Obsessional cliques: a semantic characterization of bounded time complexity

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LLL and denotational semantics

Girard (Light Linear Logic 1998): "...unfortunately this does not address the question of a denotational semantics specific to LLL which is presumably the deepest question connected with our new system..."

One looks for models of light systems which *are not* models of LL. Up to now, only two proposals:

1) Murawski-Ong (CSL00) "Discreet Games, Light Affine Logic and PTIME Computation": Full complete game model for IMAL (Intuitionistic Multiplicative fragment of Light Affine Logic)

2) Baillot (TCS04) "Stratified coherent spaces: a denotational semantics for LLL"

Both works are similar in spirit: the structures (games, coherent spaces) associated with logical formulas are modified, so that the principles (formulas) valid in LL but not in the chosen light system do not hold in the semantics.

The "relative completeness" approach

Danger of the previous approach: Beware the syntax-driven semantic analysis!

Another possibility: instead of modifying the structures associated with logical formulas, we look for a property of the elements of the structures (the interpretations of proofs) characterizing those elements which *can* interpret polytime proofs. More than a model: *a relatively complete model*.

$$\begin{array}{ccc} LL & \xrightarrow{[\![\ ]\!]} & LL - model \\ \downarrow & & \downarrow \\ LLL & \xrightarrow[{[\![\ ]\!]}]{} & LLL - model \end{array}$$

Remark: No full completeness/surjectivity nor faithfulness/injectivity is required to the model.

The idea: choose a light system and a (natural) denotational model of LL, and prove that an LL proof π is a proof of the chosen light system iff the interpretation $[\![\pi]\!]$ of π satisfies some given (semantic) property.

The result: an LL-proof π is a proof of ELL (resp. SLL) iff $[\![\pi]\!]$ is 0-obsessional (resp. t-obsessional from some t).

Denotational semantics of polytime

- I don't know what it is (difficult and interesting question)

- If S is a logical system such that f is polytime iff f is provably total in S, a denotational semantics of S *might* tell us something on polytime (depeding on the quality of S: how are the programs extracted from the totality proofs of S?)

- A possible method to find models is to produce relatively complete models (which should bring more informations on the abstract nature of polytime). This consists in the following steps:

  - take a very wide system Σ and find a syntactic property (P_{syn}) such that f is polytime iff there exists a totality proof of f in Σ satisfying (P_{syn})

  - consider the denotational semantics of Σ and find a semantic property (P_{sem}) such that f is polytime iff there exists a totality proof π_f of f in Σ satisfying (P_{syn}) iff $[\![\pi_f]\!]$ satisfies (P_{sem}).

Light systems: standard presentaton

$$\frac{}{\vdash A, A^{\perp}}\,ax \qquad \frac{\vdash \Gamma, A \quad \vdash \Delta, A^{\perp}}{\vdash \Gamma, \Delta}\,cut$$

$$\frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\otimes \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \wp B}\wp \qquad \frac{}{\vdash 1}\,1 \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \perp}\perp$$

$$\frac{\vdash \Gamma, A}{\vdash ?\Gamma, !A}\,!f \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\,?c \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?A}\,?w$$

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, \forall X\, A}\forall \quad (X \notin \Gamma) \qquad \frac{\vdash \Gamma, A[^{B}/_{X}]}{\vdash \Gamma, \exists X\, A}\exists$$

(Additive-free second order) ELL **rules.**

Certain principles are not provable: $!A \multimap A$, $!A \multimap !!A$.

$$\frac{}{\vdash A, A^{\perp}}\, ax \qquad \frac{\vdash \Gamma, A \quad \vdash \Delta, A^{\perp}}{\vdash \Gamma, \Delta}\, cut$$

$$\frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\otimes \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \wp B}\wp \qquad \frac{}{\vdash 1}\,1 \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \perp}\perp$$

$$\frac{\vdash \Gamma, A}{\vdash ?\Gamma, !A}\,!f \qquad \frac{\vdash \Gamma, A, \ldots, A}{\vdash \Gamma, ?A}\,?m$$

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, \forall X A}\forall \quad (X \notin \Gamma) \qquad \frac{\vdash \Gamma, A[{}^{B}/_{X}]}{\vdash \Gamma, \exists X A}\exists$$

(Additive-free second order) SLL **rules.**

Certain principles are not provable: $!A \multimap !A \otimes !A$, $!A \multimap !!A$.

Light systems seen from within

Implicit Computational Complexity: "complexity seen from within", find abstract properties allowing to characterize a complexity class

Danos-Joinet03: characterization of ELL within LL

Mazza02: characterization of LLL "within LL" (with paragraph)

Lafont04: SLL can be seen as a subsystem of LL

Using Danos-Joinet03 and Lafont04 one can characterize ELL and SLL in terms of *exponential trees* of LL proof-nets (and not only by "refuting" some principles as one usually does).
Much more satisfactory: the characterization tells us something on the very nature of ELL/SLL and *might* say something on the very nature of elementary/polynomial time.

Exponential trees (bundle of branches) in $MELL$

- a net is in ELL if its exponential bundles of branches are all of the shape: only branches of length 2.

- a net is in SLL if its exponential bundles of branches are all of one of the following two shapes:
  either with only branches of length 1 or the bundle containing exactly one branch of length 2.

Semantics of LL

Girard87: introduces the notion of **experiment** of a proof-net, a (multi)-labeling of the edges of the graph associating with a conclusion $a : A$ (an edge a of type A) an element $\alpha \in |A|$. If R is a proof-net with conclusion $a_1 : A_1, \ldots, a_n : A_n$ the labels $\alpha_1 \in |A_1|, \ldots, \alpha_n \in |A_n|$ constitute *the result* $(\alpha_1, \ldots, \alpha_n)$ of the experiment. $[\![R]\!] = \{\alpha$:there exists an experiment of R with result $\alpha\}$.

This "is" Linear Logic! (The interpretation of a proof is both a function and a set of points). So the of notion of experiment is a crucial one.

Obsessional experiments

In TortoradeFalco03: the notion of **obsessional experiment** is introduced and used to "rebuild" an LL proof from its interpretation in (relational or coherent) semantics. In general this is not possible: it is shown that there are different LL proofs with the same semantics.

However, the relational (resp. coherent) interpretation of a proof (and more precisely the result of an obsessional experiment of the proof) is enough to rebuild *every exponential tree*. More precisely, if e is an obsessional experiment "of degree k" of the proof-net R, and if $k \geq$ *cosize*(R) where *cosize*(R) is the maximal arity (the number of branches) of the exponential trees of R, then e is enough to determine all R's exponential trees.

This means that given $x \in \mathcal{A}$ and knowing that x is the interpretation of some LL proof π with conclusion A, by only looking at x we can find out whether or not π is an ELL (resp. SLL) proof, *provided* x *contains the results of obsessional experiments of "sufficiently big degree"*

Obsessional cliques

In order for the semantics to determine whether a given proof π is an ELL (resp. SLL) proof, it is enough that the results of the obsessional experiments of any degree are available.

A $\mathbb{N}$-*set* is given by a set A and a function $(k,a) \mapsto a^{(k)}$ from $\mathbb{N}^* \times A$ to A, called the *action*, which is an action of the monoid $(\mathbb{N}^*, \cdot, 1)$ on A (that is $a^{(1)} = a$ and $a^{(kk')} = (a^{(k)})^{(k')}$).

The constructions of $\mathbb{N}$-sets are obtained from the corresponding constructions of sets and the actions are built in the following way:
- on $1 = \{\star\}$, we use the only possible action
- the action on $A \times B$ is given by $(a,b)^{(k)} = (a^{(k)}, b^{(k)})$
- if $t \in \mathbb{N}$, we define $!_t A$ (resp. $?_t A$) as the $\mathbb{N}$-set with underlying set $\mathcal{M}_f(A)$ and the action on $!_t A$ (resp. $?_t A$) is given by:

$$[a_1, \ldots, a_n]^{(k)} = \begin{cases} [a_1^{(k)}, \ldots, a_n^{(k)}] & \text{if } n \leq t \\ [ka_1^{(k)}, \ldots, ka_n^{(k)}] & \text{if } n > t \end{cases}$$

In the spirit of coherent spaces, we call *cliques* of A the subsets of A. Let A be a $\mathbb{N}$-set, a clique c is *obsessional* if $\forall a \in c, \forall k \in \mathbb{N}^*, a^{(k)} \in c$. An obsessional clique c is *t-obsessional* if for every formula of type "$!A$" we choose $!_t A$.

Typed relatively complete models

The category $\mathcal{OREL}$ is given by:

- objects: $\mathbb{N}$-sets
- morphisms: $\mathcal{OREL}(A, B)$ is the set of obsessional cliques of $A \times B$

The category $\mathcal{SREL}$ is given by:

- objects: a set A with a family of actions $(A_n)_{n \in \mathbb{N}}$
- morphisms: $\mathcal{SREL}((A_n)_{n \in \mathbb{N}}, (B_n)_{n \in \mathbb{N}})$ is the set of cliques of $A \times B$ such that there exists some $t \in \mathbb{N}$ with for any $n \geq t$, x is an obsessional clique of $A_n \times B_n$.

Theorem: $\mathcal{OREL}$ is a model of propositional (additive-free) ELL.
$\mathcal{SREL}$ is a model of propositional (additive-free) SLL.

Relative completeness theorem: the $MELL$-proof π is an ELL (resp. SLL) proof iff $[\![\pi]\!]$ is a 0-obsessional clique (resp. a t-obsessional clique from some t).

Remark: No explicit trace of stratification! (One step further in the abstract characterization of ELL/SLL and hopefully of elementary/polynomial time).

Examples: 1) The natural clique for contraction is not t-obsessional for $t > 0$ in $!_t A \times (!_t A \times !_t A)$: we have $([2ta], ([ta], [ta]))^{(2)} = ([4ta^{(2)}], ([ta^{(2)}], [ta^{(2)}])))$
2) The natural clique for digging is not obsessional for any t in $!_t A \times !_t !_t A$: we have $([(t+1)^2 a], [(t+1)[(t+1)a]])^{(2)} = ([2(t+1)^2 a^{(2)}], [2(t+1)[2(t+1)a^{(2)}]]))$
3) The natural clique for dereliction is not obsessional in $!_0 A \times A$: we have $([a], a)^{(2)} = ([a^{(2)}, a^{(2)}], a^{(2)})$. Dereliction is obsessional in $!_t A \times A$ for any $t \geq 1$.

Untyped obsessional cliques

Problem: To represent all polynomial predicates in SLL, the second order quantification is needed (not the additives, see Mairson-Terui03)

Solution: Move from a typed to an untyped framework. Remember that the normalization bound does not depend on types (one of the main points in Girard's original work: the complexity of cut-elimination does not depend on cut-formulas, but only on the geometric structure of proofs).

1) Extend Danos-Regnier's notion of untyped net: $uELL$, $uSLL$
2) Define a space for untyped obsessionality: We define *points* by the following grammar:

$$x ::= 1 \mid \bot \mid x \otimes x \mid x \wp x \mid !\mu \mid ?\mu$$
$$\mu ::= [x_1, \ldots, x_n]$$

where $[x_1, \ldots, x_n]$ denotes a finite multiset of points. D is the set of all points.

We can also define $D_0 = \{1, \bot\}$, $D_{n+1} = D_n \cup \{(\otimes, x, y) \mid (x, y) \in D_n^2\} \cup \{(\wp, x, y) \mid (x, y) \in D_n^2\} \cup \{(!, \mu) \mid \mu \in \mathcal{M}_f(D_n)\} \cup \{(?, \mu) \mid \mu \in \mathcal{M}_f(D_n)\}$, and $D = \cup_{n=0}^{\infty} D_n$. A *clique* is a subset of D.

We define the dual $\overline{x}$ of the point x by:

$$\overline{1} = \bot \qquad \overline{\bot} = 1$$
$$\overline{x \otimes y} = \overline{x} \wp \overline{y} \qquad \overline{x \wp y} = \overline{x} \otimes \overline{y}$$
$$\overline{![x_1, \ldots, x_n]} = ?[\overline{x_1}, \ldots, \overline{x_n}]$$
$$\overline{?[x_1, \ldots, x_n]} = ![\overline{x_1}, \ldots, \overline{x_n}]$$

Untyped relatively complete models

Let $t \geq 0$ be a natural number, the *t-action* on D is the function $(k, x) \mapsto (x)_t^{(k)}$ from $\mathbb{N}^* \times D$ to D given by:

$(1)_t^{(k)} = 1 \qquad (\perp)_t^{(k)} = \perp$

$(x \otimes y)_t^{(k)} = (x)_t^{(k)} \otimes (y)_t^{(k)}$

$(x \wp y)_t^{(k)} = (x)_t^{(k)} \wp (y)_t^{(k)}$

$$(![x_1, \ldots, x_n])_t^{(k)} = \begin{cases} ![(x_1)_t^{(k)}, \ldots, (x_n)_t^{(k)}] & \text{if } n \leq t \\ ![k(x_1)_t^{(k)}, \ldots, k(x_n)_t^{(k)}] & \text{if } n > t \end{cases}$$

$$(?[x_1, \ldots, x_n])_t^{(k)} = \begin{cases} ?[(x_1)_t^{(k)}, \ldots, (x_n)_t^{(k)}] & \text{if } n \leq t \\ ?[k(x_1)_t^{(k)}, \ldots, k(x_n)_t^{(k)}] & \text{if } n > t \end{cases}$$

A clique c is *t-obsessional* if $\forall x \in c$, $\forall k \in \mathbb{N}^*$, $(x)_t^{(k)} \in c$. A clique c is *obsessional from* t if for any $t' \geq t$, c is t'-obsessional.

Theorem: 0-obsessional cliques in D are a model of $uELL$.
Cliques obsessional from some t are a model of $uSLL$.
Relative completeness theorem: An untyped $MELL$-net is a $uELL$-net (resp. a $uSLL$-net) iff $[\![R]\!]$ is 0-obsessional (resp. obsessional from some t).

Comments

- extension to the additives: no problem for ELL, more delicate for SLL

- we worked with relational semantics, because coherence is not needed. Everything extends straightforwardly to (multiset based) coherent semantics, defining the various constructions applied to the coherence relation.

- two difficulties for LLL:

  - relative completeness (as stated here) for the original LLL would fail for coherent semantics (for "bad reasons", one would be tempted to **change the syntax of LLL**...)

  - exponential trees do not characterize LLL proofs: the LLL constraint "at most one auxiliary door" is not easily captured by the semantics

- question: what about the "computational complexity" of obsessional cliques which *are not* interpretations of proofs?