Nominal Automata

Mikolaj Bojanczyk[1], Bartek Klin[12], Slawomir Lasota[1]
[1] *Warsaw University* [2] *University of Cambridge*

IFIP WG1.3, Aussois, 08/01/11

$$\lambda X.\ (\mathrm{nominal}\ X)$$

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Finite automata

An automaton:

- set Q of states
- alphabet A
- transition function $\delta : Q \times A \to Q$
- initial state $q_0 \in Q$
- accepting states $Q_a \subseteq Q$

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$A = \mathbb{N}$

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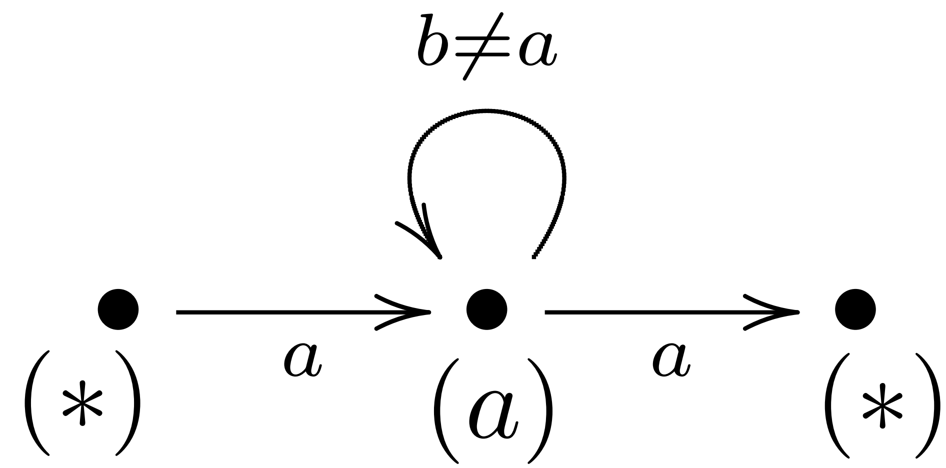

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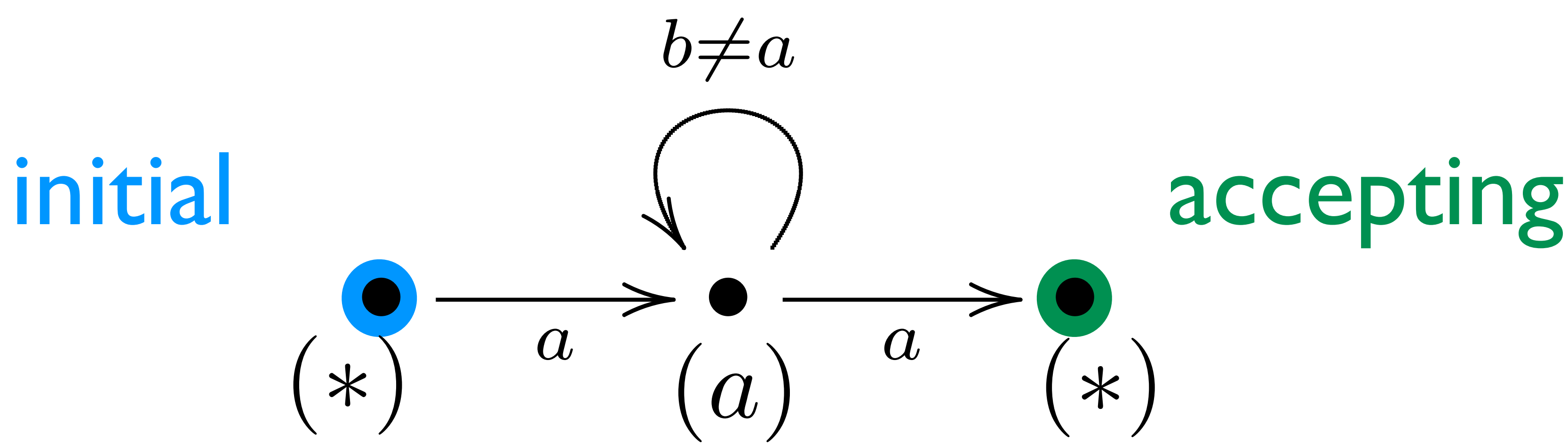

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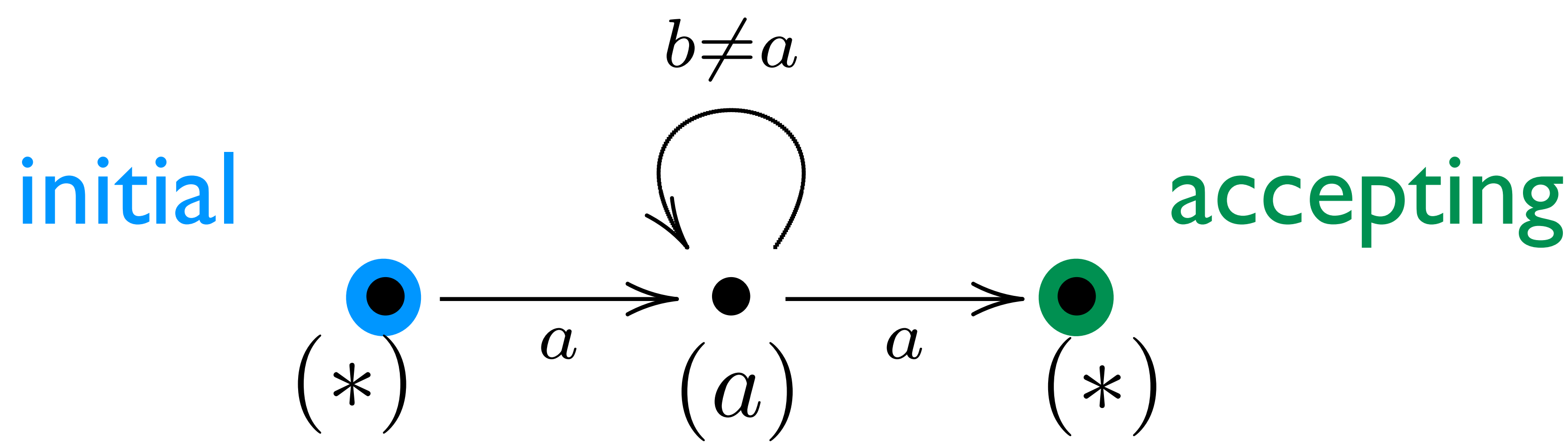

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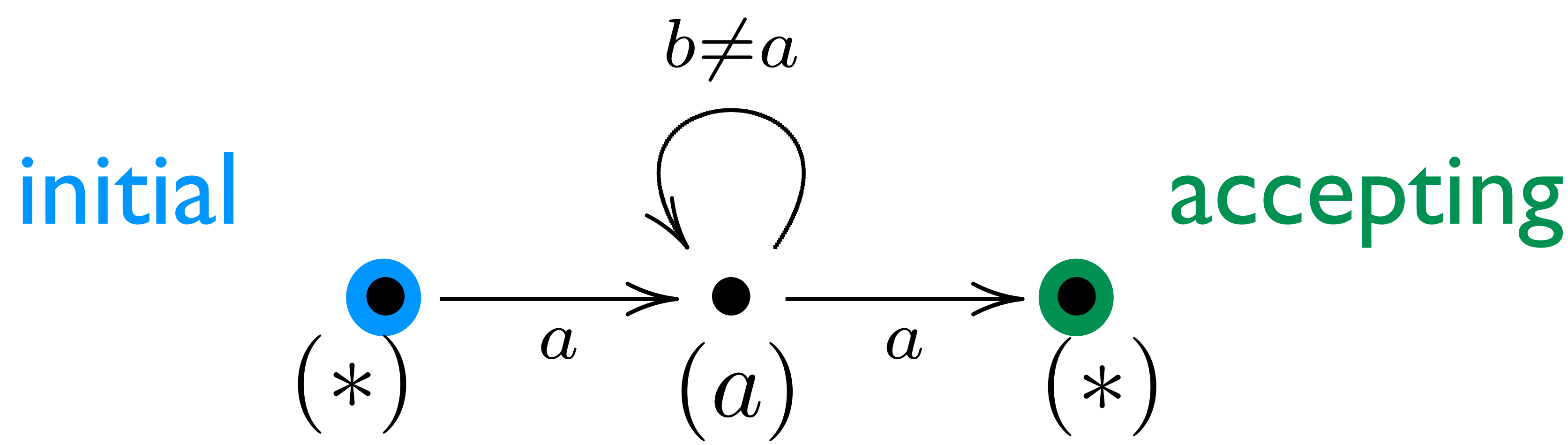

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Myhill-Nerode equivalence: $L \subseteq A^*$

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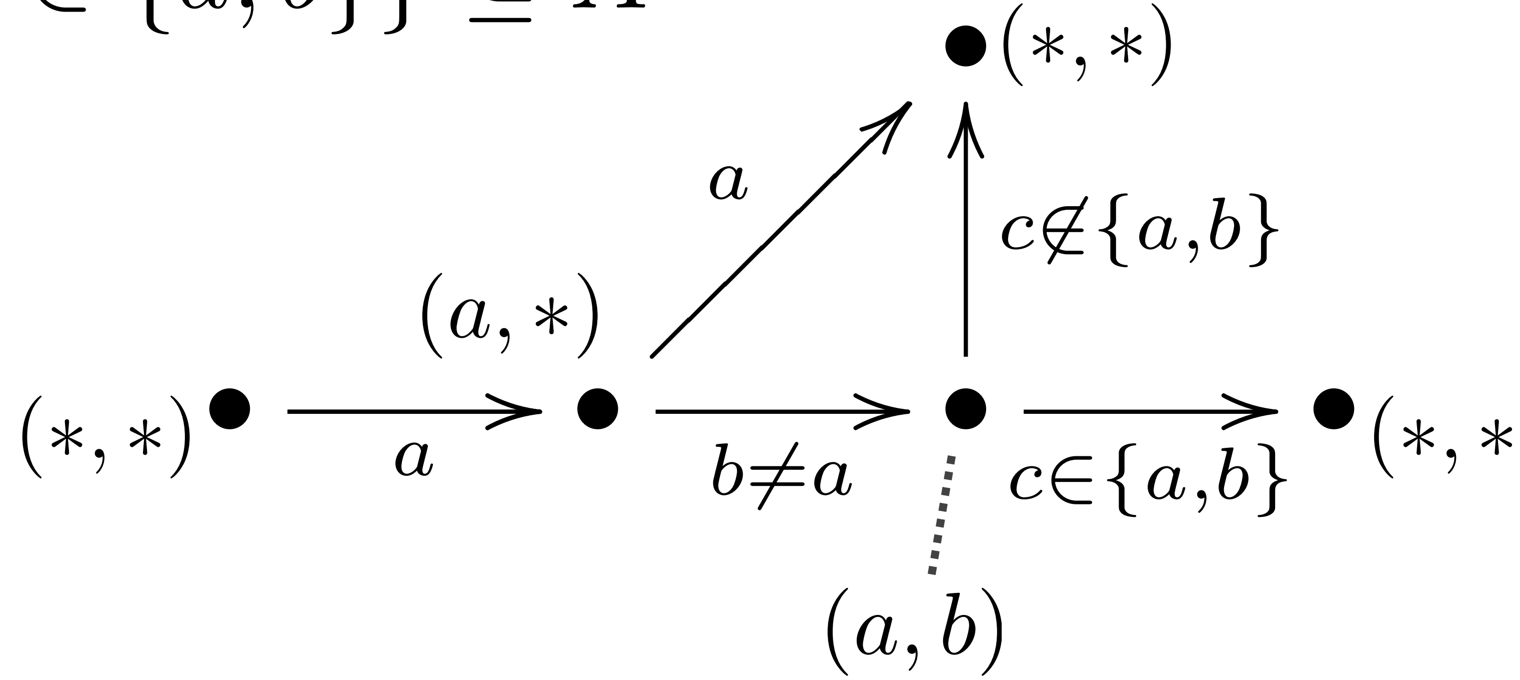

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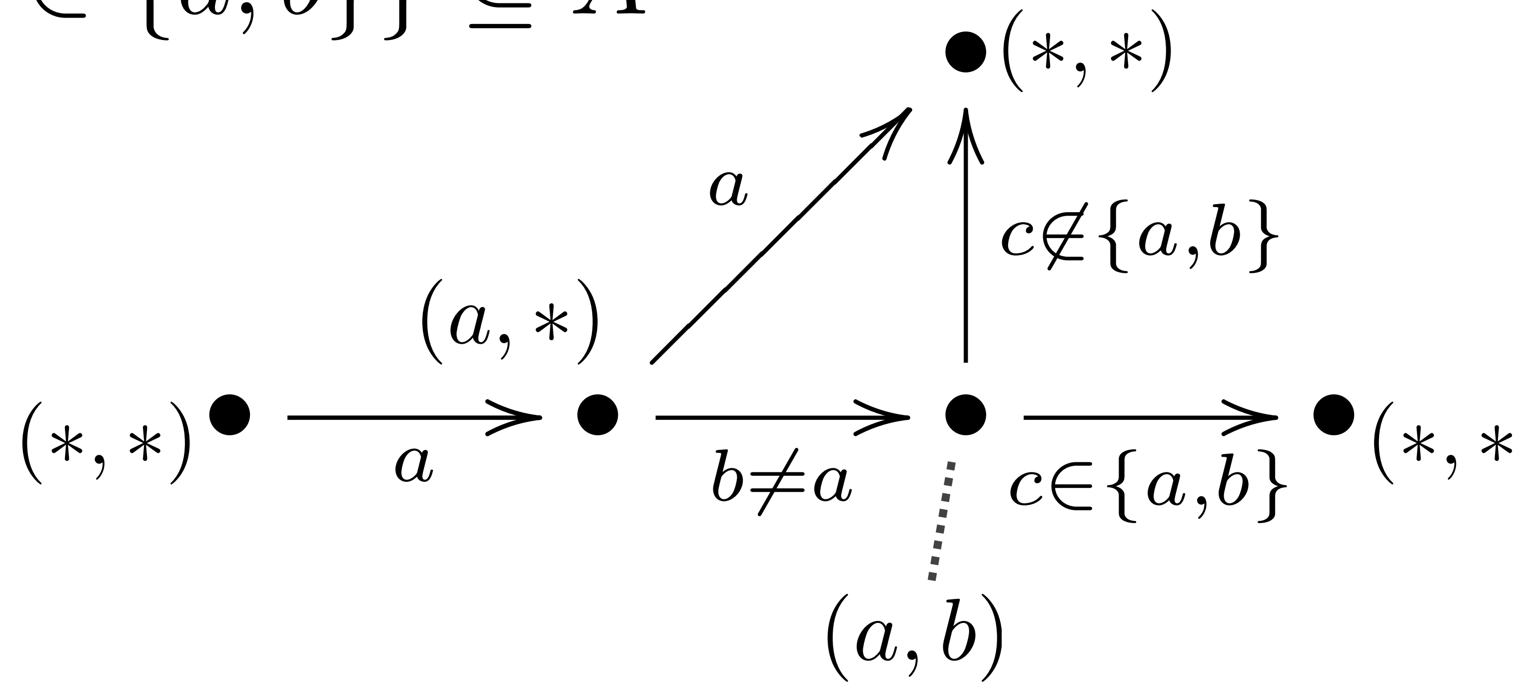

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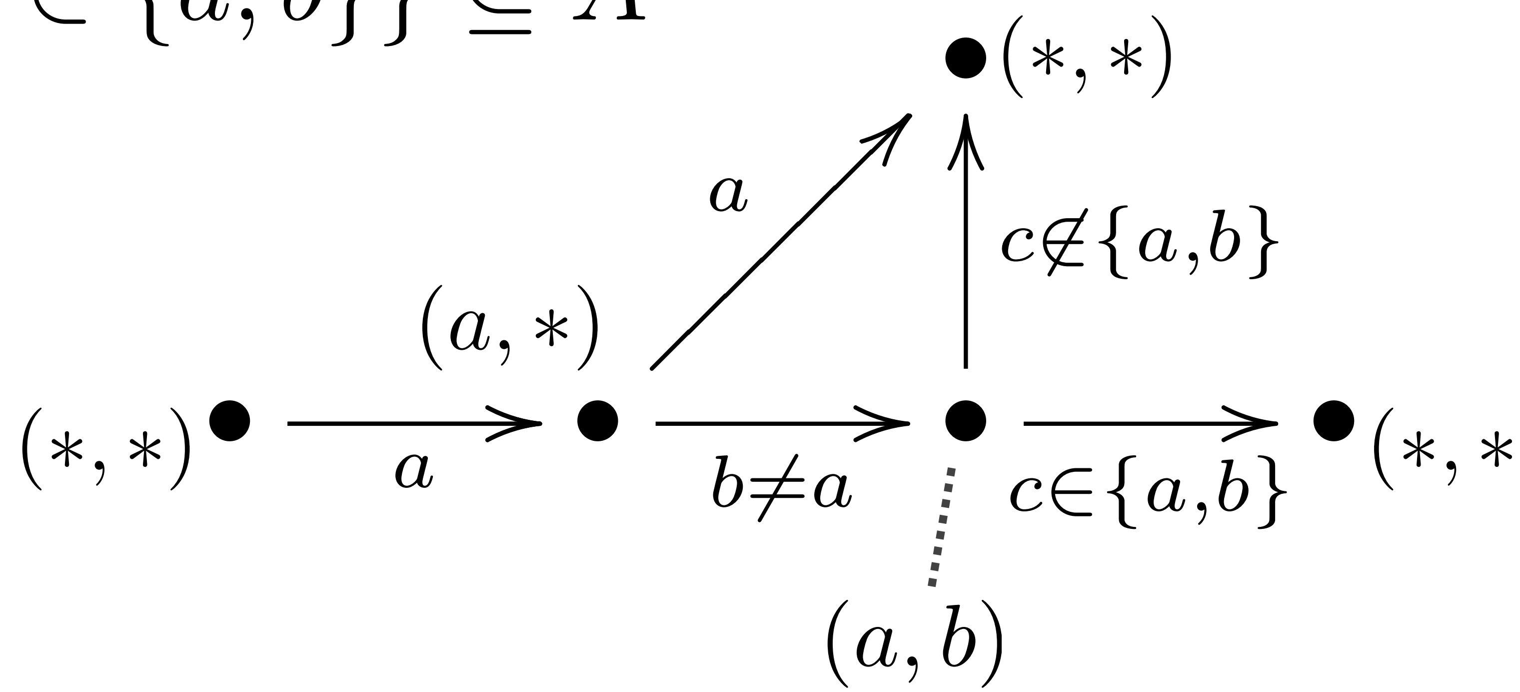

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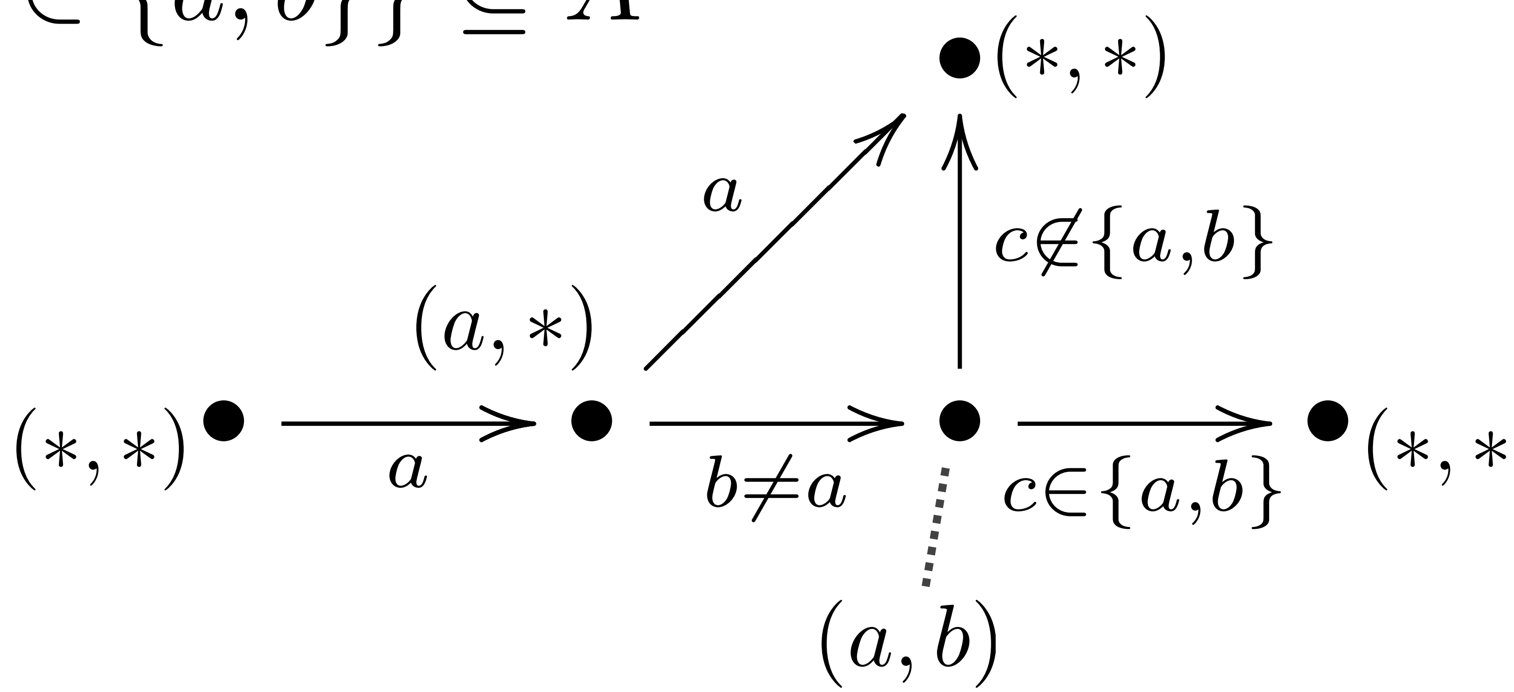

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Worse: for any $G \leq Sym(\{1,2,\ldots,n\})$,

$$L = \{a_1 \cdots a_n b_1 \cdots b_n \mid \exists \pi \in G.\ \forall i = 1..n.\ a_i = b_{\pi(i)}\}$$

F.M.A. are equivariant

Fix $G = Sym(A)$.

Defn.: A G-set is:

- a set X
- an action $\_ \cdot \_ : X \times G \to X$ (+ axioms)

Defn.: Function $f : X \to Y$ is equivariant if

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category G-**Set**

Fact: In a F.M.A.:

- configurations form a G-set
- $\delta : X \times A \to X$ is equivariant

G-set automata

Idea: study diagrams

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Can we model finiteness of the store?

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$$\mathrm{supp}(x) = \{a \in A \mid \{b \in A \mid x \cdot (ab) \neq x\} \text{ is infinite}\}$$

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$$Q = orbits(X)$$
$$R_q = \mathrm{supp}(x) \quad S_q = G_x \qquad (x \in q)$$

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Q: What if the alphabet is equipped with

- a total order,
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which we can check for?

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A: Repeat the theory with some $G \leq Sym(A)$

E.g. $G =$ monotone bijections of $\mathbb{Q}$

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Caution: least supports might not exist.

Representing G-nominal sets

Automaton:

- orbit-finite G-set X
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Idea: - $\mathbb{N}$ universally embeds finite sets

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a class of finite rel. structures closed under
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We use $G = Aut(\mathcal{U}) \leq Sym(|\mathcal{U}|)$

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- transition function: ....
- initial, accepting states

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Is
$\mathbf{P} = \mathbf{NP}$
in nominal sets?