

# Nondirected lattice paths on different lattices

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**Abstract:** We study excursions with jumps to nearest neighbors in different lattices (hexagonal, triangular, kagome, Manhattan lattices), paths in a strip, etc. We show that some families are algebraic, D-finite, and give some nice closed form formulas.

Keywords: lattice paths, algebraic functions, D-finite functions, kernel method

## 1 Introduction

In a previous work [1], we concentrated on enumeration and asymptotics of directed lattice paths (i.e. walks which goes in only 3 directions, not 4). Recent works [6, 4] showed some nice explicit formula for enumeration of nondirected walks (i.e. walks with many jumps, going to any directions) in  $\mathbb{N}^2$ . Enumeration and asymptotics of nondirected walks in  $\mathbb{Z}^2$  are in one sense solved by holonomy theory (see some examples in [2]). We concentrate in this article, first on walks in a strip, and then on enumeration of nondirected walks to nearest neighbors in different lattices, namely excursions (walks starting and ending at the origine), may it be in the whole plane or in the quarter plane (the number of such walks of length  $n$  will be  $e^{\mathbb{Z}}(n)$  and  $e^{\mathbb{N}}(n)$ ). The generating functions  $F(x, y, t)$  will enumerate walks with length encoded by  $t$ , and final position encoded by  $(x, y)$ .

## 2 Walks in a strip

**Theorem 2.1** *Non directed walks in a strip (may it be  $\mathbb{Z} \times [a, b]$  or  $\mathbb{N} \times [a, b]$ ) have an algebraic generating function  $F(x, y, t)$ .*

**Proof.** It is clear that such walks can be simulated on a nonambiguous push-down automata, (the  $b + a + 1$  possible altitudes corresponding to  $b + a + 1$  states, and use a letter for each type of jump). Going to the Greibach normal form, such walks are thus generated by a nonambiguous context free grammar, therefore the associated generating function is algebraic. Note that for a strip

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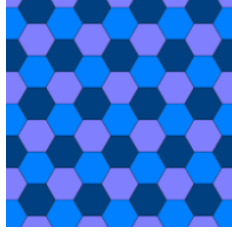
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$\mathbb{Z} \times [a, b]$ , we can use an stack isomorphic to  $\mathbb{N}$ , by adding finitely many states which mimics the border conditions (i.e., corresponding to the different ways the walks can “cross” the  $y$  axis).  $\square$

Comments: it is striking that this also holds for cases in which the iterated kernel method leads to an infinite group. The key difference is for sure that  $[x^k]F(x, y, t)$  is here a (Laurent) polynomial in  $y$ .

The nicest cases in the previous theorem are  $[a, b] = [0, h]$  or  $[a, b] = [-h, h]$ , which open the gates to the analysis of the average height of these lattice paths (similarly to [3]).

### 3 Walks in the honeycomb lattice



The honeycomb lattice (also known as hexagonal lattice, if we refer to cells, or triangular lattice if we refer to bounds) is made of hexagons. We consider here a nearest neighbor walk on the cells. This is encoded by  $P := x^2 + \frac{1}{x^2} + xy^2 + \frac{x}{y^2} + \frac{y^2}{x} + \frac{1}{xy^2}$ .

**Theorem 3.1 (honeycomb lattice)**

$$e^{\mathbb{N}}(2n) = [x^0 y^0 t^n] \frac{1}{1 - P},$$

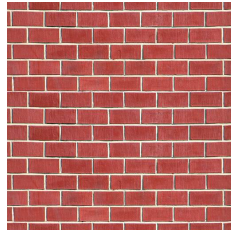
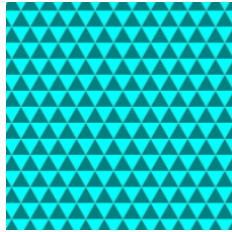
the corresponding GF satisfies a differential equation of order 5 and degree 14.

$$e^{\mathbb{Z}}(n) = [t^n] \frac{\text{HeunG}(8/9, 2/3, 1, 1, 1, 1, 8t/(6t - 1))}{1 - 6t}$$

where HeunG is the Heun general function,  $\text{HeunG}(a, q, \alpha, \beta, \gamma, \delta, t)$  being defined as the solution of the following differential equation of order 2:

$$\begin{aligned} \partial_t^2 H(t) - \frac{((\alpha + \beta + 1)t^2 + ((-\delta - \gamma)a - \alpha + \delta - \beta - 1)t + \gamma a)}{t(t - 1)(-t + a)} \partial_t H(t) \\ - \frac{(\alpha\beta t - q)}{t(t - 1)(-t + a)} H(t) = 0 \quad \text{with} \quad H(0) = 1, H'(0) = \frac{q}{\gamma a}. \end{aligned}$$

### 4 Walks in the triangular lattice



On the triangular lattice, the type jumps only depends on the parity of  $n$ : if  $n$  is even then  $P := x + 1/x + 1/y$  and if  $n$  is odd then  $P := x + 1/x + y$ .

**Theorem 4.1 (periodic lattices in dimension  $d$ )** *The generating function of excursions in a periodic lattice (in any dimension) is D-finite.*

**Proof.** The type of jumps, depending on periodicity constraints, can be encoded by making a linear combination of  $F(x\zeta_i, y\eta_j, t)$  where the  $\zeta_i$  and  $\eta_j$  are some cyclotomic numbers. This gives to a functional equation for  $F$  making numerous substitutions  $x \rightsquigarrow x\zeta_i, y \rightsquigarrow y\eta_j$  in this functional equation leads to a linear system and therefore to rationality of  $F(x, y, t)$ . However, the fastest way to prove the rationality is to see the walk as a Markov chain (finite automaton), and therefore the associated generating function is rational via the Kleene theorem. By classical closure properties on holonomic functions, it therefore implies that  $[x^k]F(x, y, t)$  or  $[y^j]F(x, y, t)$  will be algebraic (for any  $k, j$ ) and  $[x^0 y^0]F(x, y, t)$  will be D-finite.  $\square$

**Theorem 4.2 (triangular lattice)**

$$F^{\mathbb{Z}}(x, y, t) = \frac{yx((x^2 + xy + 1)t + x)}{x^2y - (x^3y^2 + x^3 + 3x^2y + x^4y + xy^2 + x + y)t^2}$$

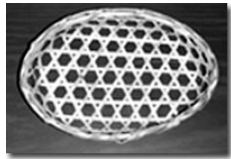
$$e^{\mathbb{Z}}(2n) = [t^{2n}] \text{HeunG}\left(\frac{1}{9}, \frac{1}{3}, 1, 1, 1, 1, t^2\right) = \sum_{k=0}^n \binom{2k}{k} \binom{n}{k}^2$$

$$e^{\mathbb{N}}(2n) = \sum_{k=0}^n \binom{2k}{k} \binom{n}{k} \binom{n+1}{k} \frac{1}{(k+1)^2}$$

$e^{\mathbb{Z}}(2n)$  is EIS A002893 (see [7]) and a lot of nice links are here mentioned: elliptic integral evaluations of Bessel moments, curves with singular fibers, cooperative phenomena in crystals, rectangular (0,1)-matrices...

$e^{\mathbb{N}}(2n)$  is EIS A128088 and is  $(n+1) \times$  number of permutations in  $S_n$  with longest increasing subsequence of length  $\leq 3$ .

## 5 Walks in the kagome lattice



The kagome lattice is a periodic lattice made of hexagons and triangles. The words “kagome” comes from Japanese and designs a basket which has this pattern. A nearest neighbor walk in the cells of the kagome lattice is therefore encoded by the following set of jumps:  $P_1 = y^2 + \frac{1}{y} + xy + x + \frac{y+1}{x}$ ,  $P_2 = x + \frac{1}{x} + \frac{1}{y^2}$ ,  $P_3 = \frac{x}{y} + \frac{1}{xy} + y$  where  $P_1$  is used if  $((x \bmod 2 = 0) \text{ and } (y \bmod 4 < 2))$  or  $((x \bmod 2 = 1) \text{ and } (y \bmod 4 \geq 2))$ , where  $P_2$  is used if  $((x \bmod 2 = 0) \text{ and } (y \bmod 4 = 2))$  or  $((x \bmod 2 = 1) \text{ and } (y \bmod 4 = 0))$ , and where  $P_3$  is used elsewhere.

**Theorem 5.1 (kagome lattice)**

$$F^{\mathbb{Z}}(x, y, t) = \frac{xy((y^3x + x + y^2x^2 + x^2y + y^2 + y)t + xy)}{y^2x^2 - (2y^4x^3 + 2x^3 + 6y^2x^2 + 2y^2x^4 + 2y^4x + 2x + 2y^2)t^2}$$

$$[x^0y^0]F^{\mathbb{Z}}(x, y, t) = \text{HeunG}\left(\frac{1}{9}, \frac{1}{3}, 1, 1, 1, 1, 2t^2\right)$$

$$e^{\mathbb{Z}}(2n) = 2^n \sum_{k=0}^n \binom{2k}{k} \binom{n}{k}^2$$

Puzzle: find a bijective proof of  $2^n$  factor between excursions in the kagome lattice and excursions in the triangular lattice.

## 6 Walks in the Manhattan lattice



The streets of Manhattan offer a quite remarkable pattern, which is one sense  $\mathbb{Z}^2$ , excepted that one-way streets are alternating.

**Theorem 6.1 ( $xy$ -Manhattan lattice)**

$$F^{\mathbb{Z}}(x, y, t) = \frac{x((y^2x^3 + x - x^2y^3 - y)t^2 + (-y^2 - xy^3)t - xy^2)}{(y^2x^4 + x^2y^4 + x^2 + y^2)t^2 - x^2y^2}$$

$$e^{\mathbb{Z}}(4n) = 2 \binom{2n-1}{n-1}^2 = [t^{4n}] \frac{1}{\pi} \text{EllipticK}(4t^2) \quad \text{where } \text{EllipticK}(z) = \int_0^1 \frac{1}{\sqrt{1-t^2}} \frac{1}{\sqrt{1-z^2t^2}} dt$$

$$e^{\mathbb{N}}(4n) = C_n^2 \quad \text{where } C_n = \frac{\binom{2n}{n}}{n+1} \text{ is the } n\text{-th Catalan number.}$$

$e^{\mathbb{Z}}(n)$  is related to EIS A060150, while  $e^{\mathbb{N}}(n)$  also enumerates bishop moves on  $\mathbb{N}^2$ , see [5].

**Theorem 6.2 (variant of Manhattan lattice)** *Each part of vertical road are alternating one-ways streets:*

$$e^{\mathbb{Z}}(2n) = \sum_{k=0}^n \binom{n}{k}^3 = [t^{2n}] \text{HeunG}(-1/8, 1/4, 1, 1, 1, 1, -t^2)$$

(the Franel numbers, EIS A000172), while  $e^{\mathbb{N}}(2n)$  satisfies a differential equation of order 3 and degree 8.

**Theorem 6.3 ( $x$ -Manhattan lattice)** *Walks on  $\mathbb{Z}^2$  with vertical jumps  $y$  at odd altitude and  $1/y$  at even altitude:*

$$F^{\mathbb{Z}}(x, y, t) = \frac{x((yx^2 - x + y)t + yx)}{yx^2 - (yx^2 + x^4y + y)t^2 + (x^2 + x^2y^2)t}$$

$$e^{\mathbb{Z}}(2n) = \frac{2}{3} \sum_{k=0}^n \binom{2k}{k} \binom{n}{k}^2$$

$$e^{\mathbb{N}}(4n) = \sum_{k=1}^n \sum_{i=k}^n \binom{n-1}{i-1} \binom{n}{i-1} \binom{i-1}{k-1} \binom{i}{k-1} \frac{1}{ik}$$

The GF of the  $e^{\mathbb{N}}(4n)$  is the antiderivative of the double Laplace transform of  $(\text{BesselI}(1, 2\sqrt{t})/\sqrt{t})^3$  (see EIS A103370).

Puzzle: find a bijective proof of the  $2/3$  factor between excursions in the  $x$ -Manhattan lattice and excursions in the triangular lattice.

## 7 Conclusion

This work is just a first step towards exact enumeration and asymptotics of walks in lattices other than  $\mathbb{Z}^2$ . The explicit formulas we gave are encouraging and suggest that numerous properties are still to be found/explained.

## References

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