem.

Theorem. *A multiplexor category provides a denotational semantics for SLL. In other words, every proof in SLL has an interpretation as an arrow in the model category, and moreover, the interpretation is preserved under the cut elimination procedure.*

Proof. Given a formula A in SLL and an assignment of atoms to objects in a model category $\mathcal{C}$, the interpretation of A is built up in the usual way. Moreover, the interpretation of a nonempty finite list of formulas $\Gamma = A_1, \ldots, A_k$ is defined to be the interpretation of the formula $l(\Gamma) = (\cdots(A_1 \otimes A_2) \otimes A_3) \cdots) \otimes A_k)$, with all pairs of parentheses starting in front; if Γ is empty, its interpretation is the tensor unit $\mathbf{1}$.

For clarity, we sometimes use the same notation for a formula and its interpretation.

The interpretation of a proof of a sequent $\Gamma \vdash C$ will be an arrow $l(\Gamma) \to C$ in $\mathcal{C}$, built inductively starting with identity arrows for the axioms. For example, we interpret the multiplexing rule with the composite arrow

$$l(\Gamma, !A) \xrightarrow{\Gamma \otimes m_n} l(\Gamma, A^n) \xrightarrow{\sim} l(\Gamma, A^{(n)}) \xrightarrow{f} C$$

where the second arrow is the canonical isomorphism, and f is given by the induction hypothesis. For the soft promotion rule we take the following composite arrow

$$l(!\Gamma) \xrightarrow{F_v} !l(\Gamma) \xrightarrow{!f} A$$

where $f : l(\Gamma) \to A$ is given by the induction hypothesis and where F_v is either F_0 or a repeated use of F_2. The other cases are similar. In this way we construct an arrow corresponding to the proof.

This proves the first part of the theorem; namely, that every proof in SLL has a denotation.

To prove the second part of the theorem, that this provides a denotational semantics for SLL, we consider the cut elimination theorem for SLL.

For each case, we must check that the interpretations of the proof before and after the reduction step are the same. For example, consider the following (principal) reduction step involving the exponential rules.

Soft promotion versus *multiplexing* :

$$\dfrac{\dfrac{\Gamma \vdash A}{!\Gamma \vdash !A} \qquad \dfrac{\Delta, A^{(n)} \vdash C}{\Delta, !A \vdash C}}{!\Gamma, \Delta \vdash C}$$

$$\Downarrow$$

$$\dfrac{\dfrac{\Gamma \vdash A \ \cdots \ \Gamma \vdash A \qquad \Delta, A^{(n)} \vdash C}{\Gamma^{(n)}, \Delta \vdash C}}{!\Gamma, \Delta \vdash C}$$

Suppose $\Gamma = C, D$ and $\Delta = \emptyset$. Let $f : C \otimes D \to A$ and $g : A^n \to B$. The top path is the interpretation before reduction, and the bottom path is the interpretation of the proof after the reduction step.

$$
\begin{array}{ccccc}
!C \otimes !D & \xrightarrow{F_2} & !(C \otimes D) & \xrightarrow{!f} & !A \\
\downarrow{\scriptstyle m_n \otimes m_n} & & \downarrow{\scriptstyle m_n} & & \downarrow{\scriptstyle m_n} \\
C^n \otimes D^n & \xrightarrow{\sim} & (C \otimes D)^n & \xrightarrow{f^n} & A^n \\
 & & & & \downarrow{\scriptstyle g} \\
 & & & & B
\end{array}
$$

The first square commutes because m_n is a monoidal natural transformation, and the second by naturality. Therefore the outer paths are the same, as desired.

The other cases are proved in a similar fashion. $\square$

Example 1

A *Linear category* in the sense of Bierman et al. These categories are models of intuitionistic multiplicative exponential linear logic (IMELL). In this case we define the multiplexor of rank $n \geq 2$ by the composite

$$!A \to (!A)^n \to A^n$$

where the first arrow is (repeated) contraction and the second is the n-fold tensor product of dereliction.

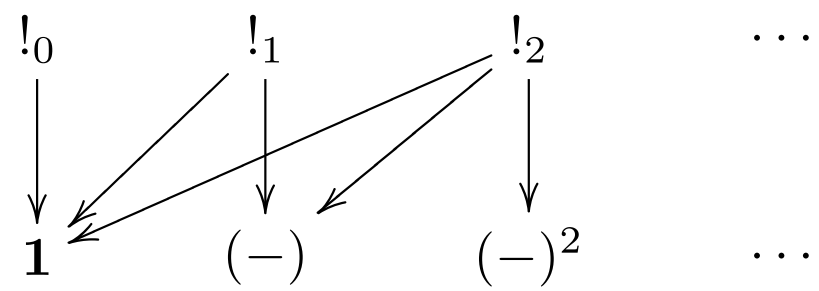
When $n = 0$ we use weakening $!A \to \mathbf{1}$.

When $n = 1$ we use dereliction $!A \to A$.

However we would like models which are specific to SLL; i.e, multiplexor categories which do not model contraction or digging. These models will not be models of IMELL.

We have the following useful theorem.

Theorem. *Suppose we have a SMCC $\mathcal{C}$ and the following diagram of symmetric monoidal endofunctors on $\mathcal{C}$ and monoidal natural transformations:*



Then if $\mathcal{C}$ has (at least) countable limits and ! denotes the limit of the above diagram, then $(\mathcal{C}, !)$ is a multiplexor category.

Proof. (Sketch.) The arbitrary small limit (if it exists) of symmetric monoidal endofunctors on $\mathcal{C}$ is again a symmetric monoidal endofunctor on $\mathcal{C}$ with projections (multiplexors) monoidal natural transformations. □

Examples 2 & 3

Let $\mathcal{C}$ be a SMCC with finite products and define:

$$!_n A = \mathbf{1} \;\&\; A \;\&\; A^2 \;\&\; \cdots \;\&\; A^n$$

for each natural number n. In this case the arrows in the above diagram are the obvious projections. If moreover, $\mathcal{C}$ has countable limits, we get the following formula for the soft exponential:

$$!A = \mathbf{1} \;\&\; A \;\&\; A^2 \;\&\; \cdots$$

Another possibility is to define:

$$!_n A = (A \;\&\; \mathbf{1})^n$$

for each natural number n. Again the arrows in the above diagram are the obvious ones. This interpretation was used by Lafont to translate proofs in SLL to proofs in IMALL (compatible with reduction).

Example 4

For this example we need the following definition due to Hofmann.

An *affine linear category* (ALC) is given by the following data:

-a category $\mathcal{C}$,

-for any objects $A, B \in \mathcal{C}$ an object $A \otimes B$, called tensor product, and morphisms $\pi : A \otimes B \to A$ and $\pi' : A \otimes B \to B$, called projections, which are jointly monomorphic in the following sense. If $f, g : C \to A \otimes B$ and $\pi \circ f = \pi \circ g$ and $\pi' \circ f = \pi' \circ g$ then $f = g$,

-for any two maps $f : A \to A'$ and $g : B \to B'$ a map $f \otimes g : A \otimes B \to A' \otimes B'$ such that $\pi \circ (f \otimes g) = f \circ \pi$ and $\pi' \circ (f \otimes g) = g \circ \pi'$,

-an isomorphism $\alpha : A\otimes(B\otimes C) \to (A\otimes B)\otimes C$ such that $\pi \circ \pi \circ \alpha = \pi$, $\pi' \circ \pi \circ \alpha = \pi \circ \pi'$, $\pi' \circ \alpha = \pi' \circ \pi'$,

-an isomorphism $\gamma : A \otimes B \to B \otimes A$ such that $\pi \circ \gamma = \pi'$ and $\pi' \circ \gamma = \pi$,

-a terminal object $\top$ such that $\pi : A \otimes \top \to A$ and $\pi' : \top \otimes A \to A$ are isomorphisms.

If we further require the tensor product functor $- \otimes B$ to have a specified right adjoint $B \multimap -$ we get what is called an *affine linear closed category* (ALCC). Note that an ALCC is a special case of a SMCC.

Let $\mathcal{C}$ be an ALCC. For each natural number n, define

$$!_n A = A^n$$

Once again the arrows in the above diagram are the obvious ones and are monoidal natural transformations.

Remarks

In each of the above examples, we may alternatively define ! to be the inverse limit (if it exists) of the endofunctors $!_0, !_1, !_2, \ldots$ with the obvious connecting monoidal natural transformations:

$$!_0 \leftarrow !_1 \leftarrow !_2 \leftarrow \cdots$$

In some cases this leads to a "tighter" interpretation of the soft exponential operator.

In any case, we have seen many examples of multiplexor categories. But in each of these examples we relied on the fact that the model category had at least countable limits. What can we say if the category has only finite limits, for example?

First let us look at a concrete example to illustrate these ideas. Later, we shall see a general construction to lift such examples to full-fledged multiplexor categories.

Application: A Realizability Model for SLL

Here we follow Hofmann and Scott's work for BLL and obtain a similar (partial) semantics for SLL. We let Λ_a denote the set of closed affine lambda terms (also see last slide).

Definition 1: A *realizability set* A is a pair $(|A|, \Vdash_A)$ where $|A|$ is a set and $\Vdash_A \subseteq \Lambda_a \times |A|$ is a binary relation. We write $t \Vdash_A a$ for $(t, a) \in \Vdash_A$. (Infix notation.)

Definition 2: A *morphism* from A to B is a function $f : |A| \to |B|$ such that there exists a term $e \in \Lambda_a$ such that

$$t \Vdash_A a \quad \text{implies} \quad et \Vdash_B f(a)$$

for each $t \in \Lambda_a$ and $a \in |A|$. In this case we say that f is realized (or witnessed) by e and we write $A \xrightarrow{f}_e B$.

This forms a category, which we denote by $\mathcal{B}$.

Indeed, the identity morphism on $A = (|A|, \Vdash_A)$ is the identity function on $|A|$ realized by the term $I = \lambda u.u$. Given $A \xrightarrow{f}_{e_1} B$ and $B \xrightarrow{g}_{e_2} C$ then the function gf is witnessed by the term $Be_2e_1 = \lambda z.e_2(e_1z)$, where $B = \lambda xyz.x(yz)$. This defines composition in $\mathcal{B}$. It is clearly associative.

$\mathcal{B}$ has an affine linear structure.

Let $A = (|A|, \Vdash_A)$ and $B = (|B|, \Vdash_B)$ be two realizability sets in $\mathcal{B}$. Their tensor product is defined by $A \otimes B = (|A| \times |B|, \Vdash_{A\otimes B})$ where

$$t \Vdash_{A\otimes B} (a, b) \quad \text{iff} \quad t = Tt_1t_2 \;\text{ where }\; t_1 \Vdash_A a \;\text{ and }\; t_2 \Vdash_B b$$

and where $Tt_1t_2 = \lambda f.ft_1t_2$ and $T = \lambda xyf.fxy$.

The projections maps are defined by $\pi(a, b) = a$ and $\pi'(a, b) = b$, and are witnessed by the terms $P_1 = \lambda p.p(\lambda xy.x)$ and $P_2 = \lambda p.p(\lambda xy.y)$, respectively. It is easy to check that $P_1(Tt_1t_2) = t_1$ and $P_2(Tt_1t_2) = t_2$, as desired.

The terminal object is defined by $\top = (\{*\}, \Vdash_\top)$ with $\mathbf{tt} = \lambda xy.x \Vdash_\top *$. The unique map $A \to \top$ is realized by $\lambda x.\mathbf{tt}$.

Moreover, it can be checked that $\mathcal{B}$ is an ALCC.

Let $A = (|A|, \Vdash_A)$ and $B = (|B|, \Vdash_B)$ be two realizability sets. Then we define $A \multimap B = (|A| \Rightarrow |B|, \Vdash_{A \multimap B})$ where

$$e \Vdash_{A \multimap B} f \quad \text{iff} \quad \forall t, a.\ t \Vdash_A a \text{ implies } et \Vdash_B f(a)$$

Therefore $\mathcal{B}$ is an ALCC.

Exponentials ($n \geq 1$)

Let A be a realizability set. We define $!_n A = (|A|, \Vdash_{!_n A})$ where

$t \Vdash_{!_n A} a$ iff $t = \lambda f.ft_1t_2 \ldots t_n$ and $\forall i.\ t_i \Vdash_A a$

For each $2 \leq k \leq n$ we define a multiplexor by the map $m_k(a) = ((\cdots((a,a),a)\cdots),a)$, k times. This map can be realized by the following term.

$$\lambda p.p(\lambda x_1 \ldots x_n.T(\cdots(T(Tx_1x_2)x_3)\cdots)x_k)$$

For $k = 1$, the map $m_1(a) = a$ can be witnessed by $\lambda p.p(\lambda x_1 \ldots x_n.x_1)$.

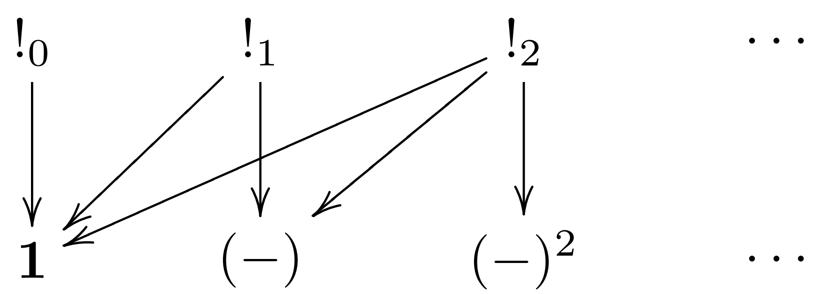
For $k = 0$, we may use the unique map to the terminal object $\top$.

Remarks

The realizability model we have just constructed may be used to give a *partial* interpretation of SLL. It is valid for proofs of rank $\leq n$, for some fixed natural number $n \geq 1$.

With a little more work, one can show that all numerical functions representable in SLL (with quantifiers) are polytime, the analog to Hofmann-Scott's result for BLL.

We have seen that our realizability model $\mathcal{B}$ is a SMCC together with the following diagram of symmetric monoidal endofunctors on $\mathcal{B}$ and monoidal natural transformations



We would like to extend to a model of SLL without taking the limit, which may not exist.

The completion *Lev*-$\mathcal{C}$

We define a "completion" of a SMCC $\mathcal{C}$, having only bounded multiplexing $!_0, !_1, !_2, \ldots$ as in the hypotheses of the theorem, to a multiplexor category *lev*-$\mathcal{C}$ as follows.

Objects: Infinite sequences of objects of $\mathcal{C}$ $(A_i) = (A_0, A_1, A_2, \ldots)$. Notation: If all the terms are the same we write (A) for the sequence $(A, A, A, \ldots)$.

Arrows: An arrow from (A_i) to (B_i) is a pair $(k; f_i) = (k; f_k, f_{k+1}, \ldots)$, where k is a natural number and for each $i \geq k$ we have $f_i : A_i \to B_i$ in $\mathcal{C}$. Notation: If all the terms are the same we may write $(k; f)$ for $(k; f, f, f, \ldots)$.

For example, the following picture represents the morphism $(2; f_i)$ from (A_i) to (B_i) in *lev*-$\mathcal{C}$.

$$\begin{array}{ccccc} A_0 & A_1 & A_2 & A_3 & \cdots \\ & & \downarrow{\scriptstyle f_2} & \downarrow{\scriptstyle f_3} & \\ B_0 & B_1 & B_2 & B_3 & \cdots \end{array}$$

Composition: Let $(k; f_i) : (A_i) \to (B_i)$ and $(m; g_i) : (B_i) \to (C_i)$ be two arrows in *lev*-$\mathcal{C}$. We define their composite to be the arrow $(\max(k, m); g_i f_i)$. Associativity follows from that in $\mathcal{C}$. The identity arrow on (A_i) is $(0; id_{A_i})$. Therefore *lev*-$\mathcal{C}$ is in fact a category.

The symmetric monoidal structure on $\mathcal{C}$ lifts to *lev*-$\mathcal{C}$ as follows. On objects (A_i) and (B_i) we define $(A_i) \otimes (B_i) = (A_i \otimes B_i)$, componentwise using the tensor product in $\mathcal{C}$. Similarly on arrows we define $(k; f_i) \otimes (m; g_i) = (\max(k, m); f_i \otimes g_i)$. The unit for the tensor is $(\mathbf{1})$. The structural maps in $\mathcal{C}$ lift in the obvious way to the following: $(0; \alpha)$, $(0; \rho)$, $(0; \lambda)$, and $(0; \gamma)$.

The closed structure extends similarly (componentwise) making *lev*-$\mathcal{C}$ into a SMCC, as desired.

The soft exponential

Let (A_i) and $(k; f_i)$ be an object and an arrow in *lev*-$\mathcal{C}$. We define:

$$\begin{aligned} !(A_i) &= (!_i A_i) \\ !(k; f_i) &= (k; !_i f_i) \end{aligned}$$

This defines a symmetric monoidal endofunctor on *lev*-$\mathcal{C}$.

But this is not the category we will ultimately be interested in. Indeed, we would like to equate arrows which "eventually" become the same.

We define an equivalence relation on the hom sets of *lev*-$\mathcal{C}$ as follows: Two maps $(k; f_i)$ and $(k'; f'_i)$ are equivalent iff there exists a natural number $m \geq k, k'$ such that for all $i \geq m$ we have $f_i = f'_i$ in $\mathcal{C}$. Moreover, the equivalence relation is preserved by composition. We denote the resulting quotient category by *Lev*-$\mathcal{C}$.

Notation: We denote the equivalence class of $(k; f_i)$ by $[(k; f_i)]$.

Then the following are well-defined for arrows $[(k; f_i)]$ and $[(m; g_i)]$ in *Lev*-$\mathcal{C}$.

$$\begin{aligned}
[(k; f_i)] \otimes [(m; g_i)] &= [(k; f_i) \otimes (m; g_i)] \\
[(k; f_i)] \multimap [(m; g_i)] &= [(k; f_i) \multimap (m; g_i)] \\
![(k; f_i)] &= [!(k; f_i)]
\end{aligned}$$

And for each (fixed) natural number n and object (A_i) of *Lev*-$\mathcal{C}$ we define:

$$m_n((A_i)) = [(n; p_{i,n}(A_i))] : \ (!_i A_i) \to (A_i^n)$$

where $p_{i,n}(A_i) : !_i A_i \to A_i^n$ in $\mathcal{C}$ for each natural numbers $i \geq n$. Note that we have the natural transformations $p_{i,n} : !_i \to (-)^n$, $i \geq n$, by assumption. This defines multiplexing.

Therefore *Lev*-$\mathcal{C}$ is a multiplexor category.

An embedding

There is a (strict monoidal) functor L from $\mathcal{C}$ to Lev-$\mathcal{C}$ defined by $L(A) = (A)$ and $L(f) = [(0; f)]$. This functor is faithful. Indeed, if $L(f) = L(g)$ then for some natural number it is the case that $f = g$. Therefore we have an embedding of $\mathcal{C}$ into the multiplexor category Lev-$\mathcal{C}$.

Again, this categorical construction allows one to lift any SMCC $\mathcal{C}$ having only bounded multiplexing $!_0, !_1, !_2, \ldots$, as in the hypotheses of the theorem, to a multiplexor category Lev-$\mathcal{C}$, regardless of whether $\mathcal{C}$ has countable limits.

It resembles in some ways Baillot's construction of the category of Stratified Coherence Spaces for LLL. (Consider the full subcategory of *stationary* sequences in Lev-$\mathcal{C}$.)

It is not a free extension, and therefore not optimal. But it does have a good representation, as we have just seen.

Future work

We would like to include in this general framework models of Intuitionistic Light Affine Logic (ILAL). For example, we would like to have an endofunctor § on *Lev*-$\mathcal{C}$ which models the following exponential rule of ILAL.

$$\frac{\Gamma, \Delta \vdash A}{\S\Gamma, !\Delta \vdash \S A}$$

Following Baillot we define a "shift" operator

$$\begin{aligned} \S((A_0, A_i, A_2, \dots)) &= (\mathbf{1}, A_0, A_1, A_2, \dots) \\ \S([(k; f_i)]) &= [(k+1; f_{i-1})] \end{aligned}$$

This is well-defined and in fact gives a (strong) symmetric monoidal endofunctor on *Lev*-$\mathcal{C}$, as can easily be checked.

The point is that there is not, in general, an arrow $\S A \to A$ for arbitrary object A in *Lev*-$\mathcal{C}$.

This motivates us to define an operator by the composite $!' = !\S$. Then we have an arrow $!'A \to (\S(A))^n$ for any natural number n (including zero), but we do not in general have dereliction $!'A \to A$ in *Lev*-$\mathcal{C}$, which is not a rule of ILAL.

The above are just some preliminary ideas, but we continue to investigate the relationship between the categorical construction *Lev*-$\mathcal{C}$ for SLL and Baillot's category of Stratified Coherence Spaces for LLL.

Lafont also considers an extension of SLL with a hierarchy of modalities $!_0, !_1, !_2, \ldots$ and a hierarchical version of *digging* :

$$\frac{\Gamma, !_n \cdots !_n A \vdash C}{\Gamma, !_{n+1} A \vdash C}$$

This leads to a system for elementary recursive computation, as in *Elementary Linear Logic.* Our models with countable limits can also model this extension.

A two level system (and beyond):

Along this direction, consider the category *Lev*-$\mathcal{C}$, where the base category is now a multiplexor category $(\mathcal{C}, !)$. Then ! extends to *Lev*-$\mathcal{C}$ in the obvious way.

$$
\begin{aligned}
!_u(A_i) &= (!A_0, !A_1, !A_2, \dots) \\
!_u[(n; f_i)] &= [(n; !f_i)]
\end{aligned}
$$

We define a new modality $!_t$ on *Lev*-$\mathcal{C}$ as follows.

$$
\begin{aligned}
!_t(A_i) &= (!A_0, !!A_1, !!!A_2, \dots) \\
!_t[(n; f_i)] &= [(n; !^{i+1} f_i)]
\end{aligned}
$$

where $!^{i+1}$ means $! \cdots !$, $i+1$ times. Then we have a morphism $!_t A \to !_u^{i+1} A$ for each $i \geq 0$. In this way we model a two level version of hierarchical digging.

Now it is easy to see how to extend to higher levels. E.g. consider the following definition:

$$
!_s(A_i) = (!^{(i+1)^2} A_i), \qquad \text{etc.}
$$

But no contraction at any level!

Finally, along another direction of research, we are working towards *fully complete* models of SLL. These are models in which any arrow $f : A \to B$ is the denotation of a proof of $A \vdash B$ in SLL. This is clearly a much stronger property than completeness. For example, recall one of our interpretations for the soft exponential in a SMCC $\mathcal{C}$ with countable products:

$$!A = \mathbf{1} \mathbin{\&} A \mathbin{\&} A^2 \mathbin{\&} \cdots$$

Then $(\mathcal{C}, !)$ is not complete even for provability: $\mathcal{C}(!a, !(a \otimes a)) \neq \emptyset$ but $!a \vdash !(a \otimes a)$ is not provable in SLL for any atom a. On the other side, one may claim that this interpretation is not faithful since certain (distinct) cut-free proofs are semantically equal under this interpretation. We have obtained some interesting partial results along this direction, but these will be reported elsewhere.

Finally, it may be instructive and interesting to develop a game semantics for SLL.

Affine lambda terms

Definition. An untyped lambda term is *affine linear* if each variable (free of bound) appears at most once (up to α-congruence).

For example, the following are affine linear terms:

$$\lambda x.x \qquad \lambda xy.x \qquad \lambda x.xy$$

But the term $\lambda x.xx$ is not.

An important property of such a term t is that it is strongly normalisable in the less than $|t|$ steps where $|t|$ is the size of the term. Therefore the runtime of the computation leading to the normal form is $O(|t|^2)$. We use the expression *affine lambda term* for an untyped affine linear lambda term which is in normal form. If s and t are affine lambda terms we define their application st as the normal form of the lambda term st. Note that application st can be computed in time $O((|s|+|t|)^2)$.

Finally, we use Λ_a to denote the set of all closed affine lambda terms.