Stratified coherent spaces:
a denotational semantics for Light Linear Logic
(extended abstract)

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Abstract

We introduce a *stratified* version of the coherent spaces model where an object is given by a sequence of coherent spaces. The intuition behind it is that each level gives a different degree of precision on the computation, an *appearance*. A morphism is required to satisfy a coherence condition at each level and this setting gives a model of Elementary Linear Logic. We then introduce a measure function on the web meant to describe the difference between the number of output and input requests in the computation. The *locally bounded* morphisms defined thanks to this measure give a subcategory which is a model of Light Linear Logic.

1 Introduction

Linear logic and time complexity. Linear logic can be seen as a typing system where one provides information about the re-usability of subprograms. This is achieved through the use of modalities called *exponentials*, distinguishing hypothesis that can be used an arbitrary number of times from hypothesis that have to be used exactly once. It follows that the exponentials typing rules constrain the information flow during execution of typed programs. Hence tame exponentials can cut down the set of typable programs to moderate time complexity classes. Indeed in [Gir95b] Girard introduced such a variant of LL called *Light Linear Logic* (LLL) which captures the class of polytime functions. A polynomial bound was given on the number of steps of the cut-elimination procedure for LLL proofs (the degree of the polynomial depending on a certain parameter of the proof called the *depth*). Conversely, any polytime function on integers was shown to be representable by a LLL proof. A simplification of this system in an affine setting (Light Affine Logic) was done by Asperti in [Asp98] and adaptations of the system to the frameworks of lambda-calculus and combinatory logic have been studied (see [Rov98] and [Abr99]).

Semantics. Still, though the motivation lying behind the restrictions on exponential rules shows quite clearly when one examines the normalization procedure, it is not so obvious to get some *positive* intuition on these new exponentials. Light linear logic is not born with a denotational semantics as its elder sister... The supply of models could facilitate the understanding of the system, give clues to fix some points where several options are possible (e.g. auto-duality of the modality §, provability of **!1**, adding of full weakening as in Asperti's LAL), possibly help to write representations of concrete functions.

A first step was done by Kanovitch *et al* who gave in [KOS97] a semantics of *provability* through phase spaces, establishing a completeness result similar to that for LL ([Gir87]). But this approach doesn't provide information on the proofs themselves. An alternative idea was to look for geometry of interaction models with intrinsic complexity bound

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on their dynamics. Such a work was undertaken in [BP99] for Elementary Linear Logic, another variant of LL corresponding to Kalmar elementary functions (see [Gir95b] and [DJ99]).

Our approach. Here we address the problem of introducing denotational semantics for LLL. But let us first make clear what we are looking for. Indeed LLL can be encoded in LL in a natural way (compatible with cut-elimination), which is no surprise since it is a refinement of LL. Therefore any model of LL yields a model of LLL. So in fact we would like a model *specific* of LLL in the sense that it should not satisfy the principles excluded by this system (for instance the dereliction principle).

Our starting point is inspired by a semantics introduced by Martin Hofmann in [Hof99] where a size is associated to values and functions are required not to increase the size. We will instead consider sizes of *computations* defined from the number of *threads* of subcomputation opened. Morphisms will be asked to offer a fixed bound on the difference between their number of output and input requests. This idea can be illustrated informally through a games semantics analogy: the AJM-like strategy ([AJM94]) associated to the canonical proof of $!A \otimes !B \multimap !(A \otimes B)$ splits each thread of computation on the right-hand-side (output) into two threads of computation on the left-hand-side (input), so here the previously mentioned difference is equal to the size of the output and therefore is not bounded.

We will develop this idea in the context of a time-free semantics derived from coherent spaces. Actually the bounding condition will have to hold for each level of nesting of subcomputations. It then turns out that to obtain a compositional model we need to require a stronger coherence condition on morphisms, which leads us to stratified spaces and stratified cliques. Surprisingly, the stratified model (without the bounding condition) is interesting by itself as it provides a specific model of Elementary Linear Logic (ELL). It bears several similarities with the fibred phase model of [KOS97].

Recently Murawski and Ong have also proposed a games semantics for Light Affine Logic yielding a full completeness result ([MO00]). It should be interesting to try to relate their approach to the present work.

Outline of the paper. In section 2 we give preliminaries on LLL, ELL and coherent spaces, then we present the stratified coherent spaces in section 3 and show that it modelizes ELL. In 4 we define the measured spaces and the subcategory of locally bounded stratified cliques which is our model of LLL.

2 Preliminaries

2.1 Light linear logic

LLL arises from restrictions on the exponential connectives of linear logic. They still enjoy the contraction $!A \vdash !A \otimes !A$ and weakening $!A \vdash 1$ principles, but:

- the *dereliction* $!A \vdash A$ and digging $!A \vdash !!A$ principles are not accepted,
- the ! is functorial, t.i.t.s. the rule (from $A \vdash B$ deduce $!A \vdash !B$) is accepted; it is not multi-functorial though and the principle $!A \otimes !B \vdash !(A \otimes B)$ is not allowed;
- an important point is that the equivalence between $!(A \& B)$ and $!A \otimes !B$ is maintained.

To compensate for the lack of dereliction a new modality § is introduced, with principles $!A \vdash \S A$ and $\S A \otimes \S B \vdash \S(A \otimes B)$. Here we will not consider § to be self-dual and we will denote its dual by $\overline{\S}$.

Girard showed how these principles could be organized into a sequent calculus ([Gir95a]) which we recall in appendix A. Proof-nets were also introduced; they offer a more convenient syntax to describe the cut-elimination procedure.

The *depth* of a proof-net is the maximal nesting of its exponential boxes. In [Gir95a] it was shown that given a fixed depth d, a polynomial P of degree depending on d could be given such that the normalization of any proof-net R of depth d can be performed in less than $P(|R|)$ steps (where $|R|$ measures the size of R). Conversely, the representation theorem states that any polytime function on integers can be represented by a LLL proof of the sequent $1^k; bint \vdash \S^k bint$, where:

- the formula 1^k stands for $! \dots !1$ with ! repeated k times,

- *bint* is a type for integers in binary representation,

- $\S^k A$ stands for $\S \dots \S A$ with $\S$ repeated k times.

2.2 Elementary linear logic

The system ELL is obtained by considering the same restrictions as before but allowing the principle $!A \otimes !B \vdash !(A \otimes B)$. The ELL ! is multifunctorial. The modality $\S$ is not introduced in this system.

Danos and Joinet showed in [DJ99] how these requirements are fulfilled by a subsystem of plain LL proofs defined thanks to proof-net notions. They established the fact that the class of functions characterized is that of Kalmar elementary functions.

2.3 Coherent spaces

We briefly recall the main definitions on coherent spaces (see [Gir95b, Gir87] for a complete exposition).

A *coherent space* is a pair $(|X|, \coh_X)$ where $|X|$ is a countable set (the *web* of X, its elements are *points*) and $\coh_X$ is a binary reflexive and symmetric relation on $|X|$. Two elements of $|X|$ that are in this relation are said to be *coherent*. We will then write $x \coh_X y$ or $x \coh y \, mod \, X$. We write $x \frown_X y$ if $x \coh_X y$ and $x \neq y$ (*strict coherence*). The complement of the relation $\coh_X$ is the *strict incoherence* $\smile_X$ and its reflexive closure is the *incoherence* relation $\asymp_X$.

A *clique* of X is a subset c of $|X|$ whose elements are pairwise coherent. A *multiclique* is a multiset whose underlying set is a clique.

The constructions needed to interprete linear logic formulas are the following ones:

- *negation*: $X^\perp = (|X|, \asymp_X)$,

- *tensor*: $X \otimes Y = (|X| \times |Y|, \coh_{X \otimes Y})$ where:

$$(x, y) \coh (x', y') \, mod \, X \otimes Y \text{ if}$$

$$x \coh x' \, mod \, X \text{ and } y \coh y' \, mod \, Y.$$

- *with*: $X \,\&\, Y = (|X| + |Y|, \coh_{X \& Y})$ where for $x, x' \in |X|$, $y, y' \in |Y|$ we have

$$\begin{array}{lcl} inl(x) & \coh & inr(y) \, mod \, X \,\&\, Y, \\ inl(x) & \coh & inl(x') \, mod \, X \,\&\, Y \text{ if } x \coh x' \, mod \, X, \\ inr(y) & \coh & inr(y') \, mod \, X \,\&\, Y \text{ if } y \coh y' \, mod \, Y. \end{array}$$

Then we set: $X \wp Y = (X^\perp \otimes Y^\perp)^\perp$, $X \oplus Y = (X^\perp \,\&\, Y^\perp)^\perp$, $X \multimap Y = X^\perp \wp Y$.

It follows that: $(x, y) \frown (x', y') \, mod \, X \multimap Y$ iff ($x \coh_X x'$ implies $y \frown_Y y'$).

We are considering the multiset version of coherent spaces (it will be important in section 4): $|!_m X|$ is the set of finite multicliques on $|X|$ and $u \coh_{!_m X} v$ if $u + v$ is a multiclique.

If f is a clique on $X \multimap Y$, then $!_m f : !_m X \multimap !_m Y$ is defined by:

$$\begin{array}{rcl} !_m f & = & \{([x_1, \dots, x_n], [y_1, \dots, y_n]) \text{ s.t.} \\ & & \forall i, \, (x_i, y_i) \in f \text{ and } [x_1, \dots, x_n] \in |!_m X|\} \end{array}$$

3 Stratified coherent spaces

3.1 Definitions

We introduce now a stratified version of the coherence semantics. Think of it as a way of managing *partial information* on the points of the web, t.i.t.s. on values/computations. From such a point x we want to be able to retrieve successive approximations x^0, x^1, ... that we shall call *appearances*, providing us with increasing information about the computation. For each level of approximation a coherence relation is specified, so that the former compatibility/incompatibility single judgement on two computations is refined into a sequence of judgements, each of them corresponding to a particular level of approximation.

Definition 1 *A* stratified coherent space *(s.c.s.) X is given by a sequence $(X^i, \phi_i)_{i \in \mathbb{N}}$ where:*

- *each X^i is a coherent space $(|X^i|, \coh_{X^i})$,*

- *each ϕ_i is a partial application from $|X^{i+1}|$ to $|X^i|$,*

- *the sequence is stationary: there exists an integer d such that:*

$$\forall i \geq d, X^i = X^d \text{ and } \phi_i = id_{|X^d|}.$$

The least d such that the last condition is satisfied will be called the *depth* of the s.c.s. and denoted by $depth(X)$. Then X^d is called the *main space* of X and X^i is its *appearance at depth* i.

For $i \leq d$, the *i-th appearance map* is the partial application $app_i : |X^d| \rightharpoondown |X^i|$ given by:

$$app_i = \phi_i \circ \phi_{i+1} \circ \ldots \circ \phi_{d-1}.$$

We say that an element x of X^d is *visible* if $app_0(x)$ is defined. Whenever there is no risk of confusion, given x in $|X^i|$ and $j \leq i$ we will write simply x^j for $\phi_j \circ \phi_{j-1} \circ \ldots \phi_i(x)$. In particular if $i = d$ we have $x^j = app_j(x)$; we call it the *appearance of x at depth j*.

We extend app_i into a partial application from $\mathcal{P}(|X^d|)$ (resp. $\mathcal{M}(|X^d|)$) to $\mathcal{P}(|X^i|)$ (resp. $\mathcal{M}(|X^i|)$) in the natural way:

$app_i(\{x_1, \ldots, x_n\})$ (resp. $app_i([x_1, \ldots, x_n])$) is defined if for all $1 \leq j \leq n$ $app_i(x_j)$ is defined, in which case

$$\begin{array}{rcl} app_i(\{x_1, \ldots, x_n\}) & = & \{x_1^i, \ldots, x_n^i\} \\ \text{(resp. } app_i([x_1, \ldots, x_n]) & = & [x_1^i, \ldots, x_n^i]). \end{array}$$

Similarly for the ϕ_is.

We will tend to identify the stratified coherence space X with its main coherent space X^d, the reason for that being that when we consider interpretations of formulas a stratified coherent space is completely defined by its main space. In particular, $|X|$ will stand for $|X^d|$.

A convenient way to represent a s.c.s. X is as a forest $\mathcal{F}_X$ displayed by levels:

- the trees have height $(d+1)$, where d is the depth of X; nodes at level $0 \leq i \leq d$ are elements of $|X^i|$;
- there is an edge betwen two elements a and b if one (say a) belongs to a $|X^{i+1}|$, the other belongs to $|X^i|$ and we have $\phi_i^X(a) = b$;
- the roots of the trees are the elements of $|X^0|$.

Therefore the leaves of the trees are the elements of $|X^d|$. Note that there are pending (or here rather floating) branches: their leaves are precisely the non-visible elements. With this representation, finding the i-th appearance of an element means going down the tree (from leaf to root) until reaching level i. We will see in the sequel that logical constructions include infinite branchings.

If x at level $i+1$ is linked to y at level i we will say that x is an *immediate ascendant* of y. The

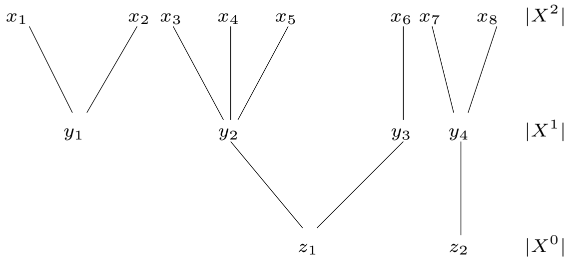

Figure 1: Example of a stratified coherent space as a forest

ascendance relation is the transitive closure of this relation.

In the example of figure 1, x_1 and x_2 are non-visible points. The point x_3 is visible and its appearances at depth 0 and 1 are respectively $x_3^0 = z_1$ and $x_3^1 = y_2$.

Definition 2 *Let X be a stratified coherent space of depth d and c be a subset of $|X^d|$. Then c is a* stratified clique *of X if all its points are visible and for all $i \leq d$, $app_i(c)$ is a clique of X^i.*

This definition can be rephrased in our tree terminology: a stratified clique is a subforest c of $\mathcal{F}_X$ whose trees have their root at level 0 (no floating branches!) and such that for each level i, the nodes of c at level i form a clique of X^i.

Coming back to the example of figure 1: assume $c = \{x_3, x_4, x_6, x_7\}$ is a clique of X^2; then c is a stratified clique of X if $\{y_2, y_3, y_4\}$ and $\{z_1, z_2\}$ are respectively cliques of X^1 and X^0.

3.2 Constructions on stratified coherent spaces

Let X, Y be two s.c.s.. For $\Box = \otimes, \&$ we define $X \Box Y$ by setting for all i's:

$$\begin{array}{rcl} (X \Box Y)^i & = & X^i \Box Y^i, \\ \phi_i^{X \otimes Y} & = & \phi_i^X \times \phi_i^Y \text{ if } \Box = \otimes, \\ \phi_i^{X \& Y} & = & \phi_i^X + \phi_i^Y \text{ if } \Box = \&. \end{array}$$

We define $X^\perp$ by setting for all i's:

$$(X^\perp)^i = (X^i)^\perp \text{ and } \phi_i^{X^\perp} = \phi_i^X.$$

Then $X \wp Y = (X^\perp \otimes Y^\perp)^\perp$ and $X \oplus Y = (X^\perp \& Y^\perp)^\perp$.

Finally $!X$ is given by:

- for $i \geq 0$, $(!X)^{i+1} = !_m(X^i)$, where the $!_m$ on the right-hand-side refers to the multiset bang construction on coherent spaces,
- $(!X)^0 = 1$ (the coherent space with singleton web $\{*\}$).

The applications $\phi_i^{!X} : |(!X)^{i+1}| \rightharpoonup |(!X)^i|$ are defined by:

- if $i \geq 1$, $\phi_i^{!X}([x_1, \ldots, x_n])$ is defined if for all $1 \leq k \leq n$, $\phi_{i-1}^X(x_k) = x'_k$ is defined and if $[x'_1, \ldots, x'_k]$ belongs to $|!(X^{i-1})|$,
- for $i = 0$, $\phi_0^{!X}$ is the constant function equal to $*$.

Note that we have: $depth(!X) = depth(X) + 1$.

The s.c.s. $?X$ is defined as expected by: $?X = (!(X^\perp))^\perp$.

These constructions on s.c.s. naturally give a notion of appearance on formulas; we only need to add that for an atomic formula α we set $\alpha^i = \alpha$ for any i.

Let us now consider an example. Let N be the stratified coherent space of depth 1 given by: $N^0 = 1$; N^1 is the set $\mathbb{N}$ equipped with the discrete coherence relation. We denote the elements of N^1 by $\underline{n}$. The choice of N being of depth 1 comes from the usual representation of the integers in LL.

It follows that $|!N^1| = \{k[\underline{n}], k, n \in \mathbb{N}\}$ where $k[\underline{n}]$ denotes $[\underline{n}, \ldots, \underline{n}]$ with $\underline{n}$ being repeated k times. We have:

$$k[\underline{n}] \frown l[\underline{m}] \, mod \, !N^1 \ \text{ iff } n = m \text{ or } k = 0 \text{ or } l = 0.$$

Let $F = !(!N \multimap N) \multimap N$. This s.c.s. has depth 3 and its appearances are given by:

$$\begin{array}{rclcll} F^3 & = & !(!N \multimap N) & \multimap & N & = \ F \\ F^2 & = & !(!1 \multimap N) & \multimap & N & \\ F^1 & = & !(1 \multimap 1) & \multimap & N & \\ F^0 & = & 1 & \multimap & 1 & \end{array}$$

Take x in $|F^3|$ given by $x = ([([\underline{0}], \underline{0}), ([\underline{2}], \underline{0})], \underline{1})$. It is visible and we have:

$$\begin{array}{rcl} x^2 & = & (\ [([*], \underline{0}), ([*], \underline{0})] \ , \ \underline{1} \ ) \\ x^1 & = & (\ [(*, *), (*, *)] \ , \ \underline{1} \ ) \\ x^0 & = & (\ * \ , \ * \ ) \end{array}$$

We now give an example of a non-visible point. Let $G = !N \multimap N$ and take on $|G^2|$ the points $y = ([\underline{0}], \underline{1})$ and $z = ([\underline{1}, \underline{1}], \underline{1})$. We have:

$$y \frown z \, mod \, G^2,$$

so $[y, z]$ belongs to $|(!G)^3|$. But besides:

$$([*], \underline{1}) \smile ([*, *], \underline{1}) \, mod \, G^1,$$

so $[y^1, z^1]$ is not a multiclique of G^1 and hence does not belong to $|(!G)^2|$. Therefore, $\phi_2^{!G}$ is not defined on $[y, z]$ and so $[y, z]$ is not a visible element of $!G$.

3.3 Composition

Let f and g be stratified cliques of s.c.s. $X^\perp \wp Y$ and $Y^\perp \wp Z$. Let d be the maximum depth of these two s.c.s. and define the composition of f and g as:

$$f; g = \{(x, z) \in |(X^\perp \wp Z)^d|, \exists y \in |Y^d| \text{ s.t.}$$

$$(x, y) \in f \text{ and } (y, z) \in g\}.$$

One easily checks that $h = f; g$ is a stratified clique. Indeed, for any (x, z) of h we clearly have that $app_i(x, z)$ is defined and as $app_i(f)$ and $app_i(g)$ are cliques and $app_i(h) \subseteq app_i(f); app_i(g)$, we know that $app_i(h)$ is a clique.

This composition is naturally associative and it has identities given by: $id_X = \{(x, x), x \text{ is visible}\}$. So we have a category $\mathbb{SCOH}$ of stratified coherent spaces and stratified cliques.

Let us stress now an important property of this composition.

Lemma 1 *If $h = f; g$ is the composition of two elements respectively of $\mathbb{SCOH}[X, Y]$ and $\mathbb{SCOH}[Y, Z]$, and if (a, c) belongs to $app_i(h)$ (where i is inferior to the depth of $X^\perp \wp Z$) then there exists a unique b in $|Y^i|$ such that (a, b) and (b, c) belong respectively to $app_i(f)$ and $app_i(g)$.*

Proof : The existence of b follows from the definition of the composition and from the fact that $app_i^{X^\perp \wp Y}$ (resp. $app_i^{Y^\perp \wp Z}$) is defined on all elements of f (resp. g). The unicity is consequence of the fact that $app_i(f)$ and $app_i(g)$ are cliques. □

This property expresses a kind of independence of each layer: roughly speaking, the i-th appearance of the "interaction" of morphisms f and g with environment (x, z) only depends on the i-th appearances of x and z. It will be crucial when later we define a subcategory of $\mathbb{SCOH}$ modelizing LLL.

3.4 A model of ELL

To the previous constructions on objects correspond constructions on morphisms. For the additive and multiplicative constructions they are done in the straightforward way, layer by layer using the constructions of $\mathbb{COH}$. For instance if f_1 and f_2 are respectively elements of $\mathbb{SCOH}[X_1, Y_1]$ and $\mathbb{SCOH}[X_2, Y_2]$ and d is the maximal depth of the s.c.s. then we have a clique $f_1 \otimes f_2$ of $((X_1 \otimes X_2) \multimap (Y_1 \otimes Y_2))^d$ and it gives a stratified clique of $(X_1 \otimes X_2) \multimap (Y_1 \otimes Y_2)$. Similarly for $\wp, \&$ and $\oplus$.

Let us consider now the exponential operations. Let f be an element of $\mathbb{SCOH}[A, B]$ and let d denote the depth. We define $!f$ as:

$$!f = \{(u, v) \in !_m f, \text{ s.t. } u \text{ is visible}\}$$

where $!_m$ is the usual multiset bang functor of coherent spaces (here used in the coherent space X^d). The following lemma ensures that $!f$ is a stratified clique:

Lemma 2 *If (u, v) belongs to $!f$ then v is visible.*

It is easy to check that $!$ is a functor. We then have natural transformations given for objects A and B by co_A, w_A and $m_{A,B} \in \mathbb{SCOH}[!A \otimes !B, !(A \otimes B)]$.

$$\begin{array}{rcl} co_A & = & \{(u+v, u, v) \in |!A^d \multimap !A^d \otimes !A^d|, \\ & & \text{ s.t. } u+v \text{ is visible } \}, \\ w_A & = & \{([\,], *)\}, \\ m_{A,B} & = & \{([a_1, .., a_n], [b_1, \ldots, b_n], u), \\ \text{where} & & u = [(a_1, b_1), \ldots, (a_n, b_n)] \in |!(A \otimes B)| \\ & & \text{and is visible } \}. \end{array}$$

If f belongs to $\mathbb{SCOH}[A_1 \otimes \ldots A_k, B]$ let us denote (abusively) by $!f$ the morphism of $\mathbb{SCOH}[!A_1 \otimes \ldots !A_k, !B]$ obtained in the obvious way from the functor $!$ and the natural transformation m.

Lemma 3 *Let $\Gamma = A_1 \otimes \ldots A_k$, f belong to $\mathbb{SCOH}[\Gamma, B]$, g belong to $\mathbb{SCOH}[B, (B_2 \otimes \ldots B_l) \multimap C]$ and let us denote by $!g$ the expected morphism in $\mathbb{SCOH}[!B, (!B_2 \otimes \ldots !B_l) \multimap !C]$, then we have:*

$$\begin{array}{rcl} !f; co_B & = & co_\Gamma; (!f \otimes !f) \\ !f; w_B & = & w_\Gamma \\ !f; !g & = & !(f; g) \end{array}$$

Proof :

These equations hold for the coherent spaces model, so here we only have to check that the constraint we added on visibility of points doesn't raise any problem. Let us just do it for the contraction case as an example, and assuming for simplicity that $\Gamma = A$.

Let us show first that $!f; co_B \subseteq co_A; (!f \otimes !f)$. Take an element of the left-hand side clique; it is of the form $(u, (v_1, v_2))$ where $(u, v_1 + v_2)$ belongs to $!f$. Then u is visible and there exist u_1, u_2 in $!A$ such that $u = u_1 + u_2$ and (u_1, v_1), (u_2, v_2) belong to $!f$. It follows that $(u, (u_1, u_2))$ belongs to co_A and $(u, (v_1, v_2))$ belongs to $co_A; (!f \otimes !f)$.

Conversely, if $(u, (v_1, v_2))$ belongs to $co_A; (!f \otimes !f)$, then there exists (u_1, u_2) in $|!A^d \otimes !A^d|$ such that $(u, (u_1, u_2))$ and $((u_1, u_2), (v_1, v_2))$ are respectively elements of co_A and $(!f \otimes !f)$. Now, to deduce that $(u_1 + u_2, v_1 + v_2)$ belongs to $!f$ we need to know that $u_1 + u_2$ is visible. This is ensured by the fact that $u = u_1 + u_2$ and $(u, (u_1, u_2))$ belongs to co_A. Hence $(v_1 + v_2)$ is also visible by lemma 2, so $(v_1 + v_2, (v_1, v_2))$ belongs to co_B, $(u, (v_1, v_2))$ belongs to $!f; co_B$ and we are done. □

Lemma 4 *There is in $\mathbb{SCOH}$ an isomorphism between $!(A \,\&\, B)$ and $!A \otimes !B$.*

Proof : The coherence semantics gives us a clique on the main space of the s.c.s. $!(A \,\&\, B) \multimap !A \otimes !B$ and one on the main space of $!A \otimes !B \multimap !(A \,\&\, B)$. One only has to check that these cliques, once restricted to visible elements, give stratified cliques and that they yield an isomorphism. It is the case. □

Let us denote respectively by $is_{A,B}$ and $is'_{A,B}$ these morphisms of $\mathbb{SCOH}[!(A \,\&\, B), !A \otimes !B]$ and $\mathbb{SCOH}[!A \otimes !B, !(A \,\&\, B)]$.

It follows now that:

Proposition 5 *The category $\mathbb{SCOH}$ is a model of Elementary Linear Logic.*

Note that the dereliction ($!A \vdash A$) and digging ($!A \vdash !!A$) principles are not valid in this semantics. So $\mathbb{SCOH}$ does not give a model of Linear Logic.

Indeed, considering the equations required and looking at the resulting cliques at depth high enough, we note that the only possible candidates would be:

$$\begin{array}{rcl} der_A & = & \{([a], a) \in |!A \multimap A| \text{ where } a \text{ is visible } \}, \\ dig_A & = & \{(u_1 + \ldots + u_n, [u_1, \ldots, u_n]) \in |!A \multimap !!A| \\ & & \text{where } (u_1 + \ldots + u_n) \text{ is visible } \}. \end{array}$$

Let us check that these are not in general stratified cliques. In fact for dig_A it is never the case if $|A|$ has at least one visible element. Indeed, given a visible element a take $x = ([a], [[a]])$ and $y = ([a, a], [[a, a]])$. Then x and y belong to dig_A and their appearances at depth 1 are: $x^1 = ([a_0], [*])$ and $y^1 = ([a_0, a_0], [*])$ in $|!A^0 \multimap !1|$.

But $([a_0], [*]) \smile ([a_0, a_0], [*]) \, mod(!A^0 \multimap !1)$, and so $app_1(dig_A)$ is not a clique.

As to der_A let us assume that : $\exists a, a' \in |A|$ s.t. $a^0 \smile a'^0 \ mod\, A^0$. Then we take $x = ([a], a)$ and $y = ([a'], a')$. These points belong to der_A and their appearances at depth 0 are: $x^0 = (*, a^0)$, $y^0 = (*, a'^0)$ in $|1 \multimap A^0|$. We have $x \smile y \, mod(1 \multimap A^0)$ and so $app_0(der_A)$ is not a clique. This argument can be extended to the case where: $\exists i, \exists a, a' \in |A|$ s.t. a and a' are visible and $a^i \smile a'^i \ mod\, A^i$. Consequently der_A is not in general a stratified clique.

3.5 Example of refutation: the iterator

As an example, we will show that the iteration principle is excluded in $\mathbb{SCOH}$. Indeed it is not provable in ELL, which is normal since the class of elementary functions is not closed under iteration.

To interprete the (tally) integers we choose the stratified coherent space N of depth 1 given by: $N^0 = 1$; N^1 is the set $\mathbb{N}$ equipped with the discrete coherence relation.

The clique $Iter_A$ would be defined on:

$$F = A \otimes !(A \multimap A) \otimes N \multimap A,$$

by:

$$Iter_A = \{(a_0, [(a_0, a_1), \ldots, (a_{n-1}, a_n)], n, a_n) \in |F|\}.$$

Let us take the example of $A = N$. Take two elements of $Iter_A$ obtained with the same integer n:

$$\begin{array}{rcl} x & = & (a_0, [(a_0, a_1), \ldots, (a_{n-1}, a_n)], n, a_n) \\ x' & = & (a'_0, [(a'_0, a'_1), \ldots, (a'_{n-1}, a'_n)], n, a'_n) \end{array}$$

Say we chose $a'_0 = a_0$, but $[(a_0, a_1), \ldots, (a_{n-1}, a_n)]$ and $[(a'_0, a'_1), \ldots, (a'_{n-1}, a'_n)]$ (i.e. the iterated functions) such that $a_n \neq a'_n$. For a concrete example one can consider $x = (0, [(0, 0)], 1, 0)$ and $x' = (0, [(0, 1)], 1, 1)$. Then at depth 1 we have:

$$\begin{array}{rcl} x^1 & = & (a_0, n[*], n, a_n) \\ x'^1 & = & (a_0, n[*], n, a'_n) \end{array}$$

(where $n[*]$ stands for $[*, \ldots, *]$ where $*$ is repeated n times). Then as $a_n \smile a'_n \, mod\, N$, we have: $x^1 \smile x'^1 \, mod\, F^1$. Hence $Iter_A$ is not a stratified clique.

The following modified iteration principle is valid in $\mathbb{SCOH}$ though and is provable in ELL:

$$\begin{array}{rcl} ItEL_A & : & !A \otimes !(A \multimap A) \otimes N \multimap !A \\ ItEL_A & = & \{(\sum_{j=1}^{m} [a_0^j], \sum_{j=1}^{m} u_j, n, \sum_{j=1}^{m} [a_n^j]) \in |F|, \\ & & \text{where } m \in \mathbb{N} \text{ and for } 1 \leq m, \\ & & u_j = [(a_0^j, a_1^j), \ldots, (a_{n-1}^j, a_n^j)]\}. \end{array}$$

4 Locally bounded stratified cliques

4.1 Measured coherent spaces

We want to enrich our semantical structure with a quantitative feature, a measuring function. As said in the introduction our goal here is to keep track of the I/O balance within a computation of the program. We mean by I/O balance the difference between the number of times the program has been called and the number of times it has requested an input. This balance will be evaluated at each level.

A *measured* coherent space X is a coherent space given together with a measuring function: $s^X : |X| \to \mathbb{Z}$. We will consider *stratified measured coherent spaces* (s.m.c.s.) (X^i, s_i, ϕ_i) adapted from definition 1. Actually spaces will be measured only starting from level 1 and s_i will denote the measure of X^{i+1}: $s_i : |X^{i+1}| \to \mathbb{Z}$ (one can think of X^0 as having constant measure s_{-1} equal to 0). The third condition is strenghtened to:

there exists an integer d such that:

$$\forall i \geq d, X^i = X^d, s_{i-1} = s_{d-1} \text{ and } \phi_i = id_{|X^d|}.$$

The reason for this mismatch on indexes is that what we really care about is the size of the multisets corresponding to exponentials, and in the interpretation of formulas these only appear from level 1. We will use upperscripts to indicate to which s.m.c.s. the measure function is relative: s_i^X is defined on $|X^{i+1}|$ where X^{i+1} is the level $i+1$ of X. We introduce the following measure functions for the various constructions:

$$\begin{array}{rcl} s_i^{X^\perp} & = & -s_i^X, \\ s_i^{X \square Y}(x,y) & = & s_i^X(x) + s_i^Y(y) \text{ for } \square = \otimes, \wp, \\ s_i^{X \square Y} & = & [s_i^X, s_i^Y] \text{ for } \square = \&, \oplus, \\ s_i^{\mathbf{1}}(*) & = & s_i^{\perp}(*) = 0 \end{array}$$

The main case is that of the ! construction, for which we set: $s_0^{!X}([x_1,\dots,x_n]) = n$ and $s_{i+1}^{!X}([x_1,\dots,x_n]) = \sum_{k=1}^n s_i^X(x_k)$.

Notice that by duality we then have $s_0^{?X}([x_1,\dots,x_n]) = -n$ and $s_{i+1}^{?X}([x_1,\dots,x_n]) = \sum_{k=1}^n s_i^X(x_k)$.

As to the new modality § we define the s.m.c.s. $\S X$ by:

- for $i \geq 0$, $(\S X)^{i+1} = X^i$,
- $(\S X)^0 = 1$.

Acually, it will be more convenient to denote elements of $|(\S X)^{i+1}|$ as singleton elements over $|X^i|$: $|(\S X)^{i+1}| = \{[x], x \in |X^i|\}$.

The measure functions are defined by: $s_0^{\S X}([x]) = 1$, and $s_{i+1}^{\S X}([x]) = s_i^X(x)$.

Note that § is not self-dual, contrarily to the definition in [Gir95b]. We shall denote its dual by $\bar{\S}$. A possibility to have a self-dual § would be to set $s_0^{\S X}$ equal to the constant zero function.

4.2 Locally bounded stratified cliques

Definition 3 *Let f be a stratified clique on a m.s.c.s. X. We say that f is* locally bounded *if for any x of f, for any i the following integers set is bounded:*

$$\{s_i(y^{i+1}), y \in f \text{ s.t. } y^i = x^i\}.$$

Recall the forest definition of a stratified clique c given in section 3.1. Now, this forest c is a locally bounded stratified clique if to each node (at level i) we can associate an integer M such that the size s_i of its immediate ascendants is bounded by M (in absolute value).

Notice that if all branchings of the forest c are finite, then the condition is trivially satisfied and so c is a locally bounded stratified clique. But this will not often be the case for cliques interpreting proofs (in the case of a cut-free proof it would mean that it doesn't make any use of !-promotion).

Another particular case of locally bounded clique is that where the measures are always zero: $\forall x \in f, \forall i \in \mathbb{N},\ s_i(y) = 0$. This condition is satisfied for instance by the identity maps, so they are bounded. It seems to be a fairly degenerate case, but surprisingly enough it turns out that it would suffice to account for all constructions of LLL ... but the §. Yet, as the § is essential to get a significative expressivity we definitely cannot stick to this particular case.

Lemma 6 *Identities, contraction and weakening morphisms and the morphisms $is_{A,B}$ and $is'_{A,B}$ are locally bounded cliques.*

Proof : One can check that they satisfy the property we just pointed out: the size of all their elements at any depth is always zero. □

Proposition 7 *Locally bounded stratified cliques are preserved by composition.*

Proof : Let f and g belong respectively to $\mathbb{SCOH}[X,Y]$ and $\mathbb{SCOH}[Y,Z]$ and assume they are locally bounded. Let i be an integer, (x,z) be an element of $f;g$ and let us show that $f;g$ satisfies the property w.r.t. (x,z) at depth i.

By definition of the composition, there exists a y in $|Y|$ such that (x,y) and (y,z) belong respectively to f and g. Let us denote:

$$\begin{array}{rcl} M_1 & = & sup\{\,|s_i(x',y')|,\ (x',y') \in f \text{ s.t.} \\ & & app_i(x',y') = app_i(x,y)\}, \\ M_2 & = & sup\{\,|s_i(y',z')|,\ (y',z') \in g \text{ s.t.} \\ & & app_i(y',z') = app_i(y,z)\}. \end{array}$$

We claim that (M_1+M_2) provides a suitable bound for $(f;g)$ and (x,z) at depth i. Indeed, let (x',z') be an element of $f;g$ such that $app_i(x',z') = app_i(x,z)$. There exists a y' such that (x',y') and (y',z') belong respectively to f and g. As

$app_i(x', z') = app_i(x, z)$ it follows from lemma 1 that $app_i(y) = app_i(y')$ and therefore we have:

$app_i(x', y') = app_i(x, y)$ and $app_i(y', z') = app_i(y, z)$.

Hence $|s_i(x', y')| \leq M_1$, $|s_i(y', z')| \leq M_2$, and as:

$$s_i(x', z') = s_i(x', y') + s_i(y', z'),$$

we conclude that: $|s_i(x', z')| \leq (M_1 + M_2)$. $\square$

Note that lemma 1 is used in a crucial way in the proof. If we were to define a stratified relational model in the lines of the stratified coherence model and then a notion of locally bounded stratified relations, these morphisms would not be preserved by composition.

4.3 The model of LLL

We denote by $\mathbb{BSCOH}$ the category of s.m.c.s. and locally bounded stratified cliques. We will consider it as a subcategory of $\mathbb{SCOH}$ even if in fact the objects are not the same (we added the measuring functions). Let us illustrate now how the quantitative condition rules out monoidality of !.

Lemma 8 *The stratified clique $m_{A,B}$ of $\mathbb{SCOH}[!A \otimes !B, !(A \otimes B)]$ is not locally bounded.*

Indeed for any n we have in $m_{A,B}$ an element of the shape:

$$x = (([a_1, \ldots a_n], [b_1, \ldots b_n]), [(a_1, b_1), \ldots, (a_n, b_n)])$$

(take for instance the a_i's equal to a single element a of $|A|$, and similarly for the b_i's).

We have: $s_0(x) = (-n) + (-n) + n = -n$. So the set$\{s_0(y), y \in m_{A,B}\}$ is not bounded and therefore $m_{A,B}$ is not a locally bounded stratified clique. Consequently, the category $\mathbb{BSCOH}$ is not a model of ELL.

Proposition 9 *The subcategory $\mathbb{BSCOH}$ is stable by $\otimes$, $\wp$, $\oplus$ and $\&$.*

For instance if f, f' belong respectively to $\mathbb{BSCOH}[X, Y]$ and $\mathbb{BSCOH}[X', Y']$ it is easy to check that the stratified clique $f \otimes f'$ of $\mathbb{SCOH}[X \otimes X', Y \otimes Y']$ is also locally bounded .

Proposition 10 *The subcategory $\mathbb{BSCOH}$ is stable by the functor !.*

The proof is given in appendix B. Note that we once again use the coherence relations in this proof.

Let us define the action of § on morphisms. Given f in $\mathbb{SCOH}[X, Y]$ we set:

$$\S f = \{([x], [y]) \in |\S X \multimap \S Y|, (x, y) \in f\}.$$

It is clear that if f is locally bounded , then $\S f$ is locally bounded . This gives us a functor on $\mathbb{BSCOH}$ and we have natural transformations mp and nd defined by:

$$\begin{aligned} mp_{A,B} &= \{((([a], [b]), [(a, b)]) \in \\ &\quad |\S A \otimes \S B \multimap \S(A \otimes B)|\}, \\ nd_A &= \{([a], [a]) \in |\S A \multimap !A|\}. \end{aligned}$$

Theorem 11 *The category $\mathbb{BSCOH}$ is a model of Light Linear Logic.*

To prove it formally we would need to use the proof-nets of [Gir95a] and show that for each case of cut-elimination step the interpretation as a stratified clique is preserved. We will not do it here but it can be checked without surprise and will be carried out in the full version of this paper. For the non-exponential cut cases it is identical to the setting of coherent spaces. The contraction and weakening cut cases are insured by lemma 3 and the commutative cut steps by the fact that ! and § are functors, by the additive/multiplicative isomorphism of lemma 4 and by the properties of the natural transformations mp and nd.

Note that certain extra principles are validated in this semantics, like $\S(A \otimes B) \multimap \S A \otimes \S B$. This principle was also considered by Hofmann and Jagadeesan in recent work.

4.4 Refutation of the ELL iterator

We want to show that the ELL iterator $ItEL$ recalled in section 3.5 is excluded in $\mathbb{BSCOH}$. We need first to set a measure for the space N. Once again guided by the LLL representation of integers we set:

$$\forall n \in |N|,\ s_0(\underline{n}) \quad = \quad 1 - n.$$

To simplify we consider elements of $ItEL_A$ of the form:

$$x = (m[a_0], mu, \underline{n}, m[a_n]),$$

(this is the particular case when all the a_0^j's are equal and so are the u_j's). We then have:

$$\begin{array}{rcl} x^0 & = & (*, *, *, *), \\ x^1 & = & (m[*], mn[*], \underline{n}, m[*]). \end{array}$$

Hence:

$$\begin{array}{rcl} s_0(x^1) & = & -m - mn - (1-n) + m \\ & = & n(1-m) - 1 \end{array}$$

So we might fix for instance $m = 2$ and choose elements x with arbitrary large n's which shows that $ItEL$ is not a locally bounded stratified clique.

Notice that if m is constrained to be 1, this argument doesn't apply. This is what happens for the LLL iterator, as the right-hand side ! is replaced by a §.

5 Conclusion and further directions

In this work we showed how the setting of stratified coherent spaces can pinpoint the characteristics of Elementary and Light Linear Logic. It is interesting to note how the multiset interpretation of exponentials is put into use to isolate a quantitative property of certain cliques. How good an account of polytime algorithms is provided by the model $\mathbb{BSCOH}$ is still to be studied. A natural question is whether this interpretation extends to second-order quantifiers; we hope this could be done by a simple adaptation of the interpretation in [Gir87]. We can wonder also if the respective properties of stratification and local bound could suffice to characterize ELL and LLL proofs among general LL proofs: if the clique interpreting an LL proof is stratified for instance, does this proof correspond to an ELL proof (in the sense of [DJ99])? Finally, another interesting question is whether a class of functions corresponds to locally bounded stratified cliques in a similar way stable functions correspond to cliques in the set version of coherent spaces. Thomas Ehrhard and Nuno Bareiro recently characterized the class of functions arising from the multiset coherence model ([BE99]) and maybe their work could be adapted to our stratified setting.

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A Light linear logic: sequent calculus

Light Linear Logic formulas are defined as Linear Logic formulas, but for the introduction of the new modality §. Here we are considering a variant of the system introduced in [Gir95b] where § is not self-dual. We shall denote its dual by $\bar{\S}$.

The negation is a defined one, by which we mean that $A^\perp$ is used as a notation for the De Morgan dual of A.

We recall briefly the sequent calculus of Light Linear Logic. A *block* is a multiset $A_1, \ldots A_n$ of formulas (to be thought of as $A_1 \oplus \ldots \oplus A_n$). A *sequent* is a multiset of blocks, denoted as $\vdash \mathbf{B_1}; \ldots; \mathbf{B_n}$ (to be thought of as $B_1 \wp \ldots \wp B_n$, where B_i is the formula associated to the block $\mathbf{B_i}$).

Identity group

$$\frac{}{\vdash A^\perp; A}\ \text{(Axiom)} \qquad \frac{\vdash A; \Gamma \quad \vdash A^\perp; \Delta}{\vdash \Gamma; \Delta}\ \text{(Cut)}$$

Structural group

$$\frac{\vdash \Gamma}{\vdash \Gamma; ?A}\ \text{(M-weakening)} \qquad \frac{\vdash \Gamma; \mathbf{A}}{\vdash \Gamma; \mathbf{A}, B}\ \text{(A-weakening)}$$

$$\frac{\vdash \Gamma; ?A; ?A}{\vdash \Gamma; ?A}\ \text{(M-contraction)} \qquad \frac{\vdash \Gamma; \mathbf{A}, B, B}{\vdash \Gamma; \mathbf{A}, B}\ \text{(A-contraction)}$$

Logical group

$$\frac{}{\vdash 1}\ \text{(One)} \qquad \frac{\vdash \Gamma}{\vdash \Gamma; \perp}\ \text{(False)}$$

$$\frac{\vdash A; \Gamma \quad \vdash B; \Delta}{\vdash A \otimes B; \Gamma; \Delta}\ \text{(Times)} \qquad \frac{\vdash A; B; \Gamma}{\vdash A \wp B; \Gamma}\ \text{(Par)}$$

$$\frac{}{\vdash \Gamma; T}\ \text{(True)}$$

$$\frac{\vdash \Gamma; A}{\vdash \Gamma; A \oplus B}\ \text{(Left Plus)} \qquad \frac{\vdash \Gamma; A \quad \vdash \Gamma; B}{\vdash \Gamma; A \,\&\, B}\ \text{(With)}$$

$$\frac{\vdash \Gamma; B}{\vdash \Gamma; A \oplus B}\ \text{(Right Plus)}$$

$$\frac{\vdash B_1, \ldots, B_n; A}{\vdash ?B_1; \ldots; ?B_n; !A}\ \text{(Of course)} \qquad \frac{\vdash B_1| \ldots |B_n; A_1; \ldots; A_m; A}{\vdash ?B_1; \ldots; ?B_n; \bar{\S}A_1; \ldots; \bar{\S}A_m; \S A}\ \text{(Neutral)}$$

In the (Neutral) rule, each | is either a "," or ";".

B Proof of proposition 10

Assume f belongs to $\mathbb{BSCOH}[X, Y]$ and let us show that then $!f : !X \multimap !Y$ is locally bounded. Take (u, v) an element of $!f$. Then u and v are of the form $u = [x_1, \ldots, x_n]$, $v = [y_1, \ldots, y_n]$, where for $1 \leq k \leq n$, (x_k, y_k) are elements of f. Let us set a depth i. As f is locally bounded we know that for each k the set:

$$\{|s_i(x', y')|, (x', y') \in f \text{ and } app_i(x', y') = app_i(x_k, y_k)\}$$

is bounded. Denote by M_k its upper bound. Now we claim that $\sum_{i=1}^n M_k$ provides a suitable bound at depth $(i+1)$ for $!f$ and (u,v). Indeed, take (u',v') in $!f$ such that $app_{i+1}(u',v') = app_{i+1}(u,v)$. Then it means that u' and v' are of the form $u' = [x'_1, \ldots, x'_n]$, $v = [y'_1, \ldots, y'_n]$ with:

$$\forall 1 \leq k \leq n,\ app_i(x_k) = app_i(x'_k), app_i(y_k) = app_i(y'_k).$$

Besides, as (u',v') belongs to $!f$ there is a permutation $\sigma : \{1, \ldots n\} \to \{1, \ldots n\}$ such that:

$$\forall 1 \leq k \leq n,\ (x'_{\sigma(k)}, y'_k) \in f.$$

Note that here we cannot assume that σ is the identity as the indexing of the multiset elements has already been chosen so that elements of same indexes in u and u' (resp. v and v') have same appearance at depth i.

Let us fix one k. As $app_i(f)$ is a clique and contains (x^i_k, y^i_k) and $(x'^i_{\sigma(k)}, y'^i_k) = (x^i_{\sigma(k)}, y^i_k)$ we have:

$$(x^i_k, y^i_k) \frown (x^i_{\sigma(k)}, y^i_k)\, mod(X^i \multimap Y^i),$$

Now, as u is visible we know that $[x^i_1, \ldots, x^i_n]$ is a multiclique of X^i, and so in particular:

$$x^i_k \frown x^i_{\sigma(k)}\, mod X^i.$$

Therefore $x^i_k = x^i_{\sigma(k)}$. So finally $(x'^i_k, y'^i_k) = (x^i_k, y^i_k)$ and consequently the bound applies:

$$|s_i(x'_k, y'_k)| \leq M_k.$$

Hence we have:

$$|s_{i+1}(u',v')| = |\sum_{k=1}^n s_i(x'_k, y'_k)| \leq \sum_{k=1}^n M_k.$$

As to the depth 0 we clearly have a bound equal to zero.